

2.2 Constructing Functions

Overview. We have learned that functions provide a map between two variables of a system. In modeling, the functions are almost always defined by formulas, with the dependent variable being equal to an expression involving only the independent variable. As we analyze these functions with calculus, the rules of computation for limits, derivatives, and integrals will depend on how a function is algebraically put together.

This section focuses on how expressions and functions are constructed. We start by reviewing elementary functions that represent basic operations on the independent variable. These will serve as the building blocks for our functions. We will then consider the basic arithmetic operations of addition, subtraction, multiplication, and division.

2.2.1 Elementary Functions

Every expression defining a function can be interpreted as a combination of various operations. Operations that act on a single expression are functions. Operations that combine multiple expressions include the binary arithmetic operations, particularly addition and multiplication. In order to characterize expressions, we first review the elementary operations that can be considered as elementary functions. We will consider an **elementary operation** to be a single operation on the variable.

The simplest operations are the constant functions and the identity function. As an operation, the constant function ignores the variable and always gives the same value for the output. The identity function, on the other hand, has no net change with the variable and returns an output that matches the input.

Definition 2.2.1 A **constant function** is a function that has the same output value for every input value, $f(x) = c$ for some constant c . \diamond

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Figure 2.2.2 The constant function $f(x) = 3$ as a map $x \xrightarrow{f} 3$.

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Figure 2.2.3 The graph of the constant function $y = f(x) = 3$ in the (x, y) plane.

Definition 2.2.4 The **identity function** is a function where the output value is the same as the input value, $f(x) = x$. \diamond

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Figure 2.2.5 The identity function $f(x) = x$ as a map $x \xrightarrow{f} x$.

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Figure 2.2.6 The graph of the identity function $y = f(x) = x$ in the (x, y) plane.

The four basic arithmetic operations of addition, subtraction, multiplication, and division can be used as functions. Because these binary operations

require two operands (the expressions being acted on), the elementary arithmetic operations will involve the variable and a particular constant.

For example, $x \mapsto x + 4$ is an elementary operation that adds the constant 4 to the independent variable. Similarly, $x \mapsto 4x$ is an elementary operation that multiplies the input by 4. Because subtraction is really addition with an additive inverse (the negation) of a number, an operation like $x \mapsto x - 4$ is equivalent to $x \mapsto x + -4$. Likewise, division is really multiplication with a multiplicative inverse (the reciprocal) of a number, so an operation like $x \mapsto x \div 4$ is equivalent to $x \mapsto \frac{1}{4}x$.

This motivates two new elementary operations: the constant sum and the constant multiple.

Definition 2.2.7 For every real number (constant) c , we can define the **constant sum** operation

$$x \mapsto x + c$$

and the **constant multiple** operation

$$x \mapsto cx.$$

◇

A constant sum represents a mapping that maintains a constant offset between the input and output. For example, the function $x \mapsto x - 3$ has an output that is always 3 units to the left of the input. We can think of the constant sum as a shift or translation. This mapping is illustrated in the following interactive figure.

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Figure 2.2.8 The constant sum $f(x) = x - 3$ as a map $x \xrightarrow{f} x - 3$.

A constant multiple represents a mapping that maintains a constant scaling or ratio between the input and output. For example, the function $x \mapsto 2x$ has an output that is always twice the value of the input. We can think of the constant multiple as stretching or squeezing by a scale. This mapping is illustrated in the following interactive figure.

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Figure 2.2.9 The constant multiple $f(x) = 2x$ as a map $x \xrightarrow{f} 2x$.

There are two more arithmetic operations possible with constants. Taking a constant and subtracting the variable, as in $x \mapsto 4 - x$, is not equivalent to a constant sum because we are not adding something to x . Similarly, dividing a constant by a variable, as in $x \mapsto 4 \div x$, is not equivalent to a constant multiple. These operations each involve two steps. The first step to each, however, introduces a new elementary operation.

Definition 2.2.10 The **negation** or additive inverse operation is the function $x \mapsto -x$, defined for all x . The **reciprocal** or multiplicative inverse operation is the function

$$x \mapsto \div x = \frac{1}{x},$$

defined for all $x \neq 0$.

◇

The negation operation maps a value x to its opposite value. This corresponds to a reflection on the numberline across zero.

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Figure 2.2.11 The negation $f(x) = -x$ as a map $x \xrightarrow{f} -x$.

The reciprocal operation maps a value x to its multiplicative inverse. The product of a number and its inverse always equals 1. We could think of this operation as a multiplicative reflection across 1 for positive values and across -1 for negative values.

