1. Let $-$ be the usual subtraction on the set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ of integers. Does $-$ have an identity?

2. Determine whether the following binary operations $*$ on $S = \{0, 1\}$ have an identity. For those that do, determine whether or not each element has an inverse.

   (a)

$*$	0	1
0	1	0
1	0	1

   (b)

$*$	0	1
0	1	0
1	1	1

   (c)

$*$	0	1
0	0	1
1	1	1

3. Below is a table for a binary operation $\circ$ on the set $S = \{a, b, c, d, f, g, h\}$.

$\circ$	a	b	c	d	f	g	h
a	g	a	h	b	d	c	f
b	a	b	c	d	f	g	h
c	h	c	d	g	a	f	b
d	b	d	g	f	h	a	c
f	d	f	a	h	c	b	g
g	c	g	f	a	b	h	d
h	f	h	b	c	g	d	a

   (a) What is the identity for $\circ$?

   (b) Pair each element of S with its inverse.

   (c) Write a table for $\circ$ using only the elements $\{b, d, h\}$.

4. $D_6 = \{I, S, S^2, S^3, S^4, S^5, R_1, R_2, R_3, R_4, R_5, R_6\}$ is the symmetry group of the regular hexagon. Recall each R_i is a reflection and $S = R_1 R_2$ is the minimal counterclockwise rotation. Since D_6 is a group, its binary operation is associative, it has an identity, and every element has an inverse. Pair each element of D_6 with its inverse.

5. Look at the following table.

$*$	a	b	c	d	f
a	c	d	a	f	b
b	f	c	b	a	d
c	a	b	c	d	f
d	b	f	d	c	a
f	d	a	f	b	c

   (a) Find the identity.

   (b) Pair every element with its inverse.

   (c) If this is *not* the table for a group, then what can you tell me about $*$?

6. Let G be a group under $*$ (so, in particular, every element of G has an inverse). If G has an even number of elements, prove there must be at least one element, other than the identity e, that is its own inverse.

1. No. The only possibility would be 0. While it is true that for any $n \in \mathbb{Z}$, $n - 0 = n$, 0 is *not* an identity because $0 - n = -n$. (So, for example, $0 - 7 = -7 \neq 7$.)

2a. 1 is the identity, so it is its own inverse. 0 is also its own inverse.
2b. There is no identity, and therefore no inverses.
2c. 0 is the identity, so it is its own inverse. 1 has no inverse.

3a. The identity is b.
3b. $b \leftrightarrow b$, $a \leftrightarrow d$, $c \leftrightarrow h$, $f \leftrightarrow g$.
3c.

$\circ$	b	d	h
b	b	d	h
d	d	f	c
h	h	c	a

Notice you need elements other than b, d, and h to fill in this table.

4. Since every reflection is its own inverse, $R_i \leftrightarrow R_i$ for each i. As always, $I \leftrightarrow I$. For the nontrivial rotations, $S \leftrightarrow S^5$, $S^2 \leftrightarrow S^4$, and $S^3 \leftrightarrow S^3$. (Notice the sum of the exponents is always 6.)

5a. The identity is c.
5b. Every element is its own inverse.
5c. $*$ cannot be associative, because if it *were* associative, then this *would* be a group.

In fact, $*$ is not associative, so this is not a group. For example, $(f * b) * d = a * d = f$, while $f * (b * d) = f * a = d$.

6. Since there are an *odd* number of elements in G other than the identity (which is its own inverse) and elements come in inverse pairs, something *else* must be paired with itself. That element is its own inverse.