1.4 Formulas as Models of Relations

1.4.1 Overview

Models are simplified abstract representations of something of interest. Airplane and automobile manufacturers create scale models to test aerodynamics in wind tunnels. Architects build models of future projects, whether a physical mock-up or a computerized 3-d representation, to see how their plan will fit together and give clients a vision of pending products. The models do not need to include every detail of the actual object of interest, just those details that are relevant to the purpose of the model.

Scientists also regularly use models. Physicists use high energy collisions of extremely fast particles to create conditions that they expect are comparable to the moments immediately after the big bang. A biologist may use mice from a well-controlled population as a model to study cancer, considering its biology to mimic that of humans at some level of approximation. A climatologist might use a computational model where a computer program tracks changes in the makeup of the air, pollutant levels and air and ocean temperatures according to known and assumed interactions.

A **mathematical model** is an abstract representation of measurable phenomena that is characterized through mathematical equations. Recall that we think of a **system** as the collection of all possible measurements associated with the objects and environment involved in the phenomenon. Each quantity is a **state variable**, even if we do not have a physical way to obtain the measurement. Many laws of science are described using mathematical equations that relate state variables. These equations are examples of mathematical models. Knowing the value of one state variable, we can use the model to predict the value of other variables.

In this section, we explore some of the most common parametrized formulas used in mathematical models. The most important concept that relates many of these families is the idea of proportionality. When we have a mathematical model, we can use values for the variables to solve for unknowns. We will introduce ways to solve equations using technology in this section and review some strategies for solving equations by hand in later sections.

1.4.2 Proportionality

The idea of proportionality occurs everytime there is a common ratio between two quantities.

Example 1.4.1 In chemistry, we know that the atomic mass of an element (daltons) represents the mass (grams) of exactly one mole of atoms of that element. The atomic mass of carbon-12 is exactly 12 Da. Thus, 1 mole of carbon-12 atoms has a mass of 12 grams, 2 moles of carbon-12 atoms has a mass of 24 grams, and 5 moles of carbon-12 atoms has a mass of 60 grams. The ratio of the number of moles to the mass is always the same constant matching the atomic mass. We say that the mass is **proportional to** the number of moles. □

So what does proportionality really mean? A **proportion** is a ratio between two quantities. The quantities are **proportional** if the ratio between the quantities always equals the same value. We sometimes say that the two quantities have a common proportion.

Definition 1.4.2 A quantity Q is **proportional to** a quantity P if the ratio Q/P is a constant, say $Q/P = k$. The value k is called the **proportionality**

constant and we can rewrite the equation as

$$Q = kP.$$

$\Diamond$

Many laws of physics are statements of proportionality. Isaac Newton discovered that the force F acting on an object due to gravity is proportional to the mass m of the object. Newton's law of gravity could be written

$$F = mg,$$

where the constant g is called the gravity acceleration constant. The French physicist Charles-Augustin de Coulomb discovered a similar law, that the electrical force F acting on a charged object is proportional to the charge of the object q. Coulomb's law can be written

$$F = qE,$$

where E is the electrical field strength at the object's location. The German physicist Georg Ohm discovered that the voltage drop V across a conductor in a circuit is proportional to the current I flowing through that conductor. Ohm's law is written

$$V = IR$$

and R is called the resistance of the conducting component.

When we know that two quantities are proportional, we can find the proportionality constant by calculating the ratio given observed data. If there are errors or uncertainties in the data, we can approximate the proportionality constant using an average of calculated ratios.

Example 1.4.3 Suppose we know that the birth rate for a population (number of births per unit time) is proportional to the number of individuals in the population. If the population has 20 births per month when it consists of 5000 individuals, find the number of births per month when the population consists of 8000 individuals.

Solution. We start by assigning variables for our quantities. Let P be the size of the population and let B be the birth rate. Because the birth rate (births per month) is proportional to the population size (individuals), we know that the ratio B/P is equal to some constant, which we will name b:

$$b = \frac{B}{P} = \frac{20}{5000} = 0.004.$$

The constant b is called a **per capita birth rate**. Rewriting the equation $B/P = b = 0.004$, we have a model

$$B = bP \qquad \Leftrightarrow \qquad B = 0.004P.$$

We now use our model. When the population has 8000 individuals, we have $P = 8000$. Substituting this value in the model, we find

$$B = 0.004(8000) = 32.$$

That is, the population is predicted by our model to have 32 births per month.

