Biological Applications Illustrating Linear Algebra Concepts

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Background

- BIO2010, National Research Council, Recommendation 2:

  > Faculty in biology, mathematics, and physical sciences must work collaboratively to find ways of integrating mathematics and physical sciences into life science courses as well as providing avenues for incorporating life science examples that reflect the emerging nature of the discipline into courses taught in mathematics and physical sciences.

Background

- Math & Bio 2010, L. A. Steen (editor), MAA; Challenges, Connection, Complexities: Educating for Collaboration, J. R. Jungck:

  "Recent recommendations for reform of undergraduate science and mathematics education have reinforced the need for more mathematics and computer science in undergraduate biology education as well as more attention to biological applications in mathematics and computer science education."

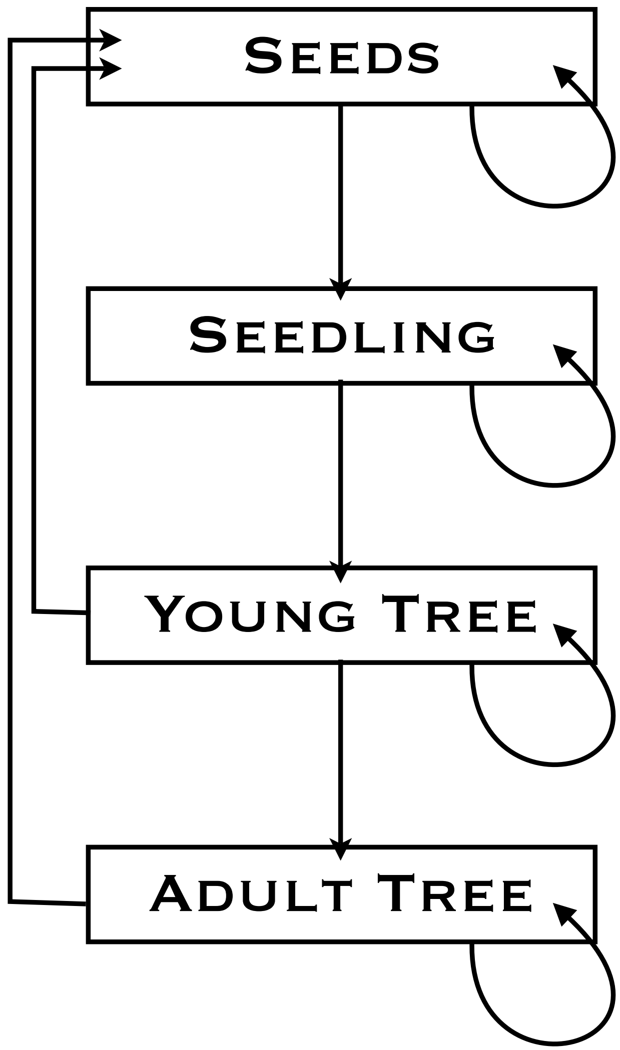
Overview of Topics

- Matrix models for population growth (see Allman and Rhodes, Mathematical Models in Biology, Chapter 2)
- Color Perception as a Vector Space (see Feynman, Leighton, and Sands, The Feynman Lectures on Physics, Vol I, Chapter 35)
- Other Possibilities

Population Growth

- Example:
  - Suppose a tree population consists of four stages: seeds, seedlings, young trees, and adult trees.
  - Each year, every individual either dies, remains in the same stage, or advances to the next stage. In addition, young and adult trees produce seeds.
  - Transitions are characterized by probabilities, and seed production is characterized by a per capita average.

