TERMS YOU SHOULD KNOW: *mutually exclusive, independent*

1. Suppose E and F are events with $P(E) = 0.29$ and $P(F) = 0.43$.
   (a) If $P(E \cap F) = 0.15$, then what is $P(E \cup F)$?
   (b) If E and F are mutually exclusive, what is $P(E \cup F)$?
   (c) If E and F are independent, what is $P(E \cup F)$?
   (d) What is $P(E')$?

2. If you roll a pair of dice, what is the probability that the sum of the numbers on the dice is greater than 3?

3. One number is chosen at random from the set $\{1, 2, 3, \ldots, 49, 50\}$. What is the probability that:
   (a) it is divisible by 6?
   (b) it is divisible by 8?
   (c) it is divisible by 6 and by 8?
   (d) it is divisible by 6 or by 8?

4. An opaque bottle contains 5 black marbles and 2 green marbles. By shaking the bottle, only one marble will fall out at a time.

   You take this bottle down to the Commons and make a bet: You let anyone shake out 3 marbles. If at least one is green, they pay you \$1.00; if not, you pay them \$1.00.

   What's the probability that you win the \$1.00?

5. Show that in a class of 23 students, the probability that at least 2 of them have the same birthday is greater than $\frac{1}{2}$. (Not the same year, just the same day. Ignore the existence of leap years.)

1a. $P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.29 + 0.43 - 0.15 = 0.57$
1b. $0.29 + 0.43 - 0 = 0.72$
1c. $0.29 + 0.43 - (0.29)(0.43) \approx 0.60$
1d. $1 - 0.29 = 0.71$.

2. Here is what we have:

$$\begin{aligned} E &: \text{ the sum of the dice is more than 3} \\ E' &: \text{ the sume of the dice is 2 or 3} \end{aligned}$$

Since $E' = \{(1,1),(1,2),(2,1)\}$, $P(E') = \frac{3}{36} = \frac{1}{12}$. So $P(E) = \frac{11}{12}$.

3a. The multiples of 6 are $6 = 6 \cdot 1, 12 = 6 \cdot 2, \ldots, 48 = 6 \cdot 8$. Since there are 8 of them, the probability is $\frac{8}{50} = \frac{4}{25}$.
3b. The multiples of 8 are $8 = 8 \cdot 1, 16 = 8 \cdot 2, \ldots, 48 = 8 \cdot 6$. Since there are 6 of them, the probability is $\frac{6}{50} = \frac{3}{25}$.
3c. If a number is divisible by 6 and by 8, then it is divisible by their lcm, which is $2^3 \cdot 3 = 24$. So the numbers divisible by both 6 and 8 are 24 and 48. So the probability is $\frac{2}{50} = \frac{1}{25}$.
3d. We could count all these, but it is easier to let

$$\begin{aligned} E &: \text{ the number is divisible by 6} \\ F &: \text{ the number is divisible by 8} \end{aligned}$$

Then

$$\begin{aligned} P(\text{ divisible by either 6 or 8 }) &= P(E \cup F) \\ &= P(E) + P(F) - P(E \cap F) \\ &= \frac{8}{50} + \frac{6}{50} - \frac{2}{50} \\ &= \frac{12}{50} \\ &= \frac{6}{25} \end{aligned}$$

4. You pay the person \$1.00 if they shake out 3 black marbles. The probability that they shake out 3 black marbles is

$$\begin{aligned}
\frac{{}_5C_3}{{}_7C_3} &= \frac{\frac{5!}{3!2!}}{\frac{7!}{3!4!}} \\
&= \frac{5\cdot 4\cdot 3}{7\cdot 6\cdot 5} \\
&= \frac{2}{7}.
\end{aligned}$$

So the probability that they shake out at least one green marble is

$$1-\frac{2}{7}=\frac{5}{7}\approx 0.71.$$

Since *you* win in this case, the probability that you win \$1.00 is 0.71.

5. First note that

$$P(\text{ at least 2 b-days are the same }) = 1-P(\text{ all 23 b-days are different }).$$

How many different ways are there to pick 23 *different* birthdays? Since we are picking different birthdays, we are picking 23 days from the 365 days in the year without replacement. If we want to pick so order matters, then we get ${}_{365}P_{23}$ different ways.

So, how many ways are there to pick 23 birthdays in general if order matters? Since we are picking with replacement (we can use the same day more than once), there are 365^{23} different ways (365 for the first, 365 for the second, 365 for the third, etc.).

So we have

$$\begin{aligned}
P(\text{ at least 2 b-days are the same }) &= 1-P(\text{ all 23 b-days are different }) \\
&= 1-\frac{{}_{365}P_{23}}{365^{23}} \\
&= 1-\frac{\frac{365!}{342!}}{365^{23}} \\
&= 1-\frac{365\cdot 364\cdot\cdots\cdot 343}{365^{23}} \\
&\approx 0.5073 \\
&> \frac{1}{2}
\end{aligned}$$