1.5 Algebra and Equivalence

1.5.1 Overview

Algebra can sometimes feel complicated. This feeling often arises when algebra is viewed as a long list of rules of manipulation. Perhaps you think of algebra as rules for moving symbols and then feel overwhelmed by the number of possible different problems. Or you might feel like you are doing the same operation in different situations and are counted correct in some situations but incorrect in others.

To be effective in algebra, we should organize our thinking around a small number of core principles. The first principle is to distinguish between expressions and equations. Our goals when working with an expression are different from when we are working with an equation. The second principle is the idea of equivalence, whether that refers to equivalent expressions or equivalent equations. Third, we want to minimize the number of rules be relating them to the fundamental properties of algebra.

In this section, we will relate the algebra principles you should have previously learned around these core principles. We will focus on the idea that algebra identities allow us to replace one expression with another equivalent expression. The operations on expressions all originate with the basic properties of real number arithmetic. Equations are relations stating that two expressions have the same value. Our operations on equations will be designed to generate simpler equations that are equivalent to the original. These strategies focus on applying the same operation to both sides of the equation to maintain a balance, with the goal of creating an equation that is more easily solved than the original.

1.5.2 Expressions and Properties of Algebra

In algebra, we use variables as placeholders for numerical values. The variable is given a symbol, usually a letter like x, that takes the place of the value. Sometimes, this is because we want to describe a calculation without referencing a specific value. Other times, we want a specific but unknown value and use a variable as a name.

Example 1.5.1 Maybe you have seen a calculation described similarly to the following.

Think of a number. Add five. Double the result. Subtract four. Divide the answer in half. Subtract your original number.

Use the variable x to represent the number chosen. Write out the formula that describes this calculation.

Solution. We use parentheses to emphasize the order of calculations. Start with x and add five to get $x+5$. Doubling this means to multiply by two to get $2(x+5)$. Subtracting four gives $2(x+5)-4$. Dividing in half means divide this by two, resulting in $\frac{2(x+5)-4}{2}$. We end by subtracting the original value x. The formula that matches this calculation would be

$$\frac{2(x+5)-4}{2} - x.$$

□

An **expression** is any *formula* involving numbers and variables, such as the formula in the previous example. An expression itself represents a numerical value. When an expression involves variables, the value of the expression itself is unknown until the values of all variables are known. Consequently, the expression itself is a **dependent variable**, and we often represent its value by another symbol.

Example 1.5.2 For the previous example, we could introduce a variable y to represent the final number. That gives a dependent variable defined by the expression

$$y = \frac{2(x+5)-4}{2} - x.$$

□

Dependent variables often depend on more than one variable.

Example 1.5.3 For a business that earns money by selling a number of items, each of which is sold for the same price, the **revenue** (money brought in through sales) is computed by multiplying the number of items sold by the price of each item. We can summarize this statement defining revenue using algebra if we represent the state variables by symbols. Let R represent the revenue, let n represent the number of items sold, and let p represent the price of each item. The product $n \cdot p$ is the expression representing the product of the number of items and the price per item. We use the equation

$$R = n \cdot p$$

to describe that the revenue is computed by this expression.

We see that the revenue R is defined here as a dependent variable based on the values of n and p. If we know the value of both n and p, then we will know the value of R. For example, if the company sells $n = 1000$ items and sets a price of $p = 1.25$ dollars, then the revenue is $R = 1000 \cdot 1.25 = 1250$ dollars. □

Different expressions can represent the same value. For example, $x + x$ and $2x$ are different expressions—they describe different calculations—but they always have the same value. We say that two expressions are **equivalent** if they result in the same value for all possible values of the involved variables. The properties of algebra describe the rules for how to create equivalent expressions.

Elementary Properties of Algebra.

For any expressions x, y, and z, the following expressions are equivalent.

- **Additive identity** (zero): $x + 0 = x$.
- **Multiplicative identity** (one): $1 \cdot x = x$.
- For every value x, there is an **additive inverse** value written $-x$ so that $x + -x = 0$. It is always the case that $-x = -1 \cdot x$.
- For every non-zero value x ($x \neq 0$), there is a **multiplicative inverse** value written $\div x$ so that $x \cdot \div x = 1$. When $x \neq 0$, we have $\div x = \frac{1}{x}$, which is why the inverse of x is also called the **reciprocal**.
- **Commutative properties**: $x + y = y + x$ and $x \cdot y = y \cdot x$.
- **Associative properties**: $(x+y)+z = x+(y+z)$ and $(x \cdot y) \cdot z = (x \cdot y) \cdot z$.
- **Distributive property** of multiplication over addition: $x \cdot (y + z) = x \cdot y + x \cdot z$.

Example 1.5.4 Our pick-a-number example with the dependent variable

$$y = \frac{2(x+5)-4}{2} - x$$

could be used as a mind-reader trick. If a volunteer from the audience chooses their own number for x and then does the math correctly, you (the mentalist) can always guess the final number y. Use algebra properties to determine what you should predict.