Population Growth



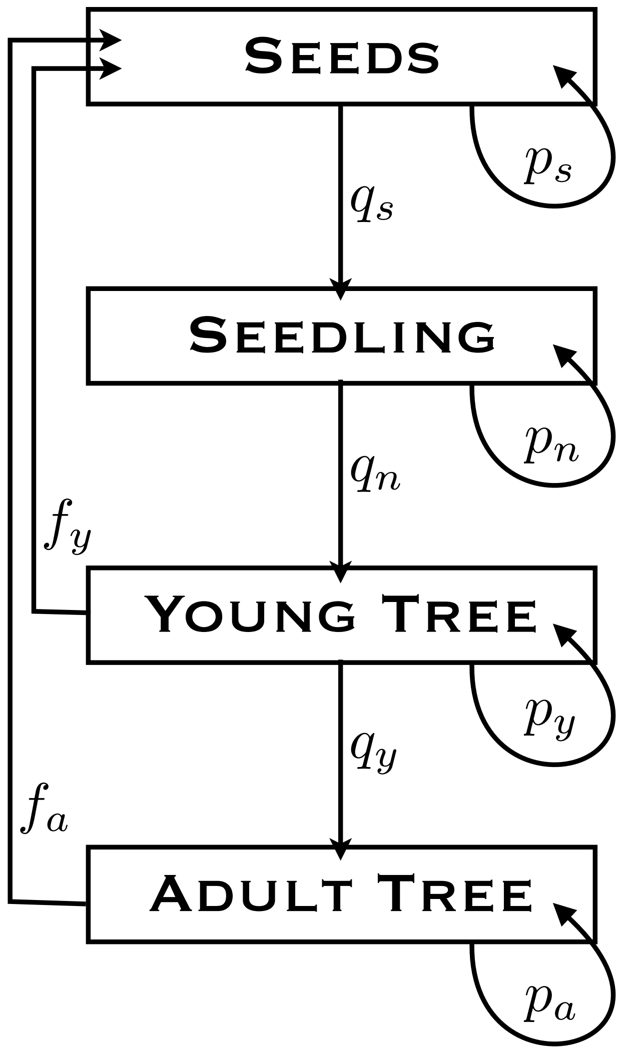
$S_t = \#$ Seeds at time t

$N_t = \#$ Seedlings at time t

$Y_t = \#$ Young Trees at time t

$A_t = \#$ Adult Trees at time t

Population Growth



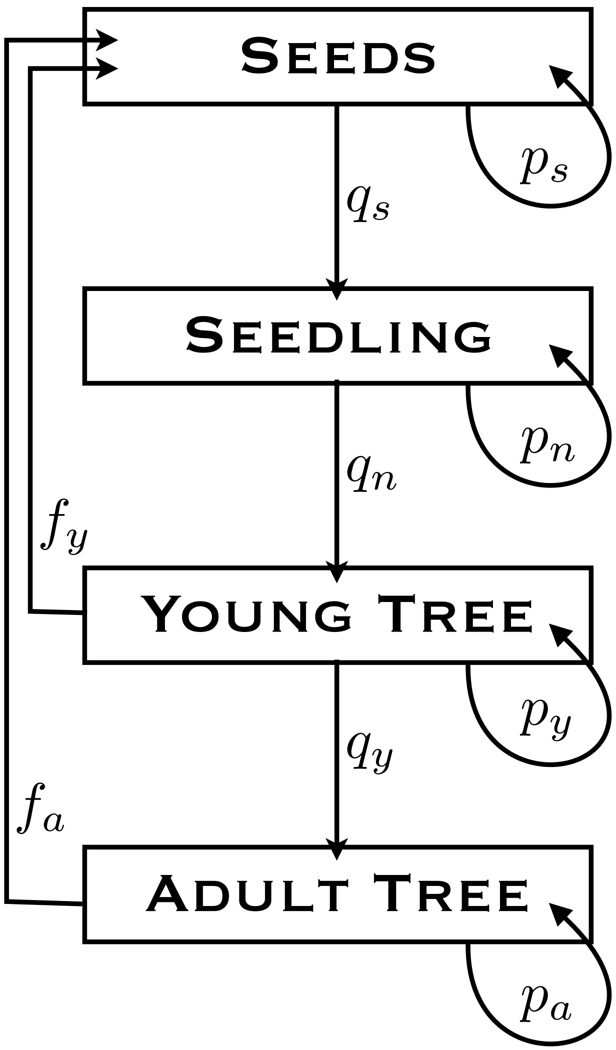
Each individual:

- Stays in stage with probability p.
- Transitions to next stage with probability q.
- or dies with probability 1-p-q.