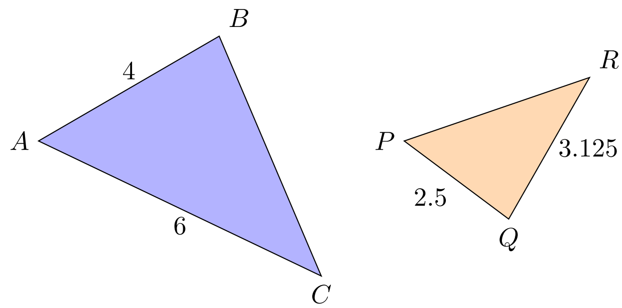
$\Box$

Mathematically, the geometric idea of similarity is one of the most common sources of proportional relations. Given two geometric polygons, we need a way to associate each vertex (points where edges meet) of one polygon with a particular vertex on the other polygon. The polygons are usually labeled by vertices, like a triangle ABC or a quadralateral $PQRS$. If we relate triangle ABC to triangle JKL, then the corresponding vertices would be

$$A \leftrightarrow J \qquad B \leftrightarrow K \qquad C \leftrightarrow L.$$

Polygons are **similar** if the ratio of distances between any pair of corresponding vertices always equal the same constant.

Example 1.4.4 Suppose triangle ABC is similar to triangle PQR, shown in the figure below. The lengths of the edges AB and AC are 4 and 6, respectively. The lengths of the edges PQ and QR are 2.5 and 3.125, respectively. Find the lengths of the other two edges.



Solution. The association between vertices of the triangles are

$$A \leftrightarrow P \qquad B \leftrightarrow Q \qquad C \leftrightarrow R.$$

Because the triangles are similar, the ratios of corresponding edges must all have the same value,

$$\rho = \frac{AB}{PQ} = \frac{AC}{PR} = \frac{BC}{QR}.$$

Because we know the lengths of both AB and PQ, we can use those values to determine the common ratio,

$$\rho = \frac{AB}{PQ} = \frac{4}{2.5} = 1.6.$$

Using this ratio, we can solve for the remaining unknowns:

$$\begin{aligned} 1.6 = \frac{AC}{PR} = \frac{6}{PR} \qquad &\Rightarrow \qquad PR = \frac{6}{1.6} = 3.75 \\ 1.6 = \frac{BC}{QR} = \frac{BC}{3.125} \qquad &\Rightarrow \qquad BC = 1.6(3.125) = 5 \end{aligned}$$

Thus, $PR = 3.75$ and $BC = 5$. □

The last example of simple proportionality arises in linear relations. Traditionally, we think of y as the dependent variable and x as the independent variable. Given any two points in the relation $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, we calculate the ordered increments of change going from P_1 to P_2 as $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$. We always calculate increments of change as the ending value minus where the starting value. A relation is **linear** if the ratio $\Delta y/\Delta x$ is always the same constant. This constant value is called the **slope**, traditionally using the symbol m,

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{1.4.1}$$

The slope represents the proportionality constant between Δx and Δy. If we know any point $(x, y) = (a, b)$ that satisfies the linear relation and the slope, then if we take any other point (x, y), we must have

$$y - b = m\,(x - a). \tag{1.4.2}$$

This equation is called the point–slope equation of a line.

Example 1.4.5 A container of ice water containing 200 grams of ice and 600 grams of water requires heat added to raise its temperature. The relation between temperature increase and required energy in heat is linear. To raise the temperature 10 degrees Celsius, we need 110.19 kJ. To raise the temperature 20 degrees Celsius, we need 133.67 kJ. How much energy is required to raise the temperature 25 degrees Celsius?