Solution. We start with the distributive property to rewrite $2(x+5) = 2x+10$. Subtracting 4 is equivalent to adding -4. We can then use the associative property to rewrite $(2x+10)+-4 = 2x+(10+-4) = 2x+6$. Dividing by two is equivalent to multiplying by $\frac{1}{2}$. This allows us to use the distributive property again,

$$\begin{aligned}\frac{2(x+5)-4}{2} = \frac{2x+6}{2} &= \frac{1}{2}(2x+6) = \frac{1}{2}(2x) + \frac{1}{2}(6) \\ &= (\frac{1}{2}\cdot 2)x + \frac{6}{2} = x+3\end{aligned}$$

The second line used the associative property and the multiplicative identity. Can you see where? The final step in the calculation is to subtract x,

$$y = \frac{2(x+5)-4}{2} - x = (x+3) - x = (x+-x)+3 = 3.$$

(What properties were used there?)

In conclusion, we have discovered $y = 3$. No matter what number is originally chosen, the final result of the described calculation will always be three. □

When finding equivalent expressions, we really are just using these basic rules. We can add zero by adding an expression and its additive inverse. We can multiply by one by multiplying by a nonzero expression and its multiplicative inverse, $1 = u/u$. The associative and commutative laws together allow us to reorder terms and operations *with respect to a single operation type*. A common mistake is to reorder terms or operations across different operations. For example, $2+3y$ might be incorrectly written as $5y$, which would be incorrectly using the idea of associativity, $2+(3\cdot y) = (2+3)\cdot y$ (false).

The distributive law is used to products of sums to sums of products and back. The reverse operation is usually called **factoring**. Adding fractions with common denominators is really about having a common factor. Because division is really multiplication by reciprocals, canceling common factors is an application of multiplicative inverses.

Example 1.5.5 Rewrite $\dfrac{3}{x} + x$ as a single fraction.

Solution. The first term $\frac{3}{x}$ has an inverse factor $\div x$. To combine terms as a fraction, this needs to be a common factor in both terms. We take x and multiply it by the inverse factors $x \div x$ and then factor:

$$3 \div x + x = 3 \div x + x\,x\ \div x = (3+x^2) \div x.$$

However, we usually do this using fraction notation:

$$\frac{3}{x} + x = \frac{3}{x} + \frac{x\cdot x}{x} = \frac{3+x^2}{x}.$$

□

Example 1.5.6 Simplify $\dfrac{3x^2y-6xy^2}{9x^2y^2}$ by canceling common factors.

Solution. A common mistake would be to cancel the x^2 in the first term and y^2 in the second term. Terms do not cancel over addition; they cancel in multiplication. We need to rewrite the numerator as multiplication rather than addition (subtraction). The distributive law allows us to identify common factors:

$$\frac{3x^2y-6xy^2}{9x^2y^2}=\frac{3xy(x-2y)}{9x^2y^2}.$$

By recognizing $9x^2y^2=(3xy)(3xy)$, we can rewrite our fraction and cancel the common factors:

$$\frac{3x^2y-6xy^2}{9x^2y^2}=\frac{3xy(x-2y)}{(3xy)(3xy)}=\frac{x-2y}{3xy}.$$

This is the simplest way to rewrite as a fraction. We could also distribute the division to get a simplified sum:

$$\frac{x-2y}{3xy}=\frac{x}{3xy}-\frac{2y}{3xy}=\frac{1}{3y}-\frac{2}{3x}.$$

□

1.5.3 Equations

An **equation** is a logical statement that two expressions are equal. As a logical statement, an equation can be **true** or **false**.

Example 1.5.7 $1+1=3$ is an equation. The two expressions are $1+1$ and 3. This equation is a false statement because the values of the expressions are different. □

When an equation involves variables, the truth of the statement depends on the values of the variables. If we specify particular values for each variable, then we calculate the exact numerical value of each expression and then test if the values are equal. For some values of the variables, the equation may be false; for other values, the equation will be true. A **solution** to the equation is a set of values for the variables in the equation that makes the statement true. The **solution set** of an equation is the set of all possible solutions. If the equation is true for all possible values of the variables, the equation is called an **identity**.

Example 1.5.8 $x+1=3$ is an equation with a variable x. When $x=1$, or when x represents the value 1, the equation is the same as our earlier example. In that case, the equation is false. However, when $x=2$, the equation corresponds to $2+1=3$, which is true. The value 2 is a solution and is in the solution set. □

Example 1.5.9 The statement $2x+5=13$ is an equation involving a single variable, x. We can test the equation using different values for x. For $x=1$, the expression $2x+5$ has a value $2(1)+5=7$. Since $7\neq 13$, the equation is false and $x=1$ is not a solution. For $x=4$, the expression $2x+5$ has a value $2(4)+5=13$. We see that $x=1$ is a solution because the expressions $2x+5$ and 13 have the same value. The value 1 is in the solution set. □

In the examples above, we took possible numbers and tested if they were solutions. Testing values to find solutions is impractical because there are infinitely many different values possible for each variable. Finding a solution by guessing would be a stroke of luck. If we did find a solution, we might use our intuition to say that we found all of the solutions. But how do we *know*? What if our intuition is wrong? Finding one solution does not tell you whether there might be more solutions.

Instead, we use algebra to find solutions by **solving** the equation. You would have learned many strategies for solving equations in an earlier algebra class. Rather than attempt to address every strategy, we will focus on the overarching principles.

Most of these strategies rely on a principle of finding **equivalent** equations. Equations are equivalent when they true or false for exactly the same values of variables. The symbol $\Leftrightarrow$ is used to say that two logical statements are equivalent.

You may have learned that an equation is like a balance or scale. The two expressions are like two masses being balanced against one another. The equation is true if the masses are in balance. We create an equivalent equation if we apply the same operation to both sides of the equation, so long as the operation is invertible.

Balanced Operations Result in Equivalent Equations.

