10.1 Antiderivatives

We have previously studied the differentiation operator. Given a function relationship between two variables $x \overset{f}{\mapsto} Q$, the derivative f' is the function relating x to the rate of change $\frac{dQ}{dx}$. Differentiation is the operation that goes maps $f \overset{\frac{d}{dx}}{\mapsto} f'$. Because f' is itself a function, we can apply differentiation again $f' \overset{\frac{d}{dx}}{\mapsto} f''$. This process can repeat indefinitely.

Consider as an example $f(x) = x^4 + 2x^2 - 3x$. There is a sequence of functions corresponding to the derivatives:

$$\begin{aligned} f(x) &= x^4 + 2x^2 - 3x, \\ f'(x) &= 4x^3 + 4x - 3, \\ f''(x) &= 12x^2 + 4, \\ f^{(3)}(x) &= 24x, \\ f^{(4)}(x) &= 24, \\ f^{(5)}(x) &= 0, \\ f^{(6)}(x) &= 0. \end{aligned}$$

This pattern continues with $f^{(n)}(x) = 0$ for $n = 5, 6, 7, \ldots$.

As the example above illustrates, given a function we can find its derivative. One of the major themes of mathematics is the idea of inverse operations. Is there an inverse operation to differentiation? That is, given $f(x)$, instead of computing $f'(x)$, can we find a function $F(x)$ so that $F(x) \overset{\frac{d}{dx}}{\mapsto} f(x)$? This inverse operation, using $f(x)$ to find $F(x)$, is called **antidifferentiation**.

In this section, we define antiderivatives. We discuss why a function has infinitely many different antiderivatives. Based on the First Part of the Fundamental Theorem of Calculus, we recognize that accumulation functions are special examples of antiderivatives for continuous rates of accumulation. Motivated by this observation, we introduce the indefinite integral as the notation for antidifferentiation. Examples will illustrate how we use our known differentiation rules to develop corresponding antidifferentiation rules.

10.1.1 Terminology

Definition 10.1.1 Antiderivatives. Given a function $f(x)$, we say that $F(x)$ is an **antiderivative** of $f(x)$ if $f(x)$ is the derivative of $F(x)$. That is, $F'(x) = f(x)$. ♢

The derivative of any constant is zero, so adding a constant to a function creates a new function that has the same derivative as the original. This means that differentiation is not one-to-one.

Example 10.1.2 Compare the following derivatives:

$$\begin{aligned} \frac{d}{dx}[x^2 + 3x] &= 2x + 3, \\ \frac{d}{dx}[x^2 + 3x - 1] &= 2x + 3, \\ \frac{d}{dx}[x^2 + 3x + 4] &= 2x + 3. \end{aligned}$$

Each of the functions have the same derivative. We say that x^2+3x, x^2+3x-1, and $x^2 + 3x + 4$ are all antiderivatives of $2x + 3$. More generally, we know

$x^2 + 3x + C$ will be an antiderivative for *any* constant value C. □

If we know that a function $F(x)$ is an antiderivative of $f(x)$, then we know that all functions of the form $F(x) + C$, where C is a constant, are also antiderivatives. This shows that infinitely many different functions have the same derivative. We call all such functions **antiderivatives**.

We will later prove the following theorem. It states that the *only* way that two antiderivatives can be different is that they differ by a constant. The proof of the theorem will use a Mean Value Theorem for derivatives.

Theorem 10.1.3 *Suppose that $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval I. That is, for all $x \in I$ we have*

$$F'(x) = G'(x) = f(x).$$

Then there is a constant C so that for all $x \in I$, $G(x) = F(x) + C$.

Consequently, knowing just one antiderivative allows us to determine all possible antiderivatives by adding some constant. Suppose $F(x)$ is an antiderivative of $f(x)$. Then any other antiderivative must be $F(x) + C$ for some constant C. If we leave the constant as an unspecified parameter, we call this the **general antiderivative**. Graphically, different antiderivatives correspond to a vertical translation of the graph. That is, all antiderivatives have the same graph shifted up or down relative to one another.

In the case that $f(x)$ is continuous on some interval I, we can define an accumulation function starting at any convenient point $a \in I$,

$$A(x) = \int_a^x f(z)\,dz.$$

By the Part One of the Fundamental Theorem of Calculus, we know that $A'(x) = f(x)$. That is, $A(x)$ is itself an antiderivative of $f(x)$ and any other antiderivative could be written $F(x) = \int_a^x f(z)\,dz + C$.

Owing to this close connection between antiderivatives and integrals, the standard notation for finding antiderivatives is with the integral symbol using an indefinite integral. An indefinite integral will not have any limits of integration, uses the same variable of integration as the independent variable, and refers to antiderivatives rather than definite integrals.

