Presentations of Subgroups of Artin Groups

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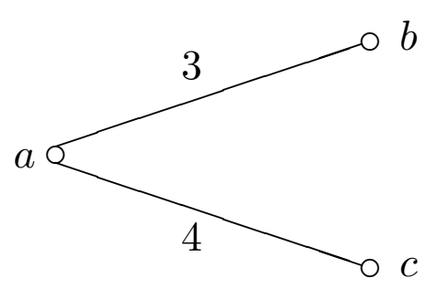
Overview

Let $A\Gamma$ be the Artin group based on the graph Γ, and let $\phi : A\Gamma \longrightarrow \mathbb{Z}$ be a homomorphism which maps each of the standard generators of $A\Gamma$ to 0 or 1. We compute an explicit presentation for $\ker \phi$ and determine graph theoretical conditions on Γ which ensure the finite presentation of $\ker \phi$ when Γ is a cone, a tree, a triangle, and a special tree-triangle combination.

Artin Groups

Let Γ be a finite simple graph whose edges are weighted with integers greater than 1. Then we can associate a group $A\Gamma$ with generators in one-to-one correspondence with the vertices of Γ and relations $[x,y]_k = [y,x]_k$ for each edge $\{x,y\}$ of weight k, where $[x,y]_k = \underbrace{xyx\ldots}_{k \text{ letters}}$. Such a group $A\Gamma$ is called an **Artin group** and Γ is its **underlying graph**.

Example:



Γ

$$A\Gamma = \langle a, b, c \mid aba = bab, acac = caca \rangle$$

Special Homomorphisms Onto $\mathbb{Z}$

Given Artin group $A\Gamma$, partition the vertices of Γ into two sets, $L = \{l_1, l_2, \ldots, l_n\}$ and $D = \{d_1, d_2, \ldots, d_n\}$. Define the homomorphism $\phi : A\Gamma \longrightarrow \langle t \rangle$ by $\phi(l_i) = t$ and $\phi(d_i) = 1$ for all i.

Definition. We call a vertex $l_i \in L$ **live** and a vertex $d_i \in D$ **dead**.

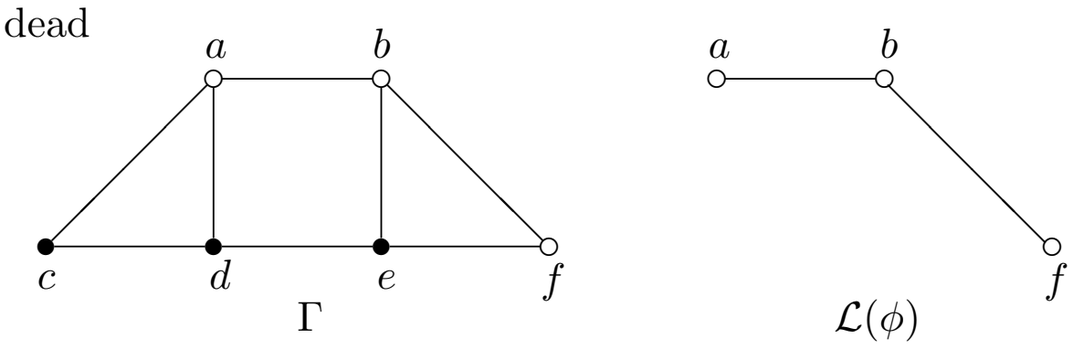
Note: For convenience, all following epimorphisms, ϕ, are of the type defined above.

Graph Terms

Definition. The **living subgraph** of Γ, denoted $\mathcal{L}(\phi)$, is the full subgraph spanned by the living vertices of Γ.

Example:

○ living
● dead



Definition. A subgraph Γ' of a graph Γ is **dominating** if every vertex in $\Gamma - \Gamma'$ is adjacent to a vertex in Γ'.

Note: $\mathcal{L}(\phi)$ above is a connected and dominating subgraph of Γ.