The following operations can be used to create equivalent equations, where each variable represents arbitrary expressions.

- Balanced Addition: $a = b$ is equivalent to $a + c = b + c$.
- Balanced Subtraction: $a = b$ is equivalent to $a - c = b - c$.
- Balanced Multiplication: $a = b$ is equivalent to $a \cdot c = b \cdot c$, so long as $c \neq 0$.
- Balanced Division: $a = b$ is equivalent to $\frac{a}{c} = \frac{b}{c}$, so long as $c \neq 0$.

Because multiplication and division include a condition $c \neq 0$, the new equation might have extra solutions corresponding to values where $c = 0$ that are not solutions to the original equation. These **extraneous** should not be confused with actual solutions.

In addition to the balanced arithmetic operations, we will later learn about invertible or one-to-one functions. An invertible function can be applied to both sides of an equation to create an equivalent equation, so long as the expressions have values in the function domain. Noninvertible functions potentially introduce extraneous solutions.

The primary strategy for solving an equation is to create an equivalent equation where the variable is isolated. If a variable appears only once in an equation, then our strategy would be to apply balanced operations until one side of the equation only has that variable. Generally, we can use the **inverse operation** for the last operation in the expression based on the order of operations. If we think about the operations involved in an expression as wrapping layers around the variable, then applying inverse operations would be like unwrapping the variable one layer at a time.

Example 1.5.10 Consider the earlier equation $2x + 5 = 13$. Use balanced operations to solve the equation.

Solution. The variable x only appears in the expression $2x+5$. Because order of operations applies multiplication before addition, the operation of addition $+5$ would be the last operation. The **inverse operation** is to add -5, which we do in a balanced way.

$$2x + 5 = 13 \quad \Leftrightarrow \quad 2x + 5 + -5 = 13 + -5 \quad \Leftrightarrow \quad 2x = 8$$

The last operation in the expression $2x$ is now multiplication by 2. The next balanced operation is to multiply by the inverse $\div 2 = \frac{1}{2}$.

$$2x + 5 = 13 \quad \Leftrightarrow \quad 2x = 8 \quad \Leftrightarrow \quad \frac{1}{2} \cdot 2x = \frac{1}{2} \cdot 8 \quad \Leftrightarrow \quad x = 4$$

The equation $x = 4$ has isolated the variable, so the only solution is $x = 4$. The solution set $\{x : 2x + 5 = 13\}$—the set of values x that make $2x + 5 = 13$ true—has a single value $\{x : 2x + 5 = 13\} = \{4\}$. □

If an equation has the variable appearing in multiple locations, we generally have two strategies to consider. One strategy—isolating a variable—is to find an equivalent equation where the variable only appears once. To do this, we use balanced operations to put terms with the variable on the same side of the equation. We then use algebra properties, if possible, to solve for that variable.

Example 1.5.11 Solve the equation $\dfrac{3x}{x+2} = 2$.

Solution. The equation has the variable x appear twice. For the expression on the left to be defined, we know $x+2 \neq 0$. We can use balanced multiplication and multiply both sides of the equation by $x+2$ to find an equivalent equation

$$3x = 2(x + 2).$$

(This is also called cross-multiplication.) The right expression can be rewritten to obtain

$$3x = 2x + 4,$$

which can be solved as

$$x = 4.$$

We check our answer by testing the truth of the original equation. With $x = 4$, our equation is

$$\frac{3(4)}{4+2} = 2.$$

Because $3(4) = 12$ and $4 + 2 = 6$ and $12 \div 6 = 2$, the equation is true. The solution set is

$$\{x : \frac{3x}{x+2} = 2\} = \{4\}.$$

□

The other common strategy—factoring—is to find an equivalent equation with one expression exactly zero and the other expression is factored. The factoring strategy is based on the properties of zero in relation to multiplication. When *non-zero* numbers are multiplied, the product is also non-zero. The only way a product can equal zero is if one of the factors is zero.

Theorem 1.5.12 Product Equals Zero. *Given any expressions A and B, the equation $A \cdot B = 0$ is equivalent to the compound statement $A = 0$ or $B = 0$.*

Consequently, when an equation is written as a product equalling zero, we can identify all solutions for each factor individually equal to zero. The solution set will then be the **union** of the solutions of these separate equations.

Example 1.5.13 Solve the equation $x^3 = 4x$.

Solution. Because the variable x appears as a cube x^3 and alone as x, isolating the variable will not be a successful strategy. We use factoring instead, which requires moving all terms to one side. The balanced operation would be to add $-4x$ to both sides,

$$x^3 - 4x = 0.$$

Now that we have an equivalent equation written as an expression equal to zero, we need to factor our expression. The expression has a common factor x in all terms, so we write

$$x(x^2 - 4) = 0.$$

The factoring principle tells us that solutions satisfy either $x = 0$ or $x^2 - 4 = 0$. We can continue to factor:

$$x(x + 2)(x - 2) = 0.$$

Now, each factor might equal zero leading to a different solution: $x = 0$, $x = -2$, or $x = 2$. Because these are the only values that make a factor equal zero, they are the only solutions. The solution set is the union of the three values,

$$\{x : x^3 - 4x\} = \{-2, 0, 2\}.$$

□

In the next section, we will explore the strategy of factoring in more depth in the context of solving polynomial equations. We will also review using the quadratic formula to solve equations. The next example reminds you to be careful about what you think will be equivalent equations.