Definition 10.1.4 Indefinite Integrals. Given a function $f(x)$, the **indefinite integral** of $f(x)$ with respect to x, written $\displaystyle\int f(x)\,dx$, is the general antiderivative of $f(x)$. That is, if $F(x)$ is any antiderivative such that $F'(x) = f(x)$, then

$$\int f(x)\,dx = F(x) + C.$$

◊

Using our earlier example, we can write the indefinite integral of $2x + 3$ as

$$\int 2x + 3\,dx = x^2 + 3x + C.$$

The indefinite integral represents the infinite family of all antiderivatives of $2x + 3$.

10.1.2 Examples

For the most part, finding antiderivatives corresponds to recognizing how a function might have been computed as a derivative. Every statement about differentiation has an equivalent statement about integrals. To check whether a proposed function is an antiderivative, we calculate its derivative and compare that with the function inside the integral.

Example 10.1.5 Find $\frac{d}{dx}\left[(3x+5)^4\right]$ and then write down the equivalent statement as an integral.

Solution. The last operation in the expression $(3x+5)^4$ is the power acting on the expression $u = 3x+5$. The derivative requires a chain rule:

$$\begin{aligned}\frac{d}{dx}\left[(3x+5)^4\right] &= 4u^3\frac{du}{dx} \\ &= 4(3x+5)^3(3) \\ &= 12(3x+5)^3.\end{aligned}$$

Once we know the derivative, we can write the equivalent integral

$$\int 12(3x+5)^3\,dx = (3x+5)^4 + C.$$

This says that $(3x+5)^4$ is an antiderivative of $12(3x+5)^3$, along with that same formula plus any constant. □

We must learn to recognize which differentiation rules would result in a particular formula for a given function. Because differentiation is a linear operator, antidifferentiation is as well.

Theorem 10.1.6 *If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$, then for any constants c_1 and c_2, $c_1F(x)+c_2G(x)$ is an antiderivative of $c_1f(x)+c_2g(x)$. We write*

$$\int [c_1f(x)+c_2g(x)]dx = c_1\int f(x)\,dx + c_2\int g(x)\,dx.$$

If the integrand $f(x)$ is expressed as a sum of terms, we typically first try to find antiderivatives of each term.

Example 10.1.7 Find $\int 4x^3 - 2e^{2x}\,dx$.

Solution. We are looking for a function $F(x)$ for which $F'(x) = 4x^3 - 2e^{2x}$. From experience computing derivatives, we know

$$\frac{d}{dx}[x^4] = 4x^3,$$
$$\frac{d}{dx}[e^{2x}] = 2e^{2x}.$$

This suggests we should use the difference $F(x) = x^4 - e^{2x}$. We verify by differentiation:

$$F'(x) = \frac{d}{dx}[x^4 - e^{2x}] = 4x^3 - 2e^{2x}.$$

This verifies that $F(x)$ is an antiderivative of $4x^3 - 2e^{2x}$. The general antiderivative is written as the indefinite integral,

$$\int 4x^3 - 2e^{2x}\,dx = x^4 - e^{2x} + C.$$

□

Most derivative rules do not result in a product of expressions. The product rule for derivatives results in the sum of two products. The quotient rule results in in difference of quotients. Only the chain rule creates a derivative by multiplying two expresions together. Consequently, if we see an integrand with expressions multiplied together, we should consider whether we would benefit from expanding the product as a sum.

Example 10.1.8 Find $\displaystyle\int x^2(x^2-3)\,dx$.

Solution. The function $f(x) = x^2(x^2-3)$ is a product that can be expanded to a sum using the distributive property.

$$f(x) = x^4 - 3x^2.$$

Our experience with the power rule suggests that we should be able to integrate this expression. We know

$$\frac{d}{dx}[x^5] = 5x^4.$$

To eliminate the unwanted constant multiple of 5, we can multiply both sides by $\frac{1}{5}$ to get

$$\frac{d}{dx}\left[\frac{1}{5}x^5\right] = x^4.$$

This suggests an antiderivative

$$F(x) = \frac{1}{5}x^5 - x^3.$$

We verify using regular differentiation rules:

$$\begin{aligned} F'(x) &= \frac{d}{dx}[\frac{1}{5}x^5 - x^3] \\ &= \frac{1}{5}(5x^4) - 3x^2 \\ &= x^4 - 3x^2 = f(x). \end{aligned}$$

We have found

$$\int x^2(x^2-3)\,dx = \frac{1}{5}x^5 - x^3 + C.$$

□

Just as it is useful to collect and learn the basic building blocks for differentiation, we can collect and learn basic building blocks for integration. Each derivative rule has its equivalent statement about antiderivatives. If we incorporate the chain rule, we extend each of the elementary rules to generalized rules.