Machinery

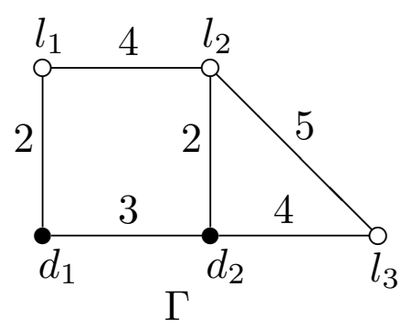
Tietze Transformations:

Given any two presentations of a group G, one can be obtained from the other by repeated applications of Tietze transformations.

Reidemeister-Schreier Rewriting Process

Given a presentation of a group G and suitable information about a subgroup $H \leq G$, the Reidemeister-Schreier method enables one to obtain a presentation for H.

Example: Let $A\Gamma$ and homomorphism ϕ be defined according to the diagram below.



Using the Reidemeister-Schreier process and Tietze transformations, one can obtain the following presentation for the homomorphism, ϕ, indicated in the graph above:

$$
\begin{aligned}
\text{generators:}\quad & \delta, \upsilon_0, \upsilon_1, \upsilon_2, \upsilon_3, \lambda_0, \lambda_1, \lambda_2, \psi_0 \\
\text{relations:}\quad & \Psi_n \delta \Psi_n = \delta \Psi_n \delta \\
& \delta \Upsilon_n \delta \Upsilon_{n+1} = \Upsilon_n \delta \Upsilon_{n+1} \delta \\
\text{where: } \Psi_n &= \begin{cases} \Lambda_n \psi_{n+1} \Lambda_n^{-1} & \text{for } n < 0 \\ \Lambda_{n-1} \psi_{n-1} \Lambda_{n-1}^{-1} & \text{for } n > 0 \end{cases} \\
\Lambda_n &= \begin{cases} \lambda_{n+1} \lambda_{n+3} \lambda_{n+2}^{-1} & \text{for } n < 0 \\ \lambda_{n-2}^{-1} \lambda_{n-3} \lambda_{n-1} & \text{for } n > 2 \end{cases} \\
\Upsilon_n &= \begin{cases} \upsilon_{n+1} \upsilon_{n+3} \upsilon_{n+4}^{-1} \upsilon_{n+2}^{-1} & \text{for } n < 0 \\ \upsilon_{n-2} \upsilon_{n-4} \upsilon_{n-3}^{-1} \upsilon_{n-1}^{-1} & \text{for } n > 3 \end{cases}
\end{aligned}
$$

Notice that $\ker \phi$ is finitely generated.

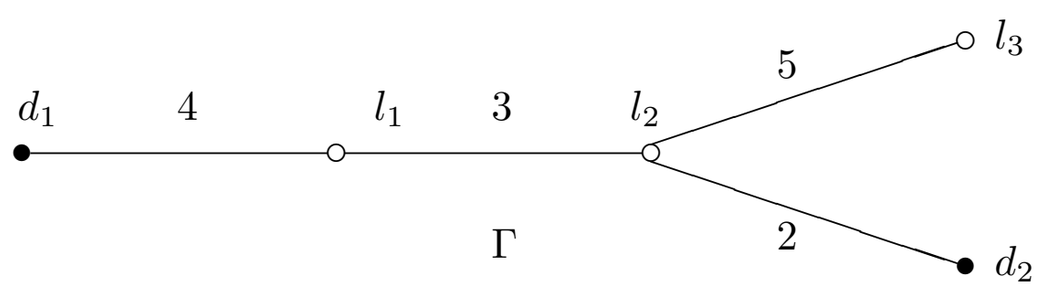
Finitely presented?

Trees

Theorem (S. Hermiller, J. Meier, 1996): Let Γ be a finite, weighted tree, and let $\phi : A\Gamma \longrightarrow \langle t \rangle$ be an epimorphism which sends each generator to t. Then $\ker \phi$ is the free group on $N = \sum_{e_i \in \Gamma}(W(e_i) - 1)$ generators where $W(e)$ represents the weight of edge e.

Theorem: Let T be a finite, weighted tree and let $\phi : AT \longrightarrow \langle t \rangle$ be an epimorphism which sends each generator to t or 1. If $\mathcal{L}(\phi)$ is connected and dominating, then $\ker \phi$ is the free group on $N = \sum_{e_i \in \mathcal{L}(\phi)} (W(e_i) - 1) + \sum_{e_j \notin \mathcal{L}(\phi)} \frac{W(e_j)}{2}$ generators, where $W(e)$ represents the weight of edge e.