Solution. Because the relation between the temperature increase and the energy added to the water is linear, we can calculate the slope as the ratio of change in these values. Let T be the amount the temperature rises and let Q be the energy in heat added. Thinking of Q as the dependent variable, the slope will be the ratio of the change in Q to the change in T,

$$m = \frac{\Delta Q}{\Delta T} = \frac{133.67 - 110.19}{20 - 10} = 2.348.$$

Knowing the slope, we can create an equation that models the relation between Q and T. The point–slope equation of a line using the point $(T, Q) = (10, 110.19)$ becomes

$$Q - 110.19 = 2.348\,(T - 10).$$

To find the energy required to raise the temperature 25 degrees Celsius, we substitute $T = 25$ and can then solve for Q.

$$\begin{aligned} Q - 110.19 &= 2.348\,(25 - 10) \\ Q - 110.19 &= 35.22 \\ Q &= 145.41 \end{aligned}$$

It will take 145.41 kJ to raise the temperature to 25 degrees Celsius. □

1.4.3 Common Models

Many common models are based on a relation of proportionality between a quantity and a dependent variable based on another quantity. Simple proportionality would be where a variable y is proportional to another variable x, $y = kx$. We are now interested in models where y is proportional to some simple function of x. We say that y is **inversely proportional to** x if y is proportional to the reciprocal of x,

$$y = k \cdot \frac{1}{x} = \frac{k}{x}.$$

This is equivalent to saying that the product $xy = k$ is constant.

A **power law** refers to a dependent variable being proportional to some power p of the independent variable,

$$y = Ax^p.$$

An **exponential relation** refers to a dependent variable being proportional to some positive base b being raised to the independent variable as the power,

$$y = A\, b^x.$$

Although these two laws look similar, they have very different behaviors because of the role of the independent variable. Find ways to train your mind to read these differently. For example, for the power law, you might read the formula as "x raised to a power p", but for the exponential relation, you would read the formula as "the exponential of base b raised to the power x". As you intentionally use language to distinguish between the two models, you will more easily remember appropriate methods for each.

We now consider how we would find the model parameters given data. Regression methods do exist. For example, in Desmos, it would be possible to put given data in a table and then compute parameters using a formula like `y1 ~ A x1^p` or `y1 ~ A b^x1`. You would get perfectly good approximate values that could be used for applications. However, our purpose is to introduce the role of equations in solving for exact values.

When we are given data that a model describes exactly, we should consider each point in the data as being a solution to the model equation. When we have multiple points, we have multiple equations, all of which need to be true simultaneously. We then treat the parameters as variables and solve the system of equations to find parameter values.

We should have a clear separation in our thinking between creating the system of equations and solving the resulting system. We first focus on creating the system of equations, and we will use a computational tool to solve for the values. We will look at graphical methods of approximating solutions as well as using computer algebra systems to find exact formulas.

Example 1.4.6 Suppose y has a power law relation with x, $y = Ax^p$. Further, suppose that we have two data points, $(x, y) = (2, 4)$ and $(x, y) = (3, 8)$. Find the equations that determine the model parameters. Graph the equations to find their values.

Solution. To find each equation, we substitute the values of x and y in the model equation. For each given point, this will leave an equation that still involves the unknown model parameters. Using the point $(x, y) = (2, 4)$, we substitute $x = 2$ and $y = 4$ in $y = Ax^p$:

$$4 = A \cdot 2^p.$$

Using the point $(x, y) = (3, 8)$, we substitute $x = 3$ and $y = 8$ in $y = Ax^p$:

$$8 = A \cdot 3^p.$$

To show that we have a system of equations that need to be solved together, we group the equations with a curly brace,

$$\begin{cases} A \cdot 2^p = 4, \\ A \cdot 3^p = 8. \end{cases}$$

Our first method for finding the values will be graphical. Because we have two parameters, A and p, we can think of these as two variables that define a plane of points (p, A). Each equation defines a curve in the plane of solutions to that equation. With two equations, we obtain two different curves. Intersection points are the points in common to both curves and are the solutions that we seek.

Some graphing tools, like Desmos and most graphing calculators, require the variables to be x and y. They also typically expect that we have solved for y as a dependent variable. If we replace $p \leftrightarrow x$ and $A \leftrightarrow y$ and then solve for y, our equations become

$$\begin{cases} y = 4/2^x, \\ y = 8/3^x. \end{cases}$$

In Desmos, you can click on the point of intersection and see approximate values $(1.71, 1.223)$. Using a handheld calculator, a menu option to find an intersection point gives a better approximation $(1.709511, 1.223055)$ meaning that $p \approx 1.7095$ and $A \approx 1.2231$. A graph showing the points and the approximate model $y = 1.2231 \cdot x^{1.7095}$ is shown below.