1. Power Rule: For any power $n \neq -1$,

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C.$$

2. Generalized Power Rule: For any power $n \neq -1$ and expression u,

$$\int u^n \cdot \frac{du}{dx}\,dx = \frac{1}{n+1}u^{n+1} + C.$$

3. Logarithm Rule:
$$\int \frac{1}{x}\,dx = \ln(|x|) + C.$$

4. Generalized Logarithm Rule: For any expression u,
$$\int \frac{u'}{u}\,dx = \ln(|u|) + C.$$

5. Elementary Exponential Rule: For any real value $k \neq 0$,
$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C.$$

6. Generalized Exponential Rule: For any expression u,
$$\int e^u \cdot \frac{du}{dx}\,dx = e^u + C.$$

Example 10.1.9 $\displaystyle\int x^2 e^{x^3}\,dx$

Solution. Because the integrand has a product of expressions, we should begin by looking to see if the problem involves the chain rule. The exponential term e^{x^3} involves the expression $u = x^3$ which has a derivative $u' = 3x^2$. Notice that the other factor in the problem, x^2, differs from u' only by a constant multiple. That is, we can recognize our problem as a generalized exponential

$$\begin{aligned}\int x^2 e^{x^3}\,dx &= \int \frac{1}{3}(3x^2)e^{x^3}\,dx \\ &\underset{u=x^3}{=} \int \frac{1}{3}e^u \cdot \frac{du}{dx}\,dx \\ &\underset{u=x^3}{=} \frac{1}{3}e^u + C \\ &= \frac{1}{3}e^{x^3} + C.\end{aligned}$$

□

10.1.3 Finding a Particular Antiderivative

Adding a constant to a function represents a graphical transformation of a vertical shift. Consequently, different antiderivatives have the same graph shifted vertically from one another. Consider the function $f(x) = x^2 - 4x$. Integration gives us

$$\int x^2 - 4x\,dx = \frac{1}{3}x^3 - 2x^2 + C.$$

The function $F(x) = \frac{1}{3}x^3 - 2x^2$ has the derivative $F'(x) = x^2 - 4x$, as does every function $F(x) + C$.

The following dynamic graph has a slider for the integration constant C. Notice that changing the value of C shifts the graph up or down. See if you can find a value so that the graph $y = F(x) + C$ goes through $(x, y) = (3, 2)$.

A deprecated JSXGraph interactive demonstration goes here in interactive output.

Figure 10.1.10 $y = F(x) + C$

We can solve for the integration constant to find a particular antiderivative passing through a given point.

Example 10.1.11 Find the constant C so that $F(x) = \frac{1}{3}x^3 - 2x^2 + C$ satisfies $F(3) = 2$.

Solution. Substitute the value $x = 3$ into the equation for $F(x)$.

$$\begin{aligned} F(3) &= \frac{1}{3}(3^3) - 2(3^2) + C \\ &= 9 - 18 + C \\ &= -9 + C \end{aligned}$$

Because we want $F(3) = 2$, we create the equation

$$-9 + C = 2$$

so that we can solve for C to get $C = 11$. □

Example 10.1.12 Find a function $P(t)$ so that $P'(t) = 20e^{-2t} + 3t$ and $P(0) = 50$.

Solution. Start by finding the general antiderivative.

$$\int [20e^{-2t} + 3t]dt = -10e^{-2t} + \frac{3}{2}t^2 + C$$

We therefore see that $P(t) = -10e^{-2t} + \frac{3}{2}t^2 + C$. Now we substitute $t = 0$ and $P(0) = 50$ to solve for C.

$$\begin{aligned} P(0) &= -10e^0 + \frac{3}{2}(0^2) + C \\ 50 &= -10 + C \\ 60 &= C \end{aligned}$$

Having found $C = 60$, we can conclude

$$P(t) = -10e^{-2t} + \frac{3}{2}t^2 + 60.$$

□

Because the derivative represents a rate of change, finding particular antiderivatives is equivalent to finding a quantity as a function of an independent variable when we know the rate of change as a function and we know an initial value.

Example 10.1.13 A cup of coffee starts at a temperature of 160 degrees Fahrenheit. The temperature changes at a rate of change (degrees per minute) modeled by the formula $-3.6e^{-0.04t}$ where t is the time in minutes. Find the temperature as a function of time.