Example 1.5.14 Solve the equation

$$\frac{x}{x-3} = \frac{3x-4}{x-3}.$$

Solution. A common strategy for this equation that two fractions are equal is to cross-multiply. That is, multiply the x in the numerator on the left by the $x - 3$ in the denominator on the right, and then multiply the $3x - 4$ in the numerator on the right by the $x - 3$ in the denominator on the left. Then we can use the factoring method.

$$\begin{aligned}
x(x - 3) &= (3x - 4)(x - 3) \\
x^2 - 3x &= 3x^2 - 9x - 4x + 12 \\
x^2 - 3x &= 3x^2 - 13x + 12 \\
0 &= 2x^2 - 10x + 12 \\
2(x^2 - 5x + 6) &= 0 \\
2(x - 2)(x - 3) &= 0
\end{aligned}$$

This final equation is factored. The equation $2 = 0$ has *no solution*. The equations $x - 2 = 0$ and $x - 3 = 0$ have solutions $x = 2$ and $x = 3$, respectively. However, because the denominators were the same, $x - 3$, our solution $x = 3$ was actually an extraneous solution.

Multiplying an equation involving fractions by an expression involving x always risks introducing extraneous solutions, particularly if it changes the domain of the expressions. Factoring is always preferable and only slightly more challenging. To use factoring, we find an equivalent equation by adding expressions to get zero on one side,

$$\frac{x}{x-3} - \frac{3x-4}{x-3} = 0.$$

The common denominator is a common inverse factor, allowing us to combine the fractions,

$$\frac{x}{x-3} - \frac{3x-4}{x-3} = \frac{x-(3x-4)}{x-3} = \frac{x-3x+4}{x-3} = \frac{-2x+4}{x-3}.$$

Consequently, our equation is equivalent to

$$\frac{-2x+4}{x-3} = 0.$$

The factors are $-2x+4$ and the multiplicative inverse of $x-3$, which can never equal zero. The only solution is the solution to $-2x+4=0$ or $x=2$. (A quotient equals zero only if the numerator equals zero and the denominator is non-zero.) □

Finally, if an equation is equivalent to an equation that is always false, then the equation has no solutions. The solution set is the empty set, $\varnothing = \{\}$.

Example 1.5.15 Find the solution set for the equation $\dfrac{c}{c+3} = 1$.

Solution. When we cross-multiply the equation by the expression $c+3$ (assuming $c \neq -3$), we get an equation

$$c = c+3$$

which is equivalent to

$$0 = 3.$$

Both of these equivalent equations are never true. There are no solutions to the original equation. The solution set is the empty set $\varnothing$. □

1.5.4 Systems of Equations

We have just discussed solving an equation for a single variable. Equations might involve multiple variables. Such an equation establishes a relation between the variables. Solutions will require that the value of one variable depends on the values of any other variables.

Example 1.5.16 The equation

$$u^2 + v^2 = 16 + 6u$$

forms a relation between variables u and v.

- Find the possible values for v when $u = -1$.
- Find the possible values for u when $v = 2$.
- Find the possible values for u and v when $u = v$.

Solution. First, to solve the equation when $u=-1$, we substitute the value of $u=-1$ and then use algebra to isolate v.

$$\begin{aligned} u^2 + v^2 &= 16 + 6u \\ (-1)^2 + v^2 &= 16 + 6(-1) \end{aligned}$$

$$\begin{aligned} 1 + v^2 &= 10 \\ v^2 &= 9 \\ v &= \pm 3 \end{aligned}$$

There are two values for v when $u = -1$. The solutions are the states $(u, v) = (-1, 3)$ and $(u, v) = (-1, -3)$.

Next, to solve the equation when $v = 2$, we substitute the value $v = 2$. However, because u appears in the equation with terms u^2 and $6u$, we can not combine terms to isolate u. Instead, we need to use the (((Unresolved xref, reference "thm-quadratic-formula"; check spelling or use "provisional" attribute)))quadratic formula.

$$\begin{aligned} u^2 + v^2 &= 16 + 6u \\ u^2 + (2)^2 &= 16 + 6u \\ u^2 + 4 &= 16 + 6u \\ u^2 - 6u - 12 &= 0 \\ u &= \frac{6 \pm \sqrt{(-6)^2 - 4(-12)}}{2} \\ u &= \frac{6 \pm \sqrt{84}}{2} = \frac{6 \pm 2\sqrt{21}}{2} \\ u &= 3 \pm \sqrt{21} \end{aligned}$$

Again, two states are solutions, $(u, v) = (3 + \sqrt{21}, 2)$ and $(u, v) = (3 - \sqrt{21}, 2)$.

Finally, the equation $u = v$ is a constraint involving both variables. Because v is shown as a dependent variable in the constraint $v = u$, we substitute u in place of v in the original equation.

$$\begin{aligned} u^2 + v^2 &= 16 + 6u \\ u^2 + (u)^2 &= 16 + 6u \\ 2u^2 - 6u - 16 &= 0 \\ u^2 - 3u - 8 &= 0 \\ u &= \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-8)}}{2} \\ u &= \frac{3 \pm \sqrt{9 + 32}}{2} = \frac{3 \pm \sqrt{41}}{2} \end{aligned}$$

For each value of u, we have $v = u$. One solution would be $(u, v) = (\frac{3+\sqrt{41}}{2}, \frac{3+\sqrt{41}}{2})$ while the other solution would be $(u, v) = (\frac{3-\sqrt{41}}{2}, \frac{3-\sqrt{41}}{2})$. □

When working with multiple variables, we often have multiple equations. For example, when we created equations for the parameters of a parametrized model from data, we created a different equation involving the parameters for each data point. A solution for one equation is a state giving values for each of the variables such that the equation is true. A solution for the *system* of equations is a state that makes all of the equations true.

