TERMS YOU SHOULD KNOW: *divisibility, prime numbers, the fundamental theorem of arithmetic.*

1. True or False, and why?

   (a) $7 \mid 84$.

   (b) $5 \mid 84$.

   (c) $7 \mid 84 \vee 5 \mid 84$.

   (d) $7 \mid 84 \wedge 5 \mid 84$.

   (e) $7 \mid 84 \Rightarrow 5 \mid 84$.

   (f) $5 \mid 84 \Rightarrow 7 \mid 84$.

   (g) $(2^3 3^{11} 7^{99}) \mid (2^5 3^{11} 5^8 7^{900})$.

   (h) $(2^3 3^{11} 7^{99}) \mid (2^5 3^{10} 5^8 7^{900})$.

   (i) $(p^3 q^{11} r^{99}) \mid (p^5 q^{11} s^8 r^{900})$, where p, q, r, and s are all primes.

   (j) $(p^3 q^{11} r^{99}) \mid (p^5 q^{10} s^8 r^{900})$, where p, q, r, and s are all primes.

2. What are the divisors of $84 = 2^2 \cdot 3 \cdot 7$?

3. The following statement is false:

   If $a \mid bc$, then $a \mid b$ or $a \mid c$.

   Find a counterexample to this statement, that is, find three positive integers a, b, and c so that $a \mid bc$ but $a \nmid b$ and $a \nmid c$.

   Hint. If a is prime, then the statement is true.

4. Find the prime factorization of each of the following numbers.

   (a) 21

   (b) 21,000

   (c) 26,460

   (d) 1,000,000

   (e) 3,000,000

1a. True, since $7 \cdot 12 = 84$.
1b. False. $5x = 84$ has no integer solution.
1c. True, since $7|84$.
1d. False, since $5 \not| 84$.
1e. False, since the hypothesis is true but the conclusion is false.
1f. True, because the hypothesis is false.
1g. True. There are at least as many of each prime in the second number as there is in the first number.
1h. False. The first number has 11 3's, while the second number has only 10 3's, so there is no way the second number is a multiple of the first number.
1i. True. This is essentially the same problem as 1g.
1j. False. This is essentially the same problem as 1h.

2. Since $84 = 2^2 \cdot 3 \cdot 7$, all the divisors of 84 must be of the form $2^a 3^b 7^c$, where $a = 0, 1, 2$, $b = 0, 1$, and $c = 0, 1$. Listing all the possibilities gives

$$\begin{aligned}
2^0 3^0 7^0 &= 1 \\
2^1 3^0 7^0 &= 2 \\
2^2 3^0 7^0 &= 4 \\
2^0 3^1 7^0 &= 3 \\
2^0 3^0 7^1 &= 7 \\
2^1 3^1 7^0 &= 6 \\
2^1 3^0 7^1 &= 14 \\
2^2 3^1 7^0 &= 12 \\
2^2 3^0 7^1 &= 28 \\
2^0 3^1 7^1 &= 21 \\
2^1 3^1 7^1 &= 42 \\
2^2 3^1 7^1 &= 84
\end{aligned}$$

So, in increasing order, the divisors of 84 are $1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84$.

3. The key is in the hint. Take any composite number for a; I'll use $a = 2 \cdot 3 = 6$. Now I want 6 to divide bc but not to divide either b or c individually.

The easiest way to do this is to take $b = 2$ and $c = 3$, but there are lots of counterexamples (infinitely many, in fact). For each, just come up with a b so that $2|b$ but $3 \not| b$ and a c so that $3|c$ but $2 \not| c$. (So, b has at least one 2 but no 3's in it, and c has at least one 3 but no 2's in it.) So, some other counterexamples are $a = 6$ and

$$\begin{aligned} b = 2 \cdot 5 = 10 &\quad \text{and} \quad c = 3 \cdot 7 = 21 \\ b = 2^2 \cdot 5 \cdot 7 = 140 &\quad \text{and} \quad c = 3 \cdot 7 = 21 \\ b = 2^7 = 128 &\quad \text{and} \quad c = 3^3 = 28, \end{aligned}$$

etc.

4a. $3 \cdot 7$
4b. $2^3 \cdot 3 \cdot 5^3 \cdot 7$
4c. $2^2 \cdot 3^3 \cdot 5 \cdot 7^2$
4d. $2^6 \cdot 5^6$
4e. $2^6 \cdot 3 \cdot 5^6$