1. Here is a table for a group G.

$\circ$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	d	c	a	f	e
c	c	f	a	d	e	b
d	d	b	f	e	c	a
f	f	c	e	b	a	d

   (a) Find all the subgroups of G.

   (b) For each of the subgroups you found in 1a, find all of its (left) cosets.

2. Below is the table for D_4. In Section 5.3, you found all the subgroups of D_4. Find all of the (left) cosets of each subgroup.

	I	S	S^2	S^3	R_1	R_2	R_3	R_4
I	I	S	S^2	S^3	R_1	R_2	R_3	R_4
S	S	S^2	S^3	I	R_4	R_1	R_2	R_3
S^2	S^2	S^3	I	S	R_3	R_4	R_1	R_2
S^3	S^3	I	S	S^2	R_2	R_3	R_4	R_1
R_1	R_1	R_2	R_3	R_4	I	S	S^2	S^3
R_2	R_2	R_3	R_4	R_1	S^3	I	S	S^2
R_3	R_3	R_4	R_1	R_2	S^2	S^3	I	S
R_4	R_4	R_1	R_2	R_3	S	S^2	S^3	I

3. If G is a group such that $|G| = 300$, *how many* possible different sizes of subgroups of G are there?

1a. By Lagrange's Theorem, the only possible sizes of subgroups are 1, 2, 3, and 6. Each must be closed, have e in them, and have all their inverses. So,

- One element subgroups: $\{e\}$
  (This is always the only one.)
- Two element subgroups: $\{e,a\}$
  (These must consist of e and an element that is its own inverse.)
- Three element subgroups: $\{e,c,d\}$
  (These must consists of e and two elements that are their own inverses or are inverses of each other, and must also be closed. Note that the subset $\{e,b,f\}$ is *not* a subgroup: although b and f are inverses of each other, $\{e,b,f\}$ is *not* closed since $b \circ b = c \notin \{e,b,f\}$.)
- Six element subgroups: G
  (This is always the only one.)

1b.

- Cosets of $\{e\}$:

$$\begin{aligned} e \circ \{e\} &= \{e\} \\ a \circ \{e\} &= \{a\} \\ b \circ \{e\} &= \{b\} \\ c \circ \{e\} &= \{c\} \\ d \circ \{e\} &= \{d\} \\ f \circ \{e\} &= \{f\} \end{aligned}$$

- Cosets of $\{e,a\}$:

$$\begin{aligned} e \circ \{e,a\} = a \circ \{e,a\} &= \{e,a\} \\ b \circ \{e,a\} = d \circ \{e,a\} &= \{b,d\} \\ c \circ \{e,a\} = f \circ \{e,a\} &= \{c,f\} \end{aligned}$$

- Cosets of $\{e,c,d\}$:

$$\begin{aligned} e \circ \{e,c,d\} = c \circ \{e,c,d\} = d \circ \{e,c,d\} &= \{e,c,d\} \\ a \circ \{e,c,d\} = b \circ \{e,c,d\} = f \circ \{e,c,d\} &= \{a,b,f\} \end{aligned}$$

- Cosets of G:

$$e \circ G = a \circ G = b \circ G = c \circ G = d \circ G = f \circ G = G$$

2. As always, there are as many cosets of $\{I\}$ as there are elements in the group, in this case 8. (See Problem 1b.) And there is only 1 coset of D_4, namely D_4 itself. Here are the cosets of the other subgroups:

- Cosets of $\{I, R_1\}$:

$$\begin{aligned}
&I \circ \{I, R_1\} = R_1 \circ \{I, R_1\} = \{I, R_1\} \\
&S \circ \{I, R_1\} = R_4 \circ \{I, R_1\} = \{S, R_4\} \\
&S^2 \circ \{I, R_1\} = R_3 \circ \{I, R_1\} = \{S^2, R_3\} \\
&S^3 \circ \{I, R_1\} = R_2 \circ \{I, R_1\} = \{S^3, R_2\}
\end{aligned}$$

- Cosets of $\{I, R_2\}$:

$$\begin{aligned}
&I \circ \{I, R_2\} = R_2 \circ \{I, R_2\} = \{I, R_2\} \\
&S \circ \{I, R_2\} = R_1 \circ \{I, R_2\} = \{S, R_1\} \\
&S^2 \circ \{I, R_2\} = R_4 \circ \{I, R_2\} = \{S^2, R_4\} \\
&S^3 \circ \{I, R_2\} = R_3 \circ \{I, R_2\} = \{S^3, R_3\}
\end{aligned}$$