Solution. Let T represent the temperature of the cup of coffee in degrees Fahrenheit. Our given information shows that

$$\frac{dT}{dt} = -3.6e^{-0.04t}.$$

The temperature T must be an antiderivative of this formula,

$$\begin{aligned} T &= \int -3.6e^{-0.04t}\,dt \\ &= \frac{-3.6}{-0.04}e^{-0.04t} + C \\ &= 90e^{-0.04t} + C. \end{aligned}$$

To find the value of C, substitute $t = 0$ and $T = 160$.

$$\begin{aligned} T &= 90e^{-0.04t} + C \\ 160 &= 90e^{0} + C \\ 160 &= 90 + C \\ 70 &= C \end{aligned}$$

Consequently, we have $T = 90e^{-0.04t} + 70$. □

10.1.4 Summary

1. An antiderivative of $f(x)$ is any function $F(x)$ so that $\frac{d}{dx}[F(x)] = f(x)$. If $F(x)$ is an antiderivative of $f(x)$, then so is $F(x) + C$ for any value of C.

2. The Fundamental Theorem of Calculus guarantees that every continuous function has an antiderivative. In particular, if $f(x)$ is continuous on an interval I with $a \in I$, then the accumulation function

$$A(x) = \int_a^x f(z)\,dz$$

is an antiderivative on the interval I.

3. We use the **indefinite integral** as the operator for antidifferentiation. For a function $f(x)$ with antiderivative $F(x)$, we write

$$\int [f(x)]\,dx = F(x) + C$$

where C (or any other chosen symbol) represents an arbitrary **constant of integration**.

4. The constant of integration graphically represents an arbitrary vertical shift of the graph of a function. Given any point representing an initial value, we can solve for the constant of integration so that there is the graph of an antiderivative which passes through the given point.

10.1.5 Exercises

Calculate the specified derivative and then write the equivalent indefinite integral.

1. $\frac{d}{dx}\left[2x^4\right]$

2. $\frac{d}{dx}\left[(2x+3)^5\right]$

3. $\dfrac{d}{dx}\left[\sqrt{x^2+3}\right]$

4. $\dfrac{d}{dx}\left[\ln(|x^2-4x|)\right]$

5. $\dfrac{d}{dx}\left[e^{3x^4}\right]$

6. $\dfrac{d}{dx}\left[x^2e^{-3x}\right]$

7. $\dfrac{d}{dx}\left[x\ln(|x|)\right]$

8. $\dfrac{d}{dx}\left[\dfrac{x-1}{x-3}\right]$

Compute the indefinite integral by finding the general antiderivative. Some integrands need to be rewritten before integration.

9. $\displaystyle\int -3x^5+2x^2+3\,dx$

10. $\displaystyle\int 2x-4x^{-1}+5x^{-3}\,dx$

11. $\displaystyle\int x^3(3x^2-4x+7)\,dx$

12. $\displaystyle\int (x+4)(x-8)\,dx$

13. $\displaystyle\int \frac{x^2+4x-5}{3x^2}\,dx$

14. $\displaystyle\int e^{2x}\,dx$

15. $\displaystyle\int 4e^{-3x}\,dx$

16. $\displaystyle\int xe^{x^2}\,dx$

17. $\displaystyle\int 2x^3e^{-x^4}\,dx$

18. $\displaystyle\int \frac{1}{x+3}\,dx$

19. $\displaystyle\int \frac{3}{2x+1}\,dx$

20. $\displaystyle\int \frac{x}{x^2+4}\,dx$

21. $\displaystyle\int \frac{e^{2x}}{e^{2x}+1}\,dx$

22. $\displaystyle\int -xe^{-x}+e^{-x}\,dx$

23. $\displaystyle\int \frac{2xe^{2x}-e^{2x}}{x^2}\,dx$

Use the given information to find the particular function.

24. Find $f(x)$ if $f'(x) = 2x - 5$ with $f(1) = 4$.

25. Find $g(x)$ if $g'(x) = 3e^{-3x}$ with $g(0) = 2$.

26. The velocity of a vehicle on track that runs left to right is $v(t) = \frac{1}{2}t^2 - 8t + 24$. If the vehicle is at a position $s = 0$ when $t = 1$, find the position $s(t)$ as a function of time.

27. A population changes at a rate defined by $R(t) = 0.24t^2 - 24t + 216$, where t is measured in years. If the population is $P = 120000$ when $t = 0$, find the population as a function of time.

28. A radiation detector absorbs radiation at a rate of $R(t) = 5e^{-0.1t}$ (grays per minute). Find the total amount of radiation absorbed by the detector as a function of time t (minutes) since $t = 0$.