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Figure 2.2.12 The reciprocal $f(x) = \div x = 1/x$ as a map $x \xrightarrow{f} \div x$.

The other elementary functions that we have studied can also be considered to be elementary operations. These include the basic powers and roots and exponentials and logarithms.

Definition 2.2.13 The **elementary power function** with power p is the function that raises the variable to a constant power,

$$\text{pow}_p(x) = x^p.$$

Because roots are also powers, roots are also elementary operations,

$$\text{pow}_p^{-1}(x) = \sqrt[p]{x} = x^{(1/p)}.$$

◇

Definition 2.2.14 The **elementary exponential function** with base b , where $b > 0$ and $b \neq 1$ is the function that raises a constant base to the power of the variable,

$$\text{exp}_b(x) = b^x.$$

Logarithms, as the inverses of exponentials, are included as elementary functions as well,

$$\text{exp}_b^{-1}(x) = \log_b(x).$$

◇

Additional elementary functions that we study later are the trigonometric functions. Trigonometric functions are used in relation to triangles as well as cyclic or periodic behavior. There are two fundamental trigonometric functions, the sine and cosine functions, from which the others are defined. We will study these functions in more depth later, but for the purpose of summary include the following table here.

$\sin(x)$	sine
$\cos(x)$	cosine
$\tan(x) = \frac{\sin(x)}{\cos(x)}$	tangent
$\sec(x) = \frac{1}{\cos(x)}$	secant
$\cot(x) = \frac{\cos(x)}{\sin(x)}$	cotangent
$\csc(x) = \frac{1}{\sin(x)}$	cosecant

The trigonometric functions are periodic, which implies that they must not be one-to-one. Inverse trigonometric functions are defined to solve equations

for a limited interval and provide additional elementary functions for our use.

$\sin^{-1}(x) = \arcsin(x)$	arcsine
$\cos^{-1}(x) = \arccos(x)$	arccosine
$\tan^{-1}(x) = \arctan(x)$	arctangent
$\sec^{-1}(x) = \operatorname{arcsec}(x)$	arcsecant

The arccotangent and arccosecant functions can be defined but are not used in practice.

2.2.2 Algebraic Combinations and Composition

Functions defined by a formula are generally formed by combining these operations and functions into more complicated expressions. One of the most valuable skills in calculus is the ability to recognize how a formula is constructed. Many rules in calculus are named according to which operation forms the expression of interest. The basic operations of combination are the arithmetic operations of addition (a **sum**), subtraction (a **difference**), multiplication (a **product**), and division (a **quotient**) along with the operation of function **composition**.

Composition occurs whenever we apply a function or operation to an expression rather than a simple variable. That is, x^4 is a simple power operation, but $(2x+1)^4$ is a composition because the power acts on the expression $2x+1$. We use the arithmetic operations when we take two expressions and combine them. We use composition when we apply a function or operation to a single expression. The expression on which a composition acts is called the **input expression** or **inner expression**.

Most formulas involve more than one operation. An expression is classified by the *last* operation that would be applied. The order of operations determines the priority with which operations are applied. In algebra, you may have learned the acronym PEMDAS, which stands for Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction. Subtraction is really the addition of an inverse, so differences can be classified as sums. The same technically applies for division being multiplication, but this is less frequently used. We will change the meaning of E to stand for *Every function*, including powers and exponentials, as all functions have higher precedence than the arithmetic operations.

Example 2.2.15 Classify each function by the last operation that is applied, and then classify each component expression. Make note of when a binary operation involves a constant instead of two variable expressions.

1. $f(x) = x^2 - 3x \sin(x)$
2. $g(x) = (2x + 1)(x - 3)$
3. $h(x) = (x^2 + 3)^4$
4. $j(x) = \frac{2xy}{\sqrt{3x-1}}$
5. $k(x) = 5e^{2x}$

Solution.

