1.2 Numbers, Measurements and Relations

1.2.1 Overview

Numbers play a fundamental role in science because they allow us to quantify observations. That is, instead of saying things are big or small, we can assign numbers to measurements. Science in large part has progressed because of our ability to determine mathematical relationships between different measurements that make prediction possible.

Mathematics admittedly views numbers themselves as objects worthy of study. A number is an exact entity—other numbers, regardless of how close they might be, are different. Consequently, a mathematical result has an exact value. Sometimes we can only approximate the value, but we should give an exact expression for that value when possible. Mathematicians classify numbers according to the complexity of their definition. Historical conventions often suggest ways in which we can simplify an expression, often so that the classification can be more easily recognized.

Measurements allow us to assign numerical values to physical attributes, such as length, temperature, mass, and speed. Instruments need to be designed so that repeated measurements will result in the same values using standard units. Measurement error results from the limited accuracy intrinsic in reading a measurement from an instrument's scale. Most often, we measure more than one attribute of an object or system under a given condition. The collection of all such measurements is called the **state** of the system. We generally wish to understand the relationships between these different quantities.

1.2.2 Numbers in Mathematics

In mathematics, numbers have precise meanings and classifications. Here, we review the basic **sets** of numbers. The **natural numbers** are the positive integers

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Including the number zero gives us all **counting numbers**

$$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}.$$

The set of all **integers** is written

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The **rational numbers** are all numbers that can be represented as a ratio of integers

$$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\}.$$

This **set-builder notation** indicates that $\mathbb{Q}$ is a set of numbers that can be written in the form p/q where p is some integer and q is some natural number.

We often visualize numbers geometrically using a number line. First, the origin of the line is specified with a value of zero. The integers are then equally spaced by a unit length counting from zero. (See Figure A.1.1.) Subdividing the unit length into a whole number of equal parts generates additional points that are rational numbers. (See Figure A.1.2.) However, even when all rational numbers are included, there are infinitely many points on the line that are never covered. These are the **irrational numbers**, which include algebraic numbers like $\sqrt{2}$ or $\sqrt{3}$, as well as transcendental numbers like π and e. The set of **real numbers** is written $\mathbb{R}$ and consists of both rational and irrational numbers.

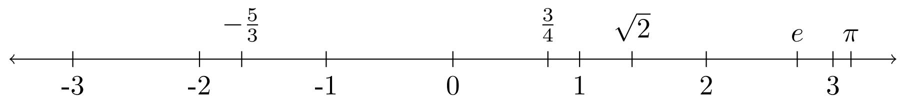


Figure 1.2.1 The number line graphically represents real numbers, both rational and irrational.

Every mathematical value represents a single point on the number line. Two values are equal only when they refer to the very same point on the line. Calculators give decimal approximations for numbers using a limited number of digits and so they can actually only represent a finite collection of the infinitely many possible numbers. In particular, irrational numbers can never be represented exactly using decimals. Thus, we usually represent mathematical values by their mathematical expression rather than decimals. When a decimal approximation is useful, we should indicate we are making an approximation using the approximation symbol ($\approx$) rather than an equals sign ($=$).

Simplification of numbers corresponds to finding a new representation of a number in a reduced form. For example, a rational number has many different representations of the form p/q with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. But there is only one representation where p and q have no common factors. Canceling any common factors to find this representation would be simplification. Other examples of simplification include simplifying a root or rationalizing a denominator.

Example 1.2.2 The fraction $\frac{126}{24}$ is not simplified. If we find the prime factorization of the numerator and denominator, we find

$$\begin{aligned} 126 &= 6 \cdot 21 = 2 \cdot 3^2 \cdot 7, \\ 24 &= 3 \cdot 8 = 2^3 \cdot 3. \end{aligned}$$

The fraction simplifies by canceling all common factors:

$$\frac{126}{24} = \frac{2 \cdot 3^2 \cdot 7}{2^3 \cdot 3} = \frac{3 \cdot 7}{2^2} = \frac{21}{4}.$$