A useful strategy for solving equations is to isolate a dependent variable in one equation and then substitute the resulting value or formula into the other equation. That equation then has one variable which can be solved using the usual methods.

Example 1.5.17 Find a model of the form $y = ax + bx^2$ that passes through the points $(x, y) = (1, 2)$ and $(x, y) = (2, 5)$.

Solution. Using the data provides us with an equation for a and b for each point:

$$\begin{aligned} (x,y) = (1,2) \quad &\Rightarrow \quad 2 = a(1) + b(1)^2 \\ (x,y) = (2,5) \quad &\Rightarrow \quad 5 = a(2) + b(2)^2 \end{aligned}$$

These equations form a system of equations that must be satisfied simultaneously,

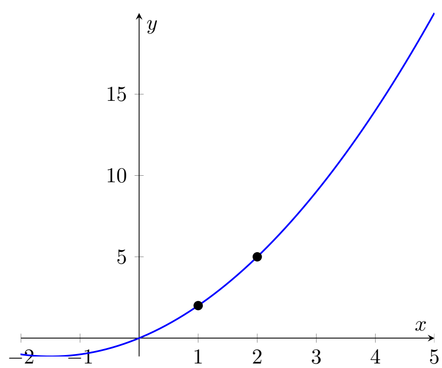
$$\begin{cases} a + b = 2, \\ 2a + 4b = 5. \end{cases}$$

This sets up the mathematical problem that we will solve.

We begin by solving one of the equations for one of the variables. If we take the equation $a + b = 2$ and solve for b, we obtain $b = 2 - a$. We can now substitute the expression $2 - a$ in place of b in the *other* equation, and then solve for a:

$$\begin{aligned} 2a + 4(2 - a) &= 5 \\ 2a + 8 - 4a &= 5 \\ -2a + 8 &= 5 \\ -2a &= -3 \\ a &= \frac{3}{2} \end{aligned}$$

Knowing that $a = \frac{3}{2}$ and that $b = 2 - a$, we find $b = \frac{1}{2}$. The model passing through the data is therefore $y = \frac{3}{2}x + \frac{1}{2}x^2$. A graph showing this solution is shown below.

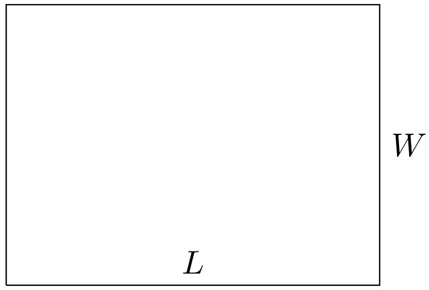


□

Systems of equations allow us to answer questions involving variables that are subject to multiple constraints. A constraint, which provides information about how the variables need to be related, is represented by an equation. Solving the system of equations finds the values that satisfy all constraints.

Example 1.5.18 Is it possible to enclose an area of 25 m^2 in a rectangle with perimeter of 18 m? If so, how?

Solution. In this problem, we need to identify the relevant variables for the system and the equations that constrain the state. We are working with a rectangle, which is characterized by a length and a width. Let us draw a figure and use variables L for the length and W for the width.



The perimeter P and the area A can be considered to be dependent variables, defined by the equations

$$P = 2L + 2W,$$
$$A = L \cdot W.$$

The problem gives us two additional pieces of information, $P = 18$ and $A = 25$. When we substitute those values of the state into the equations, we have two equations for two variables:

$$2L + 2W = 18, \qquad L \cdot W = 25.$$

In order to solve these equations, we use one equation to isolate one of the variables, say L, and then substitute the resulting expression into the other equation.

$$\begin{aligned} 2L + 2W = 18 \qquad &\Rightarrow \qquad L = 9 - W \\ L \cdot W = 25 \qquad &\Rightarrow \qquad (9 - W)W = 25 \end{aligned}$$

Then we solve the equation that only involves W.

$$\begin{aligned} (9 - W)W &= 25 \\ 9W - W^2 &= 25 \\ W^2 - 9W + 25 &= 0 \\ W &= \frac{9 \pm \sqrt{(-9)^2 - 4(25)}}{2} \\ W &= \frac{9 \pm \sqrt{81 - 100}}{2} = \frac{9 \pm \sqrt{-19}}{2} \end{aligned}$$

When solving this quadratic formula, we have the square-root of a negative number giving complex numbers.

In conclusion, we found that there are no real solutions. This means that it is not possible to create a rectangle with a perimeter of 18 m and an area of 25 m^2. □

1.5.5 Equivalence and Graphs

It is useful to think about how graphs relate to equivalent expressions, equivalent equations, and equivalent systems of equations. Understanding the interpretation relating to graphs should help us understand the concepts.

