10.3 Definite Integrals and Antiderivatives

An accumulation function with rate $f(x)$ is a function A defined as the definite integral of $f(x)$ from a fixed lower limit c to the independent variable as the upper limit,

$$A(x) = \int_c^x f(z)\,dz.$$

The motivation for such a function was that the definite integral computes the total change of a quantity when the rate of change is given. By the splitting property of integration, if $A(x)$ is continuous then

$$\int_a^b f(x)\,dx = A(b) - A(a).$$

The Fundamental Theorem of Calculus showed that if $f(x)$ is continuous, then $A(x)$ is differentiable and $A'(x) = f(x)$. That is, $A(x)$ is an antiderivative of $f(x)$. This motivates another method for computing definite integrals.

In this section, we apply the Fundamental Theorem of Calculus to evaluate definite integrals using any convenient antiderivative. This application is called the Second Part of the Fundamental Theorem of Calculus. We demonstrate these calculations with several examples.

10.3.1 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus shows that an accumulation function is an antiderivative of the integrand. The Mean Value Theorem implies that any other antiderivative will differ from the accumulation function as an antiderivative by some constant. The Second Part of the Fundamental Theorem of Calculus will then allow us to calculate definite integrals by calculating the change in any antiderivative.

Theorem 10.3.1 The Fundamental Theorem of Calculus, Part Two (FTC2). *Given any function $f(x)$ that is continuous on an interval I, let $F(x)$ be an antiderivative so that $F'(x) = f(x)$ for all $x \in I$. Then for values $a, b \in I$,*

$$\int_a^b f(x)\,dx = \big[F(x)\big]_a^b = F(b) - F(a).$$

Proof. Because $f(x)$ is continuous, we can define an accumulation function

$$A(x) = \int_a^x f(z)\,dz.$$

Because $A(x)$ and $F(x)$ are both antiderivatives, with $A'(x) = F'(x) = f(x)$ for all $x \in I$, there is some constant C so that $A(x) = F(x) + C$. Because $A(a) = 0$, we have $F(a) + C = 0$ so that $C = -F(a)$. By the splitting property of integrals, we have

$$\int_a^b f(x)\,dx = A(b) = F(b) + C = F(b) - F(a).$$

■

When evaluating definite integrals using the Fundamental Theorem of Calculus, we are *substituting* the evaluation of a definite integral, which is defined as the limit of a Riemann sum, with the change in an antiderivative. To indicate such a substitution, we should refer to the Fundamental Theorem of

Calculus, perhaps using the abbreviation FTC. We can simultaneously make the substitution and note the formula for the antiderivative by using bracket evaluation notation.

Given any function $F(x)$, the notation $\big[F(x)\big]_a^b$ means to evaluate the change of the expression when x goes from a to b:

$$\big[F(x)\big]_a^b = F(b) - F(a).$$

When the formula for $F(x)$ is simple, the brackets can be dropped and replaced by a vertical bar on the right,

$$F(x)\Big|_a^b = F(b) - F(a).$$

Example 10.3.2 Evaluate $\displaystyle\int_1^4 x^2\,dx$.

Solution. Let us consider the two steps separately. Then we will see how to represent this more compactly using evaluation notation.

First, we need an antiderivative, computed as an indefinite integral

$$\int x^2\,dx = \frac{1}{3}x^3 + C.$$

This tells us that $F(x) = \frac{1}{3}x^3$ is an antiderivative, as is $F(x) = \frac{1}{3}x^3 + 4$ (or any other constant). We'll use the first, since it is simpler; we only need one antiderivative, not all of them.

Second, the Fundamental Theorem of Calculus allows us to evaluate the definite integral as the change in $F(x)$.

$$\begin{aligned}
\int_1^4 x^2\,dx &= F(4) - F(1) \\
&= \frac{1}{3}(4)^3 - \frac{1}{3}(1)^3 \\
&= \frac{64}{3} - \frac{1}{3} \\
&= \frac{63}{3} = 21
\end{aligned}$$

This can be written more compactly by writing the formula of the antiderivative inside the evaluation notation while simultaneously indicating the use of the Fundamental Theorem of Calculus:

$$\begin{aligned}
\int_1^4 x^2\,dx &\overset{\text{FTC}}{=} \Big[\frac{1}{3}x^3\Big]_1^4 \\
&= \frac{1}{3}(4)^3 - \frac{1}{3}(1)^3 \\
&= \frac{64}{3} - \frac{1}{3} = 21
\end{aligned}$$

Notice that we did not need the constant of integration because the Fundamental Theorem of Calculus only requires one antiderivative. We generally choose the most convenient one with a zero constant. □

Evaluation of definite integrals involves recognizing antiderivatives and then evaluating their change.