In practice, we don't always have to find the prime factorization. Instead, we can find one common factor at a time until no common factors remain. For example, since 126 and 24 are both even, we could write

$$\frac{126}{24} = \frac{63}{12}.$$

Then, we look at 63 and 12 and recognize that they are both divisible by 3, allowing us to rewrite the fraction as

$$\frac{126}{24} = \frac{63}{12} = \frac{21}{4}.$$

Because $21 = 3 \cdot 7$ and $4 = 2^2$ do not have common factors, we know this is simplified. □

A square root is not simplified if there is a factor inside the root that is a perfect square. Similarly, a cube root is not simplified if there is a factor inside that is a perfect cube. We use the factors of the value inside a root to determine if we can simplify it.

Example 1.2.3 The square root $\sqrt{126}$ is not simplified. A square root is the inverse operation of squaring numbers (for non-negative numbers) so that

$\sqrt{3^2} = 3$. Because $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ (for $a, b \geq 0$), we can simplify as

$$\sqrt{126} = \sqrt{2 \cdot 3^2 \cdot 7} = \sqrt{9} \cdot \sqrt{14} = 3\sqrt{14}.$$

$\square$

Example 1.2.4 The cube root $\sqrt[3]{48}$ is not simplified. We start by factoring:

$$48 = 2 \cdot 24 = 2 \cdot 3 \cdot 8 = 2^4 \cdot 3.$$

A cube root is the inverse operation of cubing numbers so that for every perfect cube, we can simplify $\sqrt[3]{a^3} = a$. We have

$$\sqrt[3]{48} = \sqrt[3]{2^4 \cdot 3} = \sqrt[3]{2^3 \cdot 2 \cdot 3} = 2\sqrt[3]{6}.$$

$\square$

Some additional rules of simplification you may have learned were created before fast calculators and computers were available. These rules were taught so that scientists and engineers could express an answer that would be in a form where it would be faster to use the tables and slide rules available at the time. We no longer need such rules for efficiency, but they often illustrate important algebra rules.

One example of such a rule is the simplification of fractions with square roots, called rationalizing a fraction. It was much more costly to use a table or slide rule if the root was in the denominator. The practice was developed to rewrite such an answer so that the root was in the numerator. This could be accomplished by multiplying the fraction on top and bottom by a factor that would eliminate the undesired root.

Example 1.2.5 Simplify $\dfrac{4}{3\sqrt{2}}$ by rationalizing the denominator.

Solution. A square root can simplify if there is a perfect square inside. The square root in this denominator $\sqrt{2}$ would need another 2 inside to have a perfect square. Multiply numerator and denominator by the extra $\sqrt{2}$ to get a square in the denominator.

$$\frac{4}{3\sqrt{2}} = \frac{4\sqrt{2}}{3\sqrt{2}\sqrt{2}} = \frac{4\sqrt{2}}{3 \cdot 2}$$

Now we can finish simplifying the fraction by canceling common factors:

$$\frac{4}{3\sqrt{2}} = \frac{4\sqrt{2}}{6} = \frac{2\sqrt{2}}{3}.$$

$\square$

In the previous example, the two expressions $\dfrac{4}{3\sqrt{2}}$ and $\dfrac{2\sqrt{2}}{3}$ are equally simplified. The first expression has a rationalized numerator. The second expression has a rationalized denominator. You should ask your instructor whether they expect a preferred simplified form.

Although we usually use simplification for aesthetic reasons, having a standard way to write numbers can be useful to prove mathematical results. Thanks to Pythagorus, the ancient Greeks knew that $\sqrt{2}$ was a number that represented the hypotenuse of an isosceles right triangle with legs of unit length. The Greeks also originally thought that all numbers would ultimately be rational numbers. Realizing that $\sqrt{2}$ was irrational was so shocking that, according to legend, the discoverer of this fact was drowned at sea.

Example 1.2.6 The proof of the irrationality of $\sqrt{2}$ uses the idea that rational numbers have a simplified form. The basic argument is to consider a rational number that *might* represent $\sqrt{2}$ and then proceed to show that such a representation doesn't make sense. The detailed argument is shown below.