Young and Adult trees reproduce:

- creating an average of f seeds each.

Population Growth



Key assumption: Each term additive and proportional to source

$$S_{t+1} = f_a A_t + f_y Y_t + p_s S_t$$

$$N_{t+1} = q_s S_t + p_n N_t$$

$$Y_{t+1} = q_n N_t + p_y Y_t$$

$$A_{t+1} = q_y Y_t + p_a A_t$$

Matrix Formulation

$$S_{t+1} = p_s \cdot S_t + 0 \cdot N_t + f_y \cdot Y_t + f_a \cdot A_t$$

$$N_{t+1} = q_s \cdot S_t + p_n \cdot N_t + 0 \cdot Y_t + 0 \cdot A_t$$

$$Y_{t+1} = 0 \cdot S_t + q_n \cdot N_t + p_y \cdot Y_t + 0 \cdot A_t$$

$$A_{t+1} = 0 \cdot S_t + 0 \cdot N_t + q_y \cdot Y_t + p_a \cdot A_t$$

$$\mathbf{X}_{t+1} = P\mathbf{X}_t$$

$$\mathbf{X}_t = \begin{pmatrix} S_t \\ N_t \\ Y_t \\ A_t \end{pmatrix} \qquad P = \begin{pmatrix} p_s & 0 & f_y & f_a \\ q_s & p_n & 0 & 0 \\ 0 & q_n & p_y & 0 \\ 0 & 0 & q_y & p_a \end{pmatrix}$$

Sample Problems

- Given the stage distribution at time t=9, find the stage distribution at time t=10. (Matrix-vector multiplication)

- Given the stage distribution at time t=9, find the stage distribution at time t=8. (Solve a linear system, find matrix inverse)

- Given the stage distribution at time t, find a formula for the distribution at time t+s or t-s. (Integer powers of P and P^{-1})

Eigenspaces

- Instead of looking at actual population sizes, look at the relative population sizes:

$$(s_t, n_t, y_t, a_t) = \frac{1}{T_t}(S_t, N_t, Y_t, A_t)$$

$$T_t = S_t + N_t + Y_t + A_t = ||\mathbf{X}_t||_1$$

- Under basic assumptions, the distribution will converge to a distribution independent of initial conditions, called the stable stage distribution.
- This distribution is an eigenvector of P and the corresponding dominant eigenvalue is called the intrinsic growth rate.

Perception of Vision

(The Feynman Lectures on Physics, Ch 35)

- Lead-in thoughts and questions
  - A prism divides light into a spectrum of many different wavelengths, and each distinct wavelength produces a different color
  - How does a [color printer, television, computer screen] use only three colors to produce so many different colors?
  - Would it help if we added more colors? Could we do with fewer colors?
  - Why do we typically use RGB or CMYK?

Perception of Vision

(The Feynman Lectures on Physics, Ch 35)

- Human perception of color can be described as a vector space of dimension 3 (almost).
  - **Set of Vectors:** All perceivable colors and intensities of those colors created by light on a white screen.
  - **Vector:** A spotlight that creates exactly the color/intensity desired (on white screen).
  - **Vector Addition:** Given two colors, take the two corresponding spotlights and shine them simultaneously on the screen.
  - **Scalar Multiplication:** Scale the intensity of the light but keep same color.

Perception of Vision

(The Feynman Lectures on Physics, Ch 35)

- What do we mean that colors are equal (=)?
  - Indistinguishable Colors: Let X and Y be two spotlights such that the human eye cannot distinguish them.
  - Then for any other spotlight Z, the eye will not distinguish X+Z from Y+Z.
  - If X and Y are indistinguishable, we say X=Y. Then X+Z=Y+Z.
  - That is, adding another light will not allow the eye to distinguish two previously indistinguishable lights?

Perception of Vision

(The Feynman Lectures on Physics, Ch 35)

- Big Problem: What do we mean by a negative intensity or Additive Inverse?
  - Acknowledge: No physical answer!
  - Avoidance: For application, we never need to create the additive inverse. We only need to create equations relating colors involving positive intensities.
  - Let A, B, and C be colors with physical colors (positive intensity). Then we say A=B-C iff A+C=B.