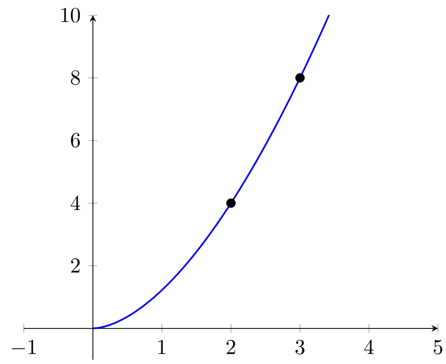


Figure 1.4.7 Graph of $y = 1.2231 \cdot x^{1.7095}$ with given points $(2, 4)$ and $(3, 8)$.

□

Example 1.4.8 Similar to the previous example, suppose y has an exponential relation with x, $y = Ab^x$ with the two data points, $(x, y) = (2, 5)$ and $(x, y) = (3, 8)$. Find the equations that determine the new model's parameters and solve the system of equations.

Solution. To find each equation, we substitute the values of x and y in the model equation. For each given point, this will leave an equation that still involves the unknown model parameters. Using the point $(x, y) = (2, 5)$, we substitute $x = 2$ and $y = 5$ in $y = Ab^x$ to obtain

$$5 = A \cdot b^2.$$

Using the point $(x, y) = (3, 8)$, we obtain the equation

$$8 = A \cdot b^3.$$

Our system of equations becomes

$$\begin{cases} A \cdot b^2 = 5, \\ A \cdot b^3 = 8. \end{cases}$$

The graphical method of solution used in the previous example results in approximate values. To find exact values, we need to perform an algebraic solution. A computational tool will expect us to provide it with our equations as well as the variables for which we are solving. In this text, we will work with the SageMath system, an open source computer algebra system. A blank interactive SageMath cell can be opened at https://sagecell.sagemath.org.

```
# Tell SageMath that A and b should be treated as variables
var('A','b')
# Define our equations
eq1 = (A*b^2 == 5)
eq2 = (A*b^3 == 8)
# Solve the system of equations for the system of variables
soln = solve([eq1, eq2], A, b)
# Display the result
show(soln)
```

When this script is executed, SageMath reports a solution

$$A = \frac{125}{64}, \quad b = \frac{8}{5}.$$

Our model becomes $y = \frac{125}{64} \cdot \left(\frac{8}{5}\right)^x$. A graph of the model with our given points is shown in the figure below.

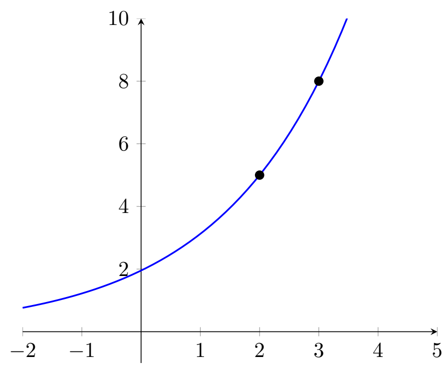


Figure 1.4.9 Graph of $y = \frac{125}{64} \cdot \left(\frac{8}{5}\right)^x$ with given points $(2, 5)$ and $(3, 8)$.

I want to note that SageMath fails to solve the similar system for the previous power law example. The change to the script above is minimal. You should try it and see what happens. This is a curious example where computer algebra systems sometimes know too much. In more advanced mathematics that account for complex numbers, there is an abiguity in that system of equations that is not obvious. You can see a glimpse of the complexity if you try to solve the system of equations using the WolframAlpha website. Try the request `solve A*2^p=4 and A*3^p=8 for A and p`. □

1.4.4 Constructing More Models

Once we have elementary models like powers and exponentials, we can construct more complicated models by using arithmetic. For example, **polynomials** are created by adding power functions where each of the powers are non-negative integers.