Example: Define $\phi : A\Gamma \longrightarrow \langle t \rangle$ by $\phi(l_i) = t$ and $\phi(d_i) = 1$ where Γ is as follows. Then by the above theorem, $\ker \phi \simeq F_9$.



Constructible Graphs

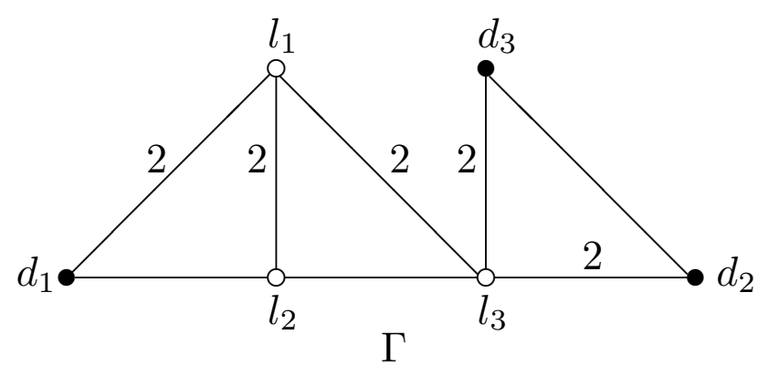
Theorem (J. Levy, C. Parker, L. VanWyk 1995): Let Γ be a finite, simple graph whose vertices are partitioned into sets $L = \{l_1, l_2, \ldots, l_r\}$ and $D = \{d_1, d_2, \ldots, d_s\}$. Let $\phi : G\Gamma \longrightarrow \langle t \rangle$ be an epimorphism defined by $\phi(l_i) = t$, $\phi(d_i) = 1$, $\forall\, i$. Then $\ker \phi$ is finitely presented if there exists a sequence of subgraphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ of Γ where Γ_1 consists of a single vertex of L and $\Gamma_n = \Gamma$ such that Γ_{j+1} can be obtained from Γ_j by either

1. adding vertex v and edge $\{v, l_i\}$ for some $l_i \in \Gamma_j$.
2. adding edge $\{a, b\}$ where $a, b, l_i \in V(\Gamma_j)$; $\{a, l_i\}, \{b, l_i\} \in E(\Gamma_j)$

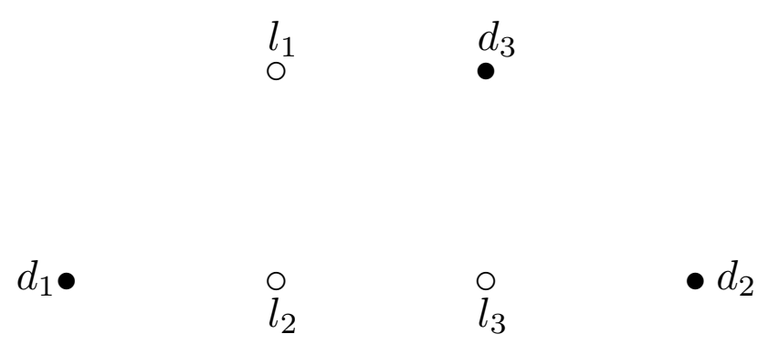
Theorem: Let Γ be a finite, simple, weighted graph whose vertices are partitioned into sets $L = \{l_1, l_2, \ldots, l_r\}$ and $D = \{d_1, d_2, \ldots, d_s\}$. Let $\phi : A\Gamma \longrightarrow \langle t \rangle$ be an epimorphism defined by $\phi(l_i) = t$ and $\phi(d_i) = 1$ for all i. Then $\ker \phi$ is finitely presented if there exists a sequence of subgraphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ of Γ where Γ_1 consists of a single vertex of L, $\Gamma_n = \Gamma$, and Γ_{j+1} can be obtained from Γ_j by either

1. adding a vertex v and an edge $\{v, l_i\}$, where $l_i \in V(\Gamma_j)$, or
2. adding an edge $\{a, b\}$, where $a, b, l_i \in V(\Gamma_j)$; $\{a, l_i\}, \{b, l_i\} \in E(\Gamma_j)$; and $k_{\{a,l_i\}} = k_{\{b,l_i\}} = 2$.