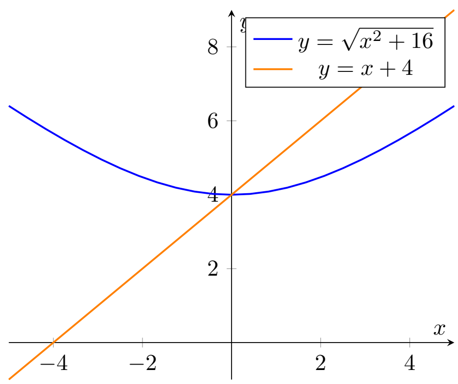
Because two expressions are equivalent if and only if they produce the identical values for any choice of the variables, the graphs of equivalent expressions should look identical. When there is only one variable involved, say x, then a simple graph suffices. Suppose u_1 and u_2 are the two expressions. In a graphing utility, if we graph the two expressions, $y = u_1$ and $y = u_2$, then the graphs should exactly overlap. It can be difficult to check if an overlap is exact or only approximate.

We could instead create a single graph that subtracts one expression from the other, $y = u_1 - u_2$. When the expressions are equivalent, the graph will show $y = 0$ for all values x. However, because computers only approximately represent numbers, computer arithmetic can introduce small errors. Consequently, we should not be surprised to see a graph with small fluctuations.

Example 1.5.19 Use a graph to test whether the following expressions are equivalent.

1. $\sqrt{x^2 + 16}$ and $x + 4$
2. $\dfrac{\sqrt{x+1} - 1}{x}$ and $\dfrac{1}{\sqrt{x+1} + 1}$

Solution. The first comparison between $\sqrt{x^2 + 16}$ and $x + 4$ is checked by graph $y = \sqrt{x^2 + 16}$ and $y = x + 4$ on the same figure.

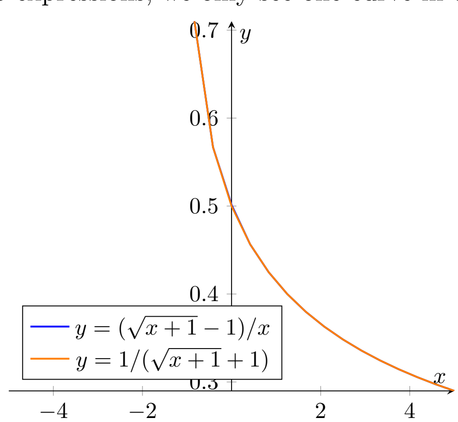


The graphs clearly are different. This is definitive evidence that the expressions are not equivalent. Even identifying just one point, like $x = 3$, and showing the formulas give different results proves that the expressions are not equivalent:

$$\begin{aligned} x = 3 \quad &\Rightarrow \quad \sqrt{x^2 + 16} = \sqrt{9 + 16} = \sqrt{25} = 5, \\ x = 3 \quad &\Rightarrow \quad x + 4 = 3 + 4 = 7. \end{aligned}$$

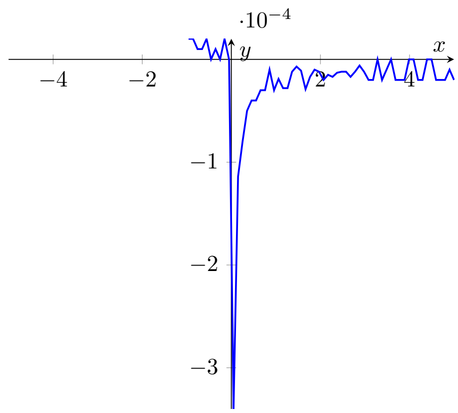
The value used provides a **counterexample** to the claim of equivalence.

The second claim is that $\dfrac{\sqrt{x+1} - 1}{x}$ and $\dfrac{1}{\sqrt{x+1} + 1}$ are equivalent. When we graph these two expressions, we only see one curve in the figure.



Is this because the graphs are the same? Or is it because one of the graphs is so different that it just doesn't appear in the window? To avoid that uncertainty, we can instead graph their difference,

$$y = \frac{\sqrt{x+1}-1}{x} - \frac{1}{\sqrt{x+1}+1}.$$



What do we see? Something interesting seems to be happening at $x = 0$. Notice that the first expression is undefined for $x = 0$ (can't divide by zero). The other expression has a value $\frac{1}{2}$. So the two expressions are not really equivalent because of this one point.

What about the other points? The rest of the graph has very small values that appear to fluctuate. We must consider the possibility that this is due to computer error. Let us try some values for which the square root will be simple.

$$\begin{aligned} x = 3 \quad &\Rightarrow \quad \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{4}-1}{3} = \frac{1}{3} \\ x = 3 \quad &\Rightarrow \quad \frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{4}+1} = \frac{1}{3} \end{aligned}$$

That's a match.

$$\begin{aligned} x = 8 \quad &\Rightarrow \quad \frac{\sqrt{x+1}-1}{x} = \frac{\sqrt{9}-1}{8} = \frac{2}{8} \\ x = 8 \quad &\Rightarrow \quad \frac{1}{\sqrt{x+1}+1} = \frac{1}{\sqrt{9}+1} = \frac{1}{4} \end{aligned}$$

Again, it's a match. We start to think that the expressions probably are equivalent. □

Graphical evidence that expressions are equivalent can give us confidence but do not provide definitive evidence. Ultimately, that needs to come from algebraic arguments. In our example, $\dfrac{\sqrt{x+1}-1}{x}$ and $\dfrac{1}{\sqrt{x+1}+1}$ appear to be equivalent for $x \neq 0$. Let us use algebra to simplify the difference between the expressions.

Example 1.5.20 Show that $\dfrac{\sqrt{x+1}-1}{x}$ and $\dfrac{1}{\sqrt{x+1}+1}$ are equivalent for $x \neq 0$.