Definition 1.4.10 A **polynomial** is a sum of terms of the form $a_k x^k$ where k is a non-negative integer, a_k is a real number, and x is the independent variable. The values a_k are called **coefficients**. The largest power k is called the **degree** of the polynomial. ◊

The equation $y = 4x^3 - x + 5$ is a polynomial with degree 3, which we call a **cubic polynomial**. The coefficients are $a_0 = 5$, $a_1 = -1$, and $a_3 = 4$. A model for y as cubic polynomial of x would look like

$$y = a_3x^3 + a_2x^2 + a_1x + a_0.$$

We usually include zero coefficients for any skipped powers smaller than the degree, so our example would also have $a_2 = 0$.

We can create a system of equations to find coefficients for a polynomial. Because a cubic polynomial has 4 coefficients, we will need four data points to find a unique solution.

Example 1.4.11 Find a quadratic polynomial (degree 2) that goes through the points $(-1, 1)$, $(1, 2)$, and $(2, 4)$.

Solution. A general degree 2 polynomial model would have the form

$$y = a_2x^2 + a_1x + a_0.$$

We substitute the values for x and y to get one equation for each point.

$$\begin{cases} 1 = a_2(-1)^2 + a_1(-1) + a_0 \\ 2 = a_2(1)^2 + a_1(1) + a_0 \\ 4 = a_2(2)^2 + a_1(2) + a_0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} 1 = a_2 - a_1 + a_0 \\ 2 = a_2 + a_1 + a_0 \\ 4 = 4a_2 + 2a_1 + a_0 \end{cases}$$

We now use a computer algebra system to solve for the coefficients.

```
# Declare the parameters as variables
var('a2','a1','a0')
# Create the equations
eq1 = 1 == a2 - a1 + a0
eq2 = 2 == a2 + a1 + a0
eq3 = 4 == 4*a2 + 2*a1 + a0
# Solve the system
soln = solve([eq1,eq2,eq3], a0, a1, a2)
show(soln)
```

The result of the algebra solution is

$$a_0 = 1, \qquad a_1 = \frac{1}{2}, \qquad a_2 = \frac{1}{2}.$$

That is, our polynomial model is

$$y = \frac{1}{2}x^2 + \frac{1}{2}x + 1.$$

A graph of the data with the model is shown in the next figure.

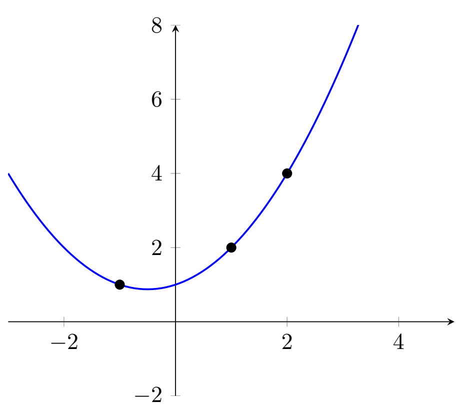


Figure 1.4.12 A graph of $y = \frac{1}{2}x^2 + \frac{1}{2}x + 1$ with the points $(-1, 1)$, $(1, 2)$, and $(2, 4)$.

□

We end this section by discussing the difference between regression and solving equations. Regression is used when we have data that we want to approximate with a model. This means that given data points will not necessarily be solutions—and most likely, they will not be. When we solve for a model passing through given data, we are finding a curve that has the given data points as solutions. Each point must actually lie on the curve. If we tried to solve for a model where we should be doing a regression, we will discover that there is no solution.

Example 1.4.13 Consider the three points $(1, -2)$, $(3, 1)$, and $(6, 6)$. Can we model these with a linear function $y = mx + b$?

Solution. Using the three data points, we can create a system of three equations for the model parameters.

$$\begin{cases} -2 = m(1) + b \\ 1 = m(3) + b \\ 6 = m(6) + b \end{cases} \quad \Leftrightarrow \quad \begin{cases} m + b = -2 \\ 3m + b = 1 \\ 6m + b = 6 \end{cases}$$

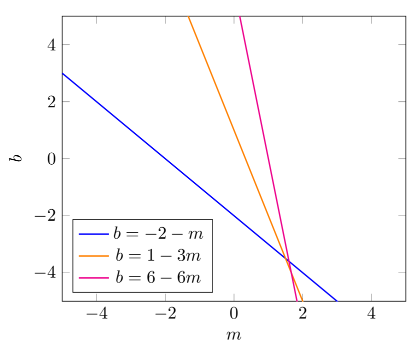
If we try to solve the system of equations, the solution set is empty.