Solution. Suppose that $\sqrt{2}$ is a rational number. Then it can be written as the ratio of two integers $\sqrt{2} = \frac{p}{q}$ in reduced form, meaning p and q do not have common factors. By definition of square roots, we must have $\frac{p^2}{q^2} = 2$ which implies

$$p^2 = 2q^2$$

so that p^2 is an even number. The only way that p^2 can be even is if p itself is even, since the product of two odd numbers is always odd.

Once we know p is even, we can factor out 2 and write $p = 2k$ where k is also an integer. Now $p^2 = 4k^2$ which implies $4k^2 = 2q^2$ or

$$q^2 = 2k^2.$$

This means q would also be an even number. This is where the contradiction occurs—since p and q were to have had no common factors, they couldn't both be even. This means that $\sqrt{2}$ can not be written as a reduced fraction, which in turn means that $\sqrt{2}$ is not a rational number. □

1.2.3 Numbers as Measurements

In science, numbers often arise from measurements. When counting objects, measurements use integers and are exact. Most measurements, however, are not exact and require the use of a scale. An instrument for measurement provides a physical tool that allows us to identify a number of units associated with the physical quantity.

The most elementary physical measurement of this type is a measurement of length. The instrument of measurement, a ruler, uses a constructed number line such that the spacing between numbers on the ruler represents distance. The standard unit for the ruler, such as an inch or centimeter, sets the spacing between integer distances. The designer of a ruler also chooses the number of subdivisions per unit. For example, many rulers with inches use either 8 or 16 subdivisions per inch while metric rulers use 10 subdivisions per centimeter. You should see a similarity between the construction of the ruler and the development of rational numbers, except that the set of rational numbers allows for all possible integer number of subdivisions of the unit.

Measured quantities generally occur between two values that an instrument can measure. Given a ruler, the observer must choose a length based on the existing rulings. Even a length that appears to be exactly on a ruling might be found to be slightly off when examined under magnification. The observer might also make judgment errors in reading off a measurement. Consequently, a measurement represents an approximation of the value of a quantity. The difference between the true and measured values of a quantity is called an **error**.

Definition 1.2.7 Given any quantity with an actual value Q and a measured value $\widehat{Q}$, the **error** or **residual** is a value, say E, that measures the difference between the actual and measured values,

$$E = Q - \widehat{Q}.$$

Equivalently, E is that quantity such that the actual value is equal to the measured value plus the error,

$$Q = \widehat{Q} + E.$$

$\diamondsuit$

Definition 1.2.8 Given any quantity with an actual value Q and a measured value $\widehat{Q}$, the **absolute error** measures the absolute value of the error:

$$|E| = |Q - \widehat{Q}|.$$

$\diamondsuit$

Note 1.2.9 Symbols that represent variables correspond to the entire symbol being used. In the previous definitions, Q and $\widehat{Q}$ are different symbols and represent different numbers even though they both use the letter "Q". Similarly, the case of a letter matters so that R and r are different symbols. Avoid the trap of thinking that the letter's name is the symbol.

In science, the error in a measurement is not known precisely but is represented by a *bound*. There are a variety of techniques used to indicate the bound for an error. We will briefly discuss how the use of significant digits or a margin of error represent error bounds.

One way that the accuracy of measurements are described is using a number of **significant digits**. The idea is that the last digit reported represents the smallest subdivision the instrument can distinguish.

Example 1.2.10 Imagine that an object has a length of 15.2772 cm, measured to the nearest micron (micrometer). (We never really know exact lengths of physical objects.) How would the length be reported with different numbers of significant digits?

If that object was measured using a ruler showing only centimeters, we would see that the length was between 15 and 16 cm but closer to 15. Our measurement would be written as 15 cm, or $\widehat{Q} = 15$, and we would have two significant digits. However, if we did not know about details of the original measurement and only saw the recorded value of 15 cm, then we would have to assume that the true length was somewhere between 14.5 cm and 15.5 cm,

$$14.5 \le Q \le 15.5$$

Subtracting $\widehat{Q} = 15$ from each term, we find

$$-0.5 \le Q - \widehat{Q} \le 0.5.$$

The absolute error is therefore *bounded* by 0.5 cm.