Example 10.3.3

$$\begin{aligned}\int_1^2 e^{3x}\,dx &\overset{\text{FTC}}{=} \Big[\frac{1}{3}e^{3x}\Big]_1^2\\ &= \frac{1}{3}e^6 - \frac{1}{3}e^3\\ &= \frac{e^6-e^3}{3}\end{aligned}$$

□

Example 10.3.4 Find $\displaystyle\int_1^3 x(3x^2+1)^4\,dx$.

Solution. The integrand has a product of x with u^4 where $u = 3x^2+1$. For the chain rule to have been the source of this product, we would need $\frac{du}{dx} = 6x$ rather than x.

$$\begin{aligned}\int_1^3 x(3x^2+1)^4\,dx &= \int_1^3 \frac{1}{6}(6x)(3x^2+1)^4\,dx\\ &\overset{\text{FTC}}{=} \left[\frac{1}{6}\cdot\frac{1}{5}u^5\right]_{x=1}^{x=3}\\ &= \left[\frac{1}{30}(3x^2+1)^5\right]_1^3\\ &= \frac{1}{30}(3(3^2)+1)^5 - \frac{1}{30}(3(1^2)+1)^5\\ &= \frac{28^5}{30} - \frac{4^5}{30}\\ &= \frac{17,210,368}{30} - \frac{1,024}{30} = \frac{17,209,344}{30} = 573,644.8\end{aligned}$$

□

We have to be careful about satisfying the hypotheses. For example, if $f(x)$ is not continuous over the interval of integration, we can not use antiderivatives to calculate the definite integral.

Example 10.3.5 Find $\displaystyle\int \frac{1}{2x-1}\,dx$. How can we use that result for the following definite integrals?

1. $\displaystyle\int_0^2 \frac{1}{2x-1}\,dx$
2. $\displaystyle\int_1^3 \frac{1}{2x-1}\,dx$

Solution. The integrand is of the form u^{-1} where $u = 2x-1$. The derivative of u is $u' = 2$. In order to antidifferentiate the chain rule, we rewrite

$$\int \frac{1}{2x-1}\,dx = \int \frac{1}{2}\frac{2}{2x-1}\,dx = \frac{1}{2}\ln(|2x-1|) + C.$$

To see if the antiderivative can be used in a definite integral, we need to see where $f(x) = \frac{1}{2x-1}$ is continuous. A discontinuity occurs at $x = \frac{1}{2}$. Consequently, a definite integral using the antiderivative can only be used for intervals that do not include $\frac{1}{2}$. Thus, $\displaystyle\int_0^2 \frac{1}{2x-1}\,dx$ can not be computed. On the other hand,

$$\begin{aligned}\int_1^3 \frac{1}{2x-1}\,dx &\overset{\text{FTC}}{=} \Big[\ln(|2x-1|)\Big]_1^3 \\ &= \ln(|2(3)-1|) - \ln(|2(1)-1|) \\ &= \ln(5) - \ln(1) = \ln(5).\end{aligned}$$

□

10.3.2 Composition with Accumulation Functions

The First Part of the Fundamental Theorem of Calculus tells us the derivative of accumulation functions. Knowing the chain rule allows us to compute derivatives of functions defined by integrals with expressions in the limits of integration.

Example 10.3.6 Compute the following derivatives.

1. $\dfrac{d}{dx}\displaystyle\int_1^x e^{-z^2}\,dz$
2. $\dfrac{d}{dx}\displaystyle\int_x^{x^2} e^{-z^2}\,dz$
3. $\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}} \frac{1}{\sqrt{z^4+1}}\,dz$

Solution. When solving problems involving definite integrals, it is often helpful to explicitly remind yourself of the concept of accumulation functions and the fundamental theorem of calculus's conclusion.