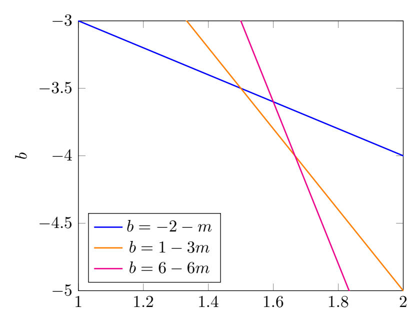
```
var('m','b')
eq1 = m + b == -2
eq2 = 3*m + b == 1
eq3 = 6*m + b == 6
soln = solve([eq1,eq2,eq3],m,b)
show(soln)
```

To visualize why there is no solution for this system, let us consider the graphical approach. We can solve each equation for b as the dependent variable and graph the resulting equations. Recall that many graphing utilities require us to rename our variables, $m \leftrightarrow x$ and $b \leftrightarrow y$:

$$\begin{cases} b = -2 - m \\ b = 1 - 3m \\ b = 6 - 6m \end{cases} \quad \leftrightarrow \quad \begin{cases} y = -2 - x \\ y = 1 - 3x \\ y = 6 - 6x \end{cases}$$

When we graph these equations, we get three lines. Although they appear close to having a single point of intersection near $(m, b) = (1.6, -3.6)$, there is not point where all three lines intersect simultaneously. This is the graphical result of no solution.



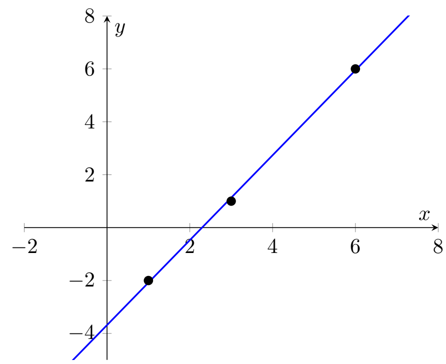


When we approach this problem as a regression model, we seek for a linear equation that is closest to all three points. Desmos reports regression coefficients $m \approx 1.60526$ and $b = -3.68421$ for a regression model

$$y = 1.60526x - 3.68421.$$

Notice that this model only approximates our points of interest:

$$x = 1 \quad \Rightarrow \quad y = -2.07895$$
$$x = 3 \quad \Rightarrow \quad y = 1.13158$$
$$x = 6 \quad \Rightarrow \quad y = 5.94737$$



□

1.4.5 Summary

- Proportionality between two variables is when the ratio of their values is always the same constant. As a parametrized model, to say y is proportional to x means that $y = kx$ for some constant k.

- Many models are generated by expressing a dependent variable as being proportional to a function of the independent variable. Common exampes are inverse proportionality, power law relations, and exponential relations.

- Given a parametrized model, a data point for the variables establishes an equation in the model parameters. With enough data points, the resulting system of equations can be solved to find the parameters.

- Solutions to a system of equations can be approximated based on the intersection of graphs. Computer algebra systems can often help solve a system of equations to find exact solutions.

1.4.6 Exercises

1. The British physicist Robert Hooke observed that when a spring is stretched, the strength of the force is proportional to the length of the stretch, at least for small to moderate lengths. When a spring is stretched 5 cm, the force exerted by the spring is 1.8 N.

(a) If F is the force of the spring and L is the length the spring is stretched, write down the general equation describing the relation that F is proportional to L. Then use the given data to find the proportionality constant.

(b) Find the force exerted by the spring if it is stretched 8 cm.

(c) What lengths should the spring be stretched for a force of 1 N? 2 N?

2. The amount of heat (a form of energy) stored in a substance is proportional to the change in temperature. When a gram of water absorbs 10 J of heat, the temperature rises by 2.389 degrees Celsius.

(a) If ΔT is the change in the temperature of the water and Q is the heat added to the water, write down the general equation describing the relation that Q is proportional to ΔT. Then use the given data to find the proportionality constant, which is called the specific heat of water.

(b) How much will the temperature change if 16 J of heat is added to the water.