Example: Define $\phi : A\Gamma \longrightarrow \langle t \rangle$ by $\phi(l_i) = t$ and $\phi(d_i) = 1$ where Γ is as follows. Note: Unlabelled edges have arbitrary weight.



Construction of subgraph sequence $\Gamma_1, \Gamma_2, \ldots, \Gamma_8, \Gamma$.



Definition. A 2-cone is a weighted graph, Γ, such that there exists a vertex, a, which is adjacent to all other vertices of Γ with an edge of weight 2.

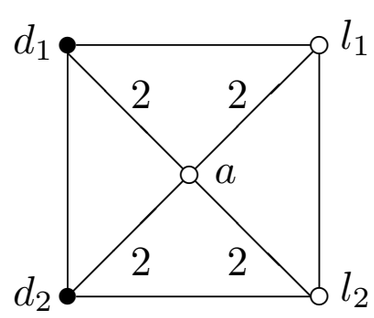
Example:

2-Cones

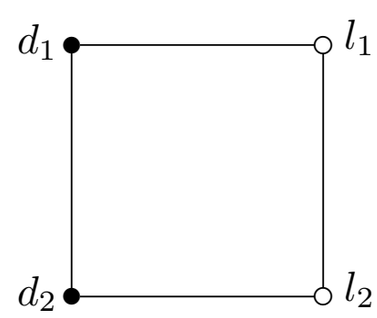
Corollary (J. Meier, L. VanWyk, 1995): Let $G\Gamma$ be a graph group and let Γ be a cone with apex a. If $\phi : G\Gamma \longrightarrow \langle t \rangle$ is an epimorphism with $\phi(a) \neq 1$, then $\ker \phi$ is finitely presented.

Theorem: Let $A\Gamma$ be an Artin group and let Γ be a 2-cone with apex a and base Γ_1. If $\phi : A\Gamma \longrightarrow \langle t \rangle$ is an epimorphism mapping all generators to either t or 1 with $\phi(a) = t$, then $\ker \phi \simeq A\Gamma_1$. In particular, $\ker \phi$ is finitely presented.

Example: Let $\phi : A\Gamma \longrightarrow \langle t \rangle$ be defined by $\phi(a) = t, \phi(l_i) = t$, and $\phi(d_i) = 1$ where Γ is as follows. (Note: unlabelled edges have arbitrary weight.) Since Γ is a cone, by the above theorem, $\ker \phi \simeq A\Gamma_1$, where Γ_1 is shown below.



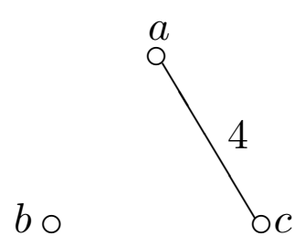
Γ



Γ_1

Note: $A\Gamma \simeq \langle a \rangle \times A\Gamma_1$.

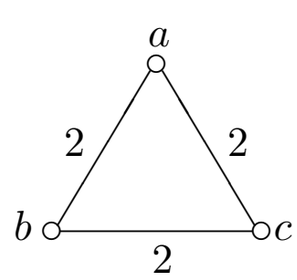
Example 2:



Γ

$$A\Gamma = \langle a, b, c \mid acac = caca \rangle$$

Example 3:

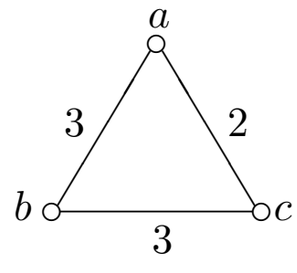


Γ

$$A\Gamma = \langle a, b, c \mid ab = ba, ac = ca, bc = cb \rangle$$

$$A\Gamma \simeq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$$

Example 4:



Γ

$$A\Gamma = \langle a, b, c \mid aba = bab, ac = ca, bcb = cbc \rangle$$

$$A\Gamma \simeq B_4$$