If the ruler showed millimeters, then our measurement would be 15.3 cm with three significant digits. Knowing only the measurement $\widehat{Q} = 15.3$ and that there are three significant digits, we can infer

$$15.25 \le Q \le 15.35$$

so that the error is bounded between

$$-0.05 \le Q - \widehat{Q} \le 0.05.$$

The absolute error based on the measurement is bounded by 0.05 cm. $\square$

An alternative to using significant digits is to state explicitly a **margin of error**. The margin of error is equivalent to providing a bound for the error. We saw that a measurement 15.3 cm with three significant digits corresponds to an inequality

$$15.25 \leq Q \leq 15.35$$

such that the absolute error is bounded by 0.05,

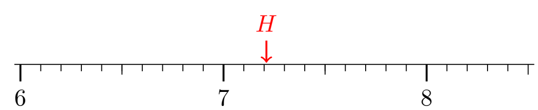
$$|Q - \widehat{Q}| \leq 0.05.$$

Using a margin of error, we write $Q = 15.3 \pm 0.05$ cm. The margin of error ± 0.05 is interpreted as $-0.05 \leq Q - \widehat{Q} \leq 0.05$.

A margin of error is more precise than significant digits. For example, if we wanted to say that a measurement was somewhere between 15.2 cm and 15.4 cm, then we would write 15.3 ± 0.1 cm. The value 15.3 was used as a central value and the margin of error gives a distance in either direction to reach the extreme values. The true value must be between the extremes.

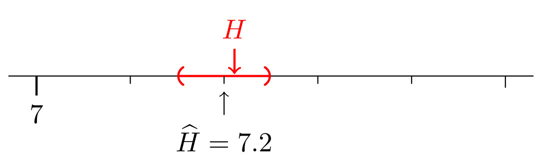
Example 1.2.11 The length of the hypotenuse of a right triangle with legs of lengths 4 cm and 6 cm is $H = \sqrt{4^2 + 6^2} = \sqrt{52} = 2\sqrt{13}$ cm. A calculator shows the decimal approximation is $H \approx 7.211103$ cm.

Now, suppose we use a ruler using centimeters but showing millimeters to measure the length. Different ways of describing the measurement with a margin of error give different information about the length.



- Write H using a margin of error to state that the measurement to the nearest millimeter is 7.2 cm.
- Write H using a margin of error to state that the measurement is between 7.2 and 7.3 cm.
- Write H using a margin of error to state that the measurement is between 7.2 and 7.25 cm.

Solution. First, we consider the nearest tick mark on the ruler. The measurement $\widehat{H} = 7.2$ cm will be the *nearest* value for any actual length satisfying $7.15 \leq H \leq 7.25$.



The spacing from $\widehat{H}$ and the edge of this interval is

$$\epsilon = |7.25 - 7.2| = |7.15 - 7.2| = 0.05.$$

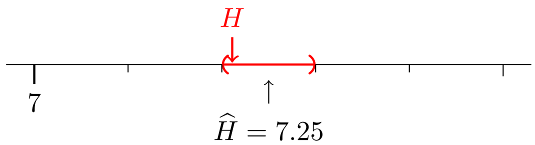
This value ϵ is the largest margin of error so that

$$|H - 7.2| \leq 0.05.$$

We write $H = 7.2 \pm 0.05$ cm.

Next, we work with the range $7.2 \leq H \leq 7.3$. We find the mid-point of this interval as our recorded measurement

$$\widehat{H} = \frac{7.2 + 7.3}{2} = 7.25.$$



Then we measure the distance from the center to the edge,

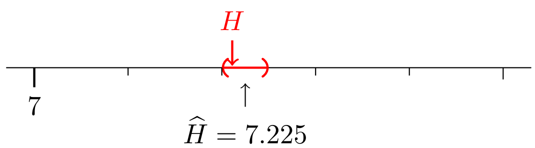
$$\epsilon = |7.3 - 7.25| = |7.2 - 7.25| = 0.05$$

