1. Prove every group with 77 elements can have at most 4 different-sized subgroups.

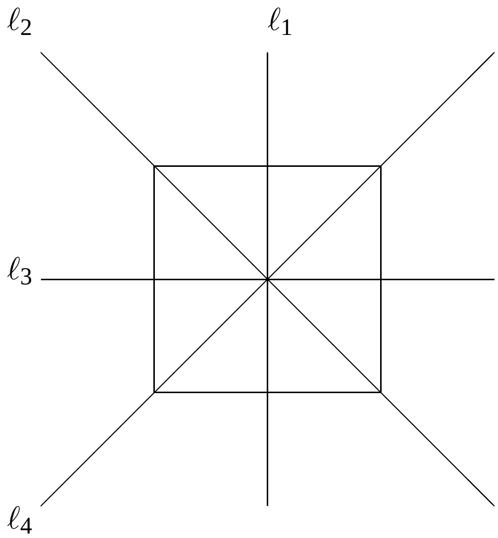
2. Let G be a group under $*$ and let $a \in G$. The *order* of a is the smallest positive number k so that

$$\underbrace{a * a * a * \cdots * a}_{k \text{ times}} = e.$$

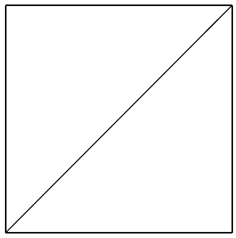
   For example, in D_4, the order of S is 4 since $S^4 = I$ but neither S^2 nor S^3 is equal to I.

   Prove the order of any element of a group G must divide $|G|$.

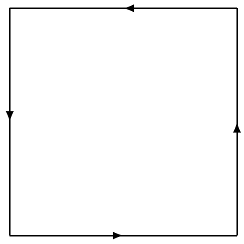
3. The symmetry group of the square is $D_4 = \{I, S, S^2, S^3, R_1, R_2, R_3, R_4\}$, where each reflection R_i is about the line ℓ_i below, and $S = R_1 R_2$.



   (a) Let H be the subgroup of D_4 that fixes the following altered figure. Find the elements of H and all its (left) cosets.



(b) Let K be the subgroup of D_4 that fixes the following altered figure. Find the elements of K and all its (left) cosets.



1. If $|G| = 77 = 7 \cdot 11$, then by Lagrange's Theorem, the only possible sizes of subgroups of G are 1, 7, 11, and 77. So there are at most 4 sizes of subgroups of G.

2. The key is that the set $\{a, a^2, a^3, \ldots a^{k-1}, a^k = e\}$ is a subgroup of G with k elements. (It is closed, has the identity, and every element has an inverse in the set.) So, by Lagrange's Theorem, k must divide $|G|$.

3a. $H = \{I, R_2, R_4, S^2\}$. The 2 cosets of H are H and $\{S, S^3, R_1, R_3\}$.
3b. $K = \{I, S, S^2, S^3\}$. The 2 cosets of K were found in Section 4: K and $\{R_1, R_2, R_3, R_4\}$.