- Cosets of $\{I, R_3\}$:

$$\begin{aligned}
&I \circ \{I, R_3\} = R_3 \circ \{I, R_3\} = \{I, R_3\} \\
&S \circ \{I, R_3\} = R_2 \circ \{I, R_3\} = \{S, R_2\} \\
&S^2 \circ \{I, R_3\} = R_1 \circ \{I, R_3\} = \{S^2, R_1\} \\
&S^3 \circ \{I, R_3\} = R_4 \circ \{I, R_3\} = \{S^3, R_4\}
\end{aligned}$$

- Cosets of $\{I, R_4\}$:

$$\begin{aligned}
&I \circ \{I, R_4\} = R_4 \circ \{I, R_4\} = \{I, R_4\} \\
&S \circ \{I, R_4\} = R_3 \circ \{I, R_4\} = \{S, R_3\} \\
&S^2 \circ \{I, R_4\} = R_2 \circ \{I, R_4\} = \{S^2, R_2\} \\
&S^3 \circ \{I, R_4\} = R_1 \circ \{I, R_4\} = \{S^3, R_1\}
\end{aligned}$$

- Cosets of $\{I, S^2\}$:

$$\begin{aligned}
&I \circ \{I, S^2\} = S^2 \circ \{I, S^2\} = \{I, S^2\} \\
&R_1 \circ \{I, S^2\} = R_3 \circ \{I, S^2\} = \{R_1, R_3\} \\
&R_2 \circ \{I, S^2\} = R_4 \circ \{I, S^2\} = \{R_1, R_4\} \\
&S \circ \{I, S^2\} = S^3 \circ \{I, S^2\} = \{S, S^3\}
\end{aligned}$$

- Cosets of $\{I, S, S^2, S^3\}$:

$$\begin{aligned} I \circ \{I, S, S^2, S^3\} &= S \circ \{I, S, S^2, S^3\} = S^2 \circ \{I, S, S^2, S^3\} \\ &= S^3 \circ \{I, S, S^2, S^3\} = \{I, S, S^2, S^3\} \end{aligned}$$

$$\begin{aligned} R_1 \circ \{I, S, S^2, S^3\} &= R_2 \circ \{I, S, S^2, S^3\} = R_3 \circ \{I, S, S^2, S^3\} \\ &= R_4 \circ \{I, S, S^2, S^3\} = \{R_1, R_2, R_3, R_4\} \end{aligned}$$

- Cosets of $\{I, S^2, R_2, R_4\}$:

$$\begin{aligned} I \circ \{I, S^2, R_2, R_4\} &= S^2 \circ \{I, S^2, R_2, R_4\} = R_2 \circ \{I, S^2, R_2, R_4\} \\ &= R_4 \circ \{I, S^2, R_2, R_4\} = \{I, S^2, R_2, R_4\} \end{aligned}$$

$$\begin{aligned} S \circ \{I, S^2, R_2, R_4\} &= S^3 \circ \{I, S^2, R_2, R_4\} = R_1 \circ \{I, S^2, R_2, R_4\} \\ &= R_3 \circ \{I, S^2, R_2, R_4\} = \{S, S^3, R_1, R_3\} \end{aligned}$$

- Cosets of $\{I, S^2, R_1, R_3\}$:

$$\begin{aligned} I \circ \{I, S^2, R_1, R_3\} &= S^2 \circ \{I, S^2, R_1, R_3\} = R_1 \circ \{I, S^2, R_1, R_3\} \\ &= R_3 \circ \{I, S^2, R_1, R_3\} = \{I, S^2, R_1, R_3\} \end{aligned}$$

$$\begin{aligned} S \circ \{I, S^2, R_1, R_3\} &= S^3 \circ \{I, S^2, R_1, R_3\} = R_2 \circ \{I, S^2, R_1, R_3\} \\ &= R_4 \circ \{I, S^2, R_1, R_3\} = \{S, S^3, R_2, R_4\} \end{aligned}$$

3. By Lagrange's Theorem, $|H|$ must divide $|G|$ if $H \leq G$. Since $300 = 2^2 \cdot 3 \cdot 5^2$, the number of divisors of $|G|$ is $d(300) = 3 \cdot 2 \cdot 3 = 18$. (Remember that from Chapter 2?) So $|H|$ could be one of 18 different sizes, one for each divisor of G.

Note. Lagrange's Theorem does not say that there *must* be a subgroup of each of these 18 sizes, only that there *could* be a subgroup of each of these 18 sizes.