to find the margin of error 0.05 cm. We can then express our measurement with a margin of error as $H = 7.25 \pm 0.05$ cm corresponding to a bounded error

$$|H - 7.25| \leq 0.05.$$

Finally, we repeat this process to indicate that H is in the range $7.2 \leq H \leq 7.25$. The mid-point gives

$$\widehat{H} = \frac{7.2 + 7.25}{2} = 7.225.$$



The margin of error is 0.025 so that our new approximate measurement with a margin of error is written $H = 7.25 \pm 0.025$ cm. Using an inequality involving absolute values, we could write

$$|H - 7.225| \leq 0.025.$$

$\square$

In general, a margin of error establishes a symmetric interval of possible values around the measurement. If we symbolically represent the margin of error by $\epsilon > 0$ (the Greek letter epsilon), then the statement $Q = \widehat{Q} \pm \epsilon$ is a statement that the absolute error is bounded by ϵ,

$$|Q - \widehat{Q}| \leq \epsilon.$$

In other words, we know from the measurement that the true value Q is in the interval

$$\widehat{Q} - \epsilon \leq Q \leq \widehat{Q} + \epsilon.$$

1.2.4 Summary

- In mathematics, numbers represent specific points on the number line. Real numbers ($\mathbb{R}$) can be classified as natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$), and irrational numbers.

- To simplify an expression is to find an expression representing the same value in a simpler form. For fractions, there should be no common factors. For roots, the power of prime factors inside should be less than the root.

- In a physical context, numbers represent measurements that have limited precision. This precision might be characterized by significant digits or by a margin of error.

- The error of approximation E for a quantity Q and an approximation $\widehat{Q}$ is defined by
$$E = Q - \widehat{Q}.$$
A symmetrical error bound $-\epsilon \le E \le \epsilon$ corresponds to the absolute value inequality $|Q - \widehat{Q}| \le \epsilon$ for a range of values
$$\widehat{Q} - \epsilon \le Q \le \widehat{Q} + \epsilon.$$

1.2.5 Exercises

Simplify the following values.

1. $\frac{42}{12}$

2. $\frac{210}{28}$

3. $\sqrt{75}$

4. $\sqrt{160}$

5. $\sqrt[3]{160}$

6. $\sqrt[4]{160}$

7. $\frac{\sqrt{72}}{4}$

8. $\frac{\sqrt{864}}{15}$

Simplify the following values by rationalizing the denominator.

9. $\dfrac{6}{5\sqrt{3}}$

10. $\dfrac{10}{\sqrt[3]{2}}$

11. $\dfrac{4\sqrt{2}}{\sqrt{3}}$

Simplify the following values by rationalizing the numerator.

12. $\dfrac{4\sqrt{2}}{\sqrt{3}}$

13. $\dfrac{5\sqrt[3]{4}}{\sqrt[3]{3}}$

14. One of your colleagues has recorded the mass of a specimen in your lab's notebook. The recording is given as 35.8 g. How much uncertainty is in this measurement? What are the possible actual masses that might correspond to that measurement?

15. One of your colleagues has recorded the mass of a specimen in your lab's notebook. The recording is given as 35.8 g. Suppose you also know that the lab instrument that was used always rounds measurements to the nearest 0.2 g. How should the recording have been written to indicate this additional information? What are the possible actual masses that might correspond to that measurement?

16. A thermometer's scale shows every 5 degrees. You observe the current temperature registers on the thermometer as being between 75 and 80 but clearly closer to 75 degrees. How would you report the temperature in order to reflect both your measurement and your uncertainty?

17. A right triangle is formed with legs measuring 5 cm and 8 cm. Express the length of the hypotenuse H to the nearest tenth of a centimeter, stating the margin of error based on a ruler showing millimeters. Rewrite your statement about margin of error as an inequality involving absolute values.

18. A right triangle is formed with one leg measuring 10 cm and the hypotenuse measuring 18 cm. Express the length of the other leg L to the nearest tenth of a centimeter, stating the margin of error based on a ruler showing millimeters. Rewrite your statement about margin of error as an inequality involving absolute values.