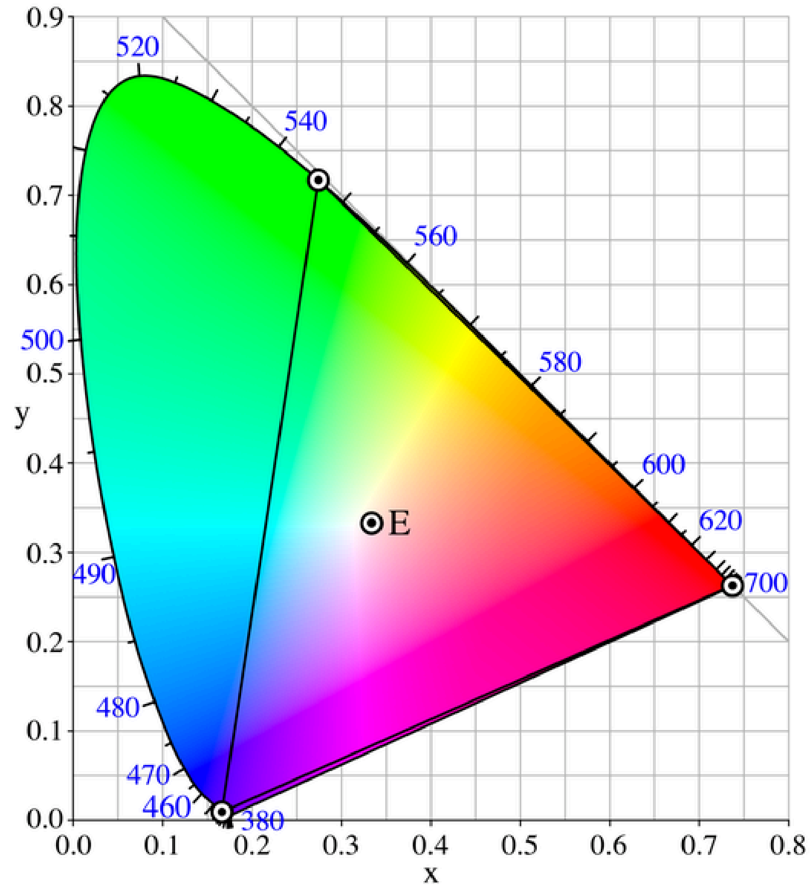
Perception of Vision

(The Feynman Lectures on Physics, Ch 35)

- All visible colors may be described as a linear combination of three colors, for example, Red, Green and Blue (figure on right):

$$C = r \cdot R + g \cdot G + b \cdot B$$

- Colors outside of the triangle require negative intensities.
- Any three non-collinear colors will form a basis.



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Perception of Vision

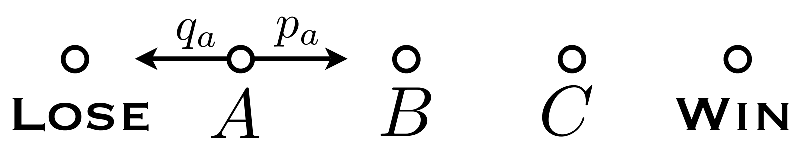
(The Feynman Lectures on Physics, Ch 35)

- Different color methods choose different bases to represent the set of all colors.
- CMYK is used for printing and is a subtractive mode (pigments absorb light rather) starting with white paper. Black ink (K) is also required.

Other Possibilities

- Markov chains and transition matrices: for example, applications to DNA mutations.

- Success probabilities: (Markov chain)



  - For any starting point {a,b,c}, there is a probability to win.
  - Use a recurrence relation to compute these probabilities:

$$
\begin{aligned}
P_A &= r_a P_A + p_a P_B \\
P_B &= q_b P_A + r_b P_B + p_b P_C \\
P_C &= q_c P_B + r_c P_C + p_a
\end{aligned}
$$