(c) How much energy in added heat is required to raise a gram of water from 20 degrees to 100 degrees?

3. The surface area of a cube is proportional to the square of the length of one side.

(a) If A is the surface area and s is the length of a side, write down the general equation describing the relation that A is proportional to the square of s.

(b) Using geometrical reasoning, what is the proportionality constant?

(c) What is the surface area of a cube whose sides are each 5 cm in length?

(d) What the length is the side of a cube whose surface area is exactly 1 cm^2 in length?

4. The mass of a raindrop is proportional to the cube of its diameter. A raindrop with a diameter of 3 mm has a mass of 14.137 mg

(a) If m is the mass of a raindrop and d is the diameter, write down the general equation describing the relation that m is proportional to the cube of d.

(b) Find the constant of proportionality using the given data.

(c) What is the mass of a raindrop with a diameter of 5 mm?

(d) What is the diameter of a raindrop with a mass of 25 mg?

5. The time to complete a large manual labor job is inversely proportional to the number of people performing the labor. Suppose a job will take 20 days when 5 people are working.

(a) If T the time required to complete the job and L is the number of laborers, write down the general equation describing the relation that T is inversely proportional to L.

(b) Find the constant of proportionality using the given data. What is the physical interpretation of this constant?

(c) How long will the job take if there are 16 people working?

(d) What is the fewest number of people that can complete the job in 8 days?

6. The intensity of radiation from the sun is inversely proportional to the square of the distance from the sun. The earth, which is 1 AU (atronomical unit) from the sun, receives radiation from the sun at an intensity of 1367 W/tothe2.

(a) If I the radiation intensity and r is the distance fom the sun, write down the general equation describing the relation that I is inversely proportional to the square of r.

(b) Find the constant of proportionality using the given data.

(c) Find the intensity of radiation at Mercury, which is 0.387 AU from the sun.

(d) Find the distance at which the sun's radiation is half the intensity as compared to earth.

7. A right triangle ABC has legs $AC = 4$ and $BC = 3$.

(a) Find the length of the hypotenuse AB by applying the Pythagorean theorem.

(b) Find the lengths of a triangle PQR that is similar to ABC whose hypotenuse has length $PQ = 1$.

(c) Find the lengths of a triangle STU that is similar to ABC whose leg SU has length $SU = 1$.

8. A right triangle ABC has a leg AC with length $AC = 2$ and a hypotenuse AB with length $AB = 3$.

(a) Find the length of the other leg BC by applying the Pythagorean theorem.

(b) Find the lengths of a triangle PQR that is similar to ABC whose hypotenuse has length $PQ = 1$.

(c) Find the lengths of a triangle STU that is similar to ABC whose leg SU has length $SU = 1$.

9. Consider the parametrized model $y = ax^2 + b$ and data points $(x, y) = (1, 3)$ and $(x, y) = (2, 9)$.

(a) Determine the system of equations in terms of a and b.

(b) Graph the system of equations for the parameters in the (a, b)-plane.

(c) Solve for a and b, using a computer to assist if needed.

(d) Find the value of y when $x = 4$.

10. Consider the parametrized model $y = ax^2 + bx$ and data points $(x, y) = (1, 3)$ and $(x, y) = (2, 9)$.

(a) Determine the system of equations in terms of a and b.

(b) Graph the system of equations for the parameters in the (a, b)-plane.

(c) Solve for a and b, using a computer to assist if needed.

(d) Find the value of y when $x = 4$.

11. Consider the parametrized power law model $y = a \cdot x^b$ and data points $(x, y) = (1, 3)$ and $(x, y) = (2, 9)$.

(a) Determine the system of equations in terms of a and b.

(b) Graph the system of equations for the parameters in the (a, b)-plane.

(c) Solve for approximate values for a and b using a graphing utility.

(d) Find the value of y when $x = 4$.

12. Consider the polynomial model $y = ax^2 + bx + c$ and data points $(x, y) = (1, 3)$, $(x, y) = (3, 6)$, and $(x, y) = (-2, 4)$.

(a) Determine the system of equations in terms of a, b, and c.

(b) Solve the system of equations to find exact values for the model parameters

(c) Find the value of y when $x = 6$.