Solution. We will simplify the difference by finding a common denominator. Notice how we have to use the distributive property as we FOIL out the product.

$$\begin{aligned}
\frac{\sqrt{x+1}-1}{x} - \frac{1}{\sqrt{x+1}+1} &= \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)} - \frac{x}{x(\sqrt{x+1}+1)} \\
&= \frac{(\sqrt{x+1})^2 + \sqrt{x+1} - \sqrt{x+1} - 1 - x}{x(\sqrt{x+1}+1)} \\
&= \frac{x+1-1-x}{x(\sqrt{x+1}+1)} \\
&= \frac{0}{x(\sqrt{x+1}+1)} = 0
\end{aligned}$$

This proves that the expressions are equivalent for $x \neq 0$. $\square$

The graphical interpretation of equivalent equations is not the same as equivalent expressions. That should make sense because expressions and equations are not the same type of objects. Recall that equations are equivalent if they have the same solution sets. Consequently, we need to understand the graphical meaning of solution sets.

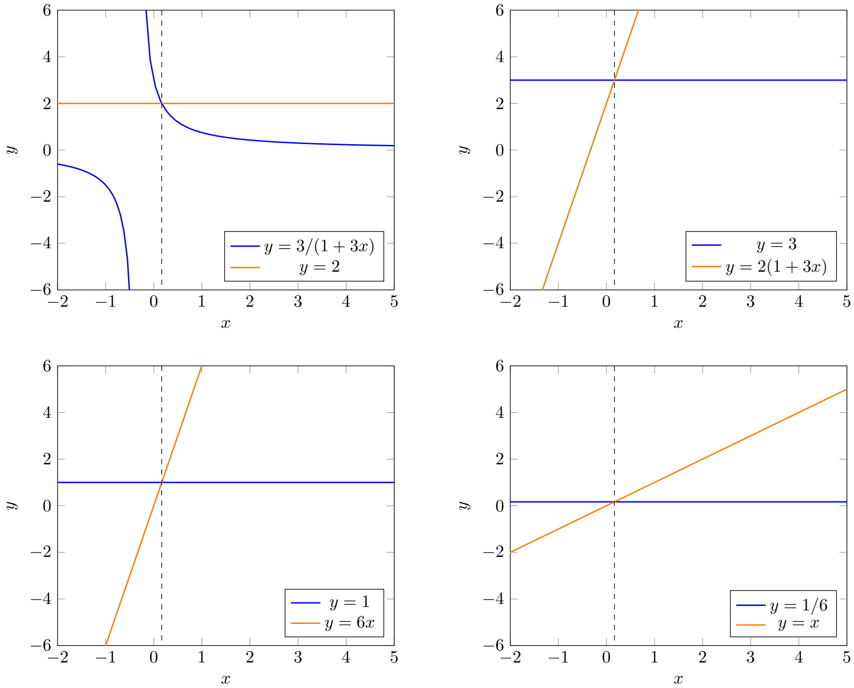
An equation is a statement that two expressions have the same value. We can graph the value of an expression in terms of its independent variable. Since an equation has two expressions, we consider two different graphs. A solution to an equation is a value for the variable where the expressions have the same value. This corresponds to a point where the graphs intersect. Consequently, two equations are equivalent if the values of the variable where the expressions agree are the same.

Example 1.5.21 Solve the equation $\dfrac{3}{1+3x} = 2$ using equivalent equations. For each stage of the solution, graph the expressions in the equations to illustrate the equivalence.

Solution. To solve the equation, we will multiply both sides by $1+3x$ and then isolate the variable x. This gives us the following sequence of equivalent equations.

$$\begin{aligned}
\frac{3}{1+3x} &= 2 \\
3 &= 2(1+3x) \\
3 &= 2+6x \\
1 &= 6x \\
\frac{1}{6} &= x
\end{aligned}$$

For each equation, we should see the expression intersect exactly at $x = \frac{1}{6}$. We graph the two expressions for each equation and the vertical line at $x = \frac{1}{6}$. Each figure shows the graphs intersect at the same x-value. The y-values for the intersection points are changing because the expressions are changing.



□

When we visualized one equation as a graph of two expressions, we introduced a new variable y so that the graph could be shown in the (x, y) plane. The value of y at the point of intersection is not necessarily the same when we change to an equivalent equation. However, when we work with systems of equations, all of the variables are essential. An equivalent system of equations needs solutions to keep the same values for all variables at the point of intersection.