1. Define $A(x) = \displaystyle\int_1^x e^{-z^2}\,dz$, which is the accumulation function with integrand $f(z) = e^{-z^2}$. The Fundamental Theorem of Calculus tells us that $A'(x) = f(x) = e^{-x^2}$. The following work would communicate these results:

$$\begin{aligned}A(x) &= \int_1^x e^{-z^2}\,dz \\ A'(x) &\overset{\text{FTC}}{=} e^{-x^2} \\ \frac{d}{dx}\int_1^x e^{-z^2}\,dz &= A'(x) = e^{-x^2}\end{aligned}$$

2. To compute $\dfrac{d}{dx}\displaystyle\int_x^{x^2} e^{-z^2}\,dz$, we first need the accumulation function $A(x) = \displaystyle\int_1^x e^{-z^2}\,dz$ with rate $f(x) = e^{-x^2}$. The integral that defines our function involves a composition by the splitting property,

$$\int_x^{x^2} e^{-z^2}\,dz = A(x^2) - A(x).$$

When we differentiate this, we must use the chain rule knowing

$$\frac{d}{dx}\Big[A(u)\Big] = A'(u)\frac{du}{dx} \overset{\text{FTC}}{=} f(u)\frac{du}{dx}.$$

The following work would communicate these results:

$$\begin{aligned} A(x) &= \int_1^x e^{-z^2}\,dz \\ \frac{d}{dx}\int_x^{x^2} e^{-z^2}\,dz &= \frac{d}{dx}\Big[A(x^2) - A(x)\Big] \\ &\overset{\text{FTC}}{=} f(x^2)\cdot\frac{d}{dx}[x^2] - f(x) \\ &= e^{-(x^2)^2}\cdot 2x - e^{-x^2} \\ &= 2xe^{-x^4} - e^{-x^2} \end{aligned}$$

3. To compute $\dfrac{d}{dx}\left[\displaystyle\int_0^{\sqrt{x}} \frac{1}{\sqrt{z^4+1}}\,dz\right]$, we will need to define an accumulation function and then apply the Fundamental Theorem of Calculus to find the derivative required.

$$\begin{aligned} A(x) &= \int_0^x \frac{1}{\sqrt{z^4+1}}\,dz \\ A'(x) &\overset{\text{FTC}}{=} \frac{1}{\sqrt{x^4+1}} \\ \frac{d}{dx}\left[\int_0^{\sqrt{x}} \frac{1}{\sqrt{z^4+1}}\,dz\right] &= \frac{d}{dx}[A(\sqrt{x})] \\ &= A'(\sqrt{x})\frac{d}{dx}[\sqrt{x}] \\ &= \frac{1}{\sqrt{(\sqrt{x})^4+1}}\cdot(\frac{1}{2}x^{-1/2}) \\ &= \frac{1}{\sqrt{x^2+1}}\cdot\frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x(x^2+1)}} \end{aligned}$$

□

10.3.3 Summary

1. The Fundamental Theorem of Calculus, Part 1, together with the Mean Value Theorem, imply that for any continuous function $f(x)$, an accumulation function and any other antiderivative will differ by a constant value.

2. The Fundamental Theorem of Calculus, Part 2, states that the definite integral of a function that is continuous on the interval of integration can be substituted for the change in *any* antiderivative of the rate. That is, if $F(x)$ is an antiderivative of $f(x)$ and $f(x)$ is continuous on the interval containing a and b,
$$\int_a^b f(x)\,dx \overset{\text{FTC}}{=} \Big[F(x)\Big]_a^b = F(b) - F(a).$$

3. The Fundamental Theorem of Calculus, Part 1, together with the chain rule, allows us to compute the derivative of functions where the limits

of integration are expressions involving the independent variable. Let u and w be expressions involving x and suppose that $f(x)$ is a continuous function.

$$\frac{d}{dx}\left[\int_u^w f(z)\,dz\right] \overset{\text{FTC}}{=} f(w)\frac{dw}{dx} - f(u)\frac{du}{dx}$$

10.3.4 Exercises