1. Let $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ be the group of integers under addition. Give an example of:

   (a) A finite subset that is closed.

   (b) A finite subset that isn't closed.

   (c) An infinite subset that is closed.

   (d) An infinite subset that isn't closed.

2. Complete the following table so that the set has a chance of being a *commutative* group.

$\circ$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d			c
b	b					e
c	c	f		d		
d	d		f	e		
f	f					

3. Let $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ be the usual group under "clock addition."

   (a) Make a table for $\mathbb{Z}_7$.

   (b) Show that each of the elements $1, 2, 3, 4, 5, 6$ "generates" all of $\mathbb{Z}_7$ by adding each to itself over and over until you get the whole set. (For example, $2+2=4$, $2+2+2=6$, $2+2+2+2=1$, etc.)

   (c) Conclude that the only subgroups of $\mathbb{Z}_7$ are $\{0\}$ and all of $\mathbb{Z}_7$.

4. We have seen that there are 16 different binary operation on the set $S = \{0, 1\}$ (one for each of the possible 2×2 multiplication tables – see your notes). If I tell you that exactly two of these are groups, which two are they?

5. Below is the table for D_4. Find all the subgroups you can. (As you will see in the next section, a group with 8 elements cannot have subgroups with 3,

5, 6, or 7 elements, so there is no need to check subsets of those sizes.)

	I	S	S^2	S^3	R_1	R_2	R_3	R_4
I	I	S	S^2	S^3	R_1	R_2	R_3	R_4
S	S	S^2	S^3	I	R_4	R_1	R_2	R_3
S^2	S^2	S^3	I	S	R_3	R_4	R_1	R_2
S^3	S^3	I	S	S^2	R_2	R_3	R_4	R_1
R_1	R_1	R_2	R_3	R_4	I	S	S^2	S^3
R_2	R_2	R_3	R_4	R_1	S^3	I	S	S^2
R_3	R_3	R_4	R_1	R_2	S^2	S^3	I	S
R_4	R_4	R_1	R_2	R_3	S	S^2	S^3	I

1a. $\{0\}$ is the only one.
1b. Any set other than $\{0\}$ isn't closed. If a set contains some $k \neq 0$, then (if it is closed), it must also contain $k+k=2k$, $k+k+k=3k$, etc., so it must contain $\{k, 2k, 3k, 4k, \ldots\}$, so it must be infinite.
1c. The set of even integers. Or, more generally, the set of multiples of a given integer k. (For example, multiples of 5.)
1d. The set of even integers except for 4 is one example. Generally, take an infinite set that is closed and toss something out.

2.

$\circ$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	d	c	a	f	e
c	c	f	a	d	e	b
d	d	b	f	e	c	a
f	f	c	e	b	a	d

3a.

$\circ$	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

3b. Just grab an element and keep adding it to itself; they all generate the whole group. For example, starting with 3: $3=3$, $3+3=2\cdot 3=6$, $3+3+3=3\cdot 3=2$, $4\cdot 3=5$, $5\cdot 3=1$, $6\cdot 3=4$, $7\cdot 3=0$.
3c. If any closed subset contains something other than zero, then by the previous problem we know it will contain all of $\mathbb{Z}_7$. Since the only closed subsets are $\{0\}$ and all of Z_7, those are the only possible subgroups. (And, as always, the set consisting of the identity alone and the whole group *are* subgroups.)

4. There are only two that have each element in each row and each column:

$*$	0	1
0	0	1
1	1	0

$*$	0	1
0	1	0
1	0	1

Each of these is structurally the same as $\mathbb{Z}_2$: the first one *is* $\mathbb{Z}_2$, and the second one is $\mathbb{Z}_2$ if the roles of 0 and 1 are interchanged. So, there is essentially only one group with two elements.

5. Here are all the subgroups:

- One element subgroups: $\{I\}$
- Two element subgroups: $\{I, R_i\}$ for each i and $\{I, S^2\}$
- Four element subgroups: $\{I, S, S^2, S^3\}$, $\{I, S^2, R_2, R_4\}$, and $\{I, S^2, R_1, R_3\}$.
- Eight element subgroups: D_4