Example 1.5.22 Solve the system of equations

$$\begin{cases} x + 4y = 6 \\ 3x - y = 5 \end{cases}$$

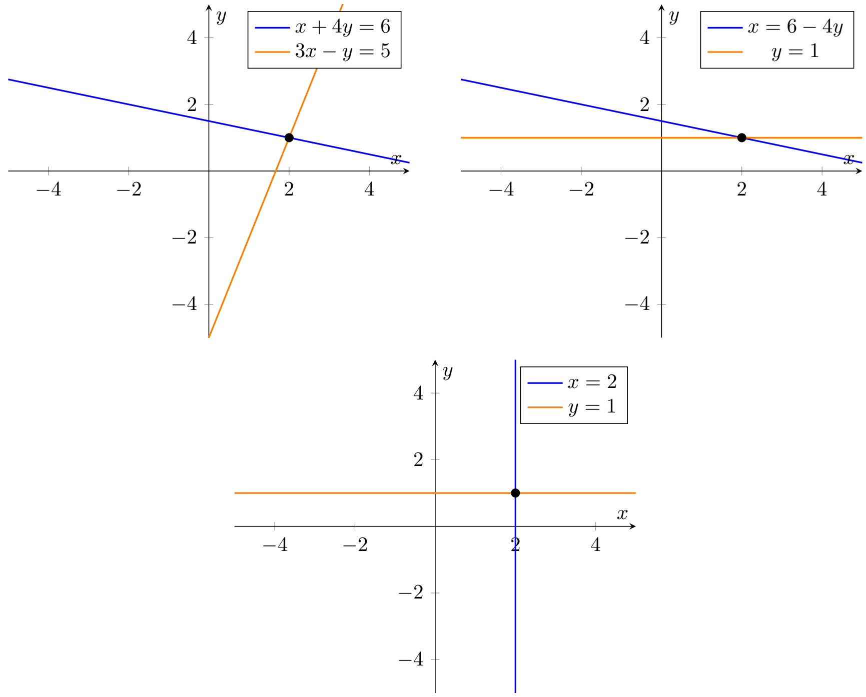
and graph the equivalent systems at each stage.

Solution. Solving the first equation for x, we get $x = 6 - 4y$. When we substitute this expression into the second equation, we have a new system.

$$\begin{cases} x = 6 - 4y \\ 3(6 - 4y) - y = 5 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = 6 - 4y \\ 18 - 13y = 5 \end{cases}$$

The second equation in this system simplifies to $-13y = -13$ or $y = 1$. When we substitute that value back into the first equation, we get $x = 6 - 4(1)$ or $x = 2$. The solution to the system is $(x, y) = (2, 1)$.

When graphing the equations in the equivalent systems, we note that each equation is the graph of a line. We can graph the lines quickly by plotting their intercepts found by setting $x = 0$ or $y = 0$ and solving for the other value. The equation $x + 4y = 6$ has a y-intercept $(x = 0)$ at $y = \frac{3}{2}$ and an x-intercept $(y = 0)$ at $x = 6$. The equation $3x - y = 5$ has a y-intercept $(x = 0)$ at $y = -5$ and an x-intercept $(y = 0)$ at $x = \frac{5}{3}$. In our second system, the equation $y = 1$ is a horizontal line because x can have any value. In the final system (the solution), the equation $x = 2$ corresponds to a vertical line where y can have any value.



□

1.5.6 Summary

- An expression is any value or a formula that represents a value. Expressions are equivalent if they have the same value for all possible assignments of the variables. Simplifying an expression is to find an equivalent expression in a form that meets an established convention.

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- An equation is a logical statement that two expressions are equal. A solution to an equation is a state (values specified for all variables) that makes the equation true. The solution set is the set of all possible solutions. Equations are equivalent if they have exactly the same solution sets.

- The primary method for solving an equation is to find an equivalent equation that isolates the variable.

- A key fact about arithmetic is that the only way a product can equal zero is if one factor is zero. This fact is used to solve equations written as an expression equal to zero by factoring.

- A system of equations has solutions described by states with all variables having values that make every equation true. Equivalent systems of equations have the same solutions.

1.5.7 Exercises

1. Show using the elementary properties of addition and multiplication why $2(x + 3) - 1 = 2x + 5$.

2. Show using the elementary properties of addition and multiplication why $(x + 3)(x - 1) = x^2 + 2x - 3$.

3. A student made a mistake writing $\dfrac{3x+1}{x} = \dfrac{3+1}{1} = 4$. What did the student do? Why was it incorrect?

4. A student made a mistake writing $x \cdot \dfrac{2x+1}{x+3} = \dfrac{2x^2+x}{x^2+3x}$. What did the student do? Why was it incorrect?

Rewrite each of the following expressions as an equivalent sum instead of as a product.

5. $4(x+3)$

6. $3(x-2)(x+4)$

7. $(x-1)(x-2)(x-3)$

Rewrite each of the following expressions as an equivalent factored expression.

8. $3x - 15$

9. $4x^2 - 6x$

Without solving the equation, which of the following values are in the solution set?

10. $x^2 - 2x = x + 4$

(a) $x = -2$

(b) $x = -1$

(c) $x = 0$

(d) $x = 1$

(e) $x = 2$

11. $z^3 - 5z = 2z^2 - 6$

(a) $z = -2$

(b) $z = -1$

(c) $z = 0$

(d) $z = 1$

(e) $z = 2$

Find the solution set for each equation.

12. $2(x+5) - 3 = 7$

13. $3t + 5 = t - 2$

14. $\dfrac{4u}{u+5} = 3$

15. $\dfrac{4u}{u+5} = 4$

16. $\dfrac{2y}{y-1} = 2$

17. $\dfrac{2y}{y-1} = 3$

18. $(2x-3)(x+2) = 0$

19. $t(5t-1)(3-t) = 0$

20. $\dfrac{p(p+1)}{p+2} = 0$

Find the solutions to the system of equations.

21. $3x + 2y = 15$ and $y = 3$.

22. $2x - 5y = 7$ and $x + 2y = 9$.

23. $x^2 - y = 4x$ and $2x + y = 3$.

24. Is it possible to enclose an area of 25 m^2 using a rectangle with perimeter of 25 m? If so, how?

25. Is it possible to enclose an area of 50 m^2 in two congruent rectangles that share an edge such that the total length of edges is 40 m (counting the shared edge only once)? If so, how?