TERMS YOU SHOULD KNOW: *the number of divisors of a number, gcd's, lcm's.*

1. Find the *number* of divisors of each of the following numbers. *Note:* You don't have to find the divisors themselves. Some of these numbers look remarkably similar to some in the last section.
   (a) 21
   (b) 21,000
   (c) 26,460
   (d) 1,000,000
   (e) 3,000,000
   (f) p^{111}, where p is prime.
   (g) $p^3q^4r^5$, where p, q, and r are all prime.
   (h) n^3m^4, where n and m are any positive integers.
2. A number has exactly 18 divisors.
   (a) What are the possible forms of its prime factorization?
   (b) What is the smallest number with exactly 18 divisors?
   (c) What is the largest number with exactly 18 divisors?
3. Find the least common multiple (lcm) and the greatest common divisor (gcd) of each pair of numbers.
   (a) 21 and 14
   (b) 2^35^9 and $2^43^75^2$
   (c) 1,000,000 and 3,000,000
   (d) 1 and n, where n is any positive integer.
   (e) p^{111} and p^{112}, where p is prime.
   (f) $p^3q^4r^5$ and pq^9r^2s, where p, q, r, and s are all prime.
   (g) 21,000 and 26,460

1a. Since $21 = 3 \cdot 7$, this has $(1+1)(1+1) = 4$ divisors.
1b. Since $21,000 = 2^3 \cdot 3 \cdot 5^3 \cdot 7$, this has $(4)(2)(4)(2) = 64$ divisors.
1c. Since $26,460 = 2^2 \cdot 3^3 \cdot 5 \cdot 7^2$, this has $(3)(4)(2)(3) = 72$ divisors.
1d. Since $1,000,000 = 2^6 \cdot 5^6$, this has $(7)(7) = 49$ divisors.
1e. Since $3,000,000 = 2^6 \cdot 3 \cdot 5^6$, this has $(7)(2)(7) = 98$ divisors.
1f. $111 + 1 = 112$.
1g. $(4)(5)(6) = 120$.
1h. Since we don't know if n and m are prime, there is no way to know how many divisors $n^3 m^4$ has.

2a. Well, $18 = 2 \cdot 3^2$, so we can figure out the different ways to partition one 2 and two 3's:

$$\begin{aligned}
(2 \cdot 3^2) &= (17+1) \\
(2)(3^2) &= (1+1)(8+1) \\
(2 \cdot 3)(3) &= (5+1)(2+1) \\
(2)(3)(3) &= (1+1)(2+1)(2+1)
\end{aligned}$$

So, using the formulat for $d(n)$ and the partitions above, the four possible prime factorizations of a number with exactly 18 divisors are, respectively,

$$p^{17}$$
$$p \cdot q^8$$
$$p^5 \cdot q^2$$
$$p \cdot q^2 \cdot r^2$$

2b. Using the previous answer, just plug in the smallest possible primes into each form and see which gives the smallest number. So, we compare 2^{17}, $3 \cdot 2^8$, $2^5 \cdot 3^2$, and $5 \cdot 2^2 \cdot 3^2$. The smallest is the last one, which equals 180.
2c. There is no such largest number, since there is no largest prime.

3a. Since $21 = 3 \cdot 7$ and $14 = 2 \cdot 7$, their lcm is $2 \cdot 3 \cdot 7 = 42$ and their gcd is 7.
3b. Their lcm is $2^4 3^7 5^9$ and their gcd is $2^3 5^2$.

3c. Since 1,000,000 divides 3,000,000, their lcm is 3,000,000 and their gcd is 1,000,000.
3d. Their lcm is n and their gcd is 1.
3e. Since p^{111} divides p^{112}, their lcm is p^{112} and their gcd is p^{111}.
3f. Their lcm is $p^3q^9r^5s$ and their gcd is pq^4r^2.
3g. Since $21,000 = 2^3 \cdot 3 \cdot 5^3 \cdot 7$ and $26,460 = 2^2 \cdot 3^3 \cdot 5 \cdot 7^2$, their lcm is $2^3 \cdot 3^3 \cdot 5^3 \cdot 7^2$ and their gcd is $2^2 \cdot 3 \cdot 5 \cdot 7$.