1. The function $f(x) = x^2 - 3x \sin(x)$ is a *difference* of the expressions x^2 and $3x \sin(x)$. The first component expression x^2 is a power function

- ($p = 2$) of x ; the second component expression $3x \sin(x)$ is a product of $3x$ and $\sin(x)$. (We could also have used a sum of x^2 and $-3x \sin(x)$.)
- The function $g(x) = (2x + 1)(x - 3)$ is a *product* of the expressions $2x + 1$ and $x - 3$. The first expression $2x + 1$ is the constant sum of $2x$ and 1 while the second expression is the constant sum of x and -3 .
 - The function $h(x) = (x^2 + 3)^4$ has the power ($p = 4$) as its last operation. Because we treat powers as functions, this is a composition. The inner expression is $u = x^2 + 3$, and the operation is the elementary power $\text{pow}_4(u) = u^4$. The inner expression is a sum of x^2 and the constant 3 .
 - The function $j(x) = \frac{2xe^x}{\sqrt{3x-1}}$ is a *quotient* of expressions $2xe^x$ and $\sqrt{3x-1}$. The first expression $2xe^x$ is a product of $2x$ and e^x ; the second expression $\sqrt{3x-1}$ is a square root (a function) of the expression $u = 3x - 1$, meaning this is a composition with the operation would be \sqrt{u} . We could also think of the square root as an elementary power function, $\sqrt{u} = \text{pow}_{1/2}(u) = u^{1/2}$.
 - The expression $5e^{2x}$ is a *constant multiple* of 5 with e^{2x} . the expression e^{2x} is a natural exponential function (base e) in composition, e^u , with the expression $u = 2x$.

□

Although binary operations like addition and multiplication are defined in terms of two operands, we often see them in expressions involving more than two terms, such as $a+b+c$ or $3xy$. By convention, the operations are performed left to right as $(a + b) + c$ or $(3x)y$. Because addition and multiplication are commutative and associative, this order doesn't matter; we act as if it were one sum or one product. In calculus, however, all of the rules are based on the binary nature of the operations. When classifying the structure of a formula, we should identify exactly two operands.

One of the most common ways to combine expressions in mathematics is to create a sum of constant multiples of those expressions. Such a combination is called a **linear combination**. The calculus operations of limits, integrals, and derivatives all satisfy a linearity in that they preserve linear combinations. It is therefore useful to recognize them.

Definition 2.2.16 Given a finite set of expressions, $u = (u_1, u_2, \dots, u_n)$, and the same number of constants, $c = (c_1, c_2, \dots, c_n)$, the **linear combination** of the expressions u with **coefficients** c is the sum of constant multiples of the expressions

$$c_1u_1 + c_2u_2 + \dots + c_nu_n.$$

◇

The non-negative integer powers of x are the powers $x^0 = 1$, $x^1 = x$, x^2 , x^3 , etc. Linear combinations of non-negative integer powers establish a family of functions called polynomials.

Definition 2.2.17 Let n be a non-negative integer. A **polynomial** of **degree** n is a function that can be written in the form

$$f(x) = a_nx^n + \dots + a_2x^2 + a_1x + a_0,$$

where a_0, a_1, \dots, a_n are constants called the **coefficients**. The term with the highest power a_nx^n is called the **leading term** and a_n is called the **leading coefficient**. A single term a_kx^k is called a **monomial** of degree k . A

polynomial with exactly two terms is called a **binomial**. \diamond

Example 2.2.18 The polynomial $f(x) = 3x^3 + x^2 - 5x + 8$ is a linear combination of the powers $(1, x, x^2, x^3)$. The degree of the polynomial is $n = 3$, and the coefficients are $(c_0, c_1, c_2, c_3) = (8, -5, 1, 3)$. The leading coefficient is $c_3 = 3$. \square

Example 2.2.19 Write down the polynomial $f(x)$ of degree $n = 4$ with coefficients $(c_0, c_1, c_2, c_3, c_4) = (16, 0, -8, 0, 1)$.

Solution. Because $c_1 = 0$ and $c_3 = 0$, we skip the terms with powers $x^1 = x$ and x^3 . We usually write polynomials in decreasing powers, so we have

$$\begin{aligned} f(x) &= 1x^4 + 0x^3 + -8x^2 + 0x^1 + 16x^0 \\ &= x^4 - 8x^2 + 16. \end{aligned}$$

\square

2.2.3 Models From Arithmetic

Understanding how functions are constructed also helps us develop models. When a quantity has contributions from multiple sources, we might create a model for each source and then add the contributions. Multiplication often combines factors that affect a single contribution. Division is used when the quantity of interest is defined as a ratio.

Example 2.2.20 Suppose that a population of an diploid organism has a trait characterized by a single gene. That gene has two alleles, a dominant allele A and a recessive allele a . The dominant trait will be present in two possible ways. Either the individual has two copies of the dominant allele (homozygous dominant) or the individual has one copy of each allele (heterozygous). If the population is subject to random mating that is independent of this trait, then the probability that an individual in the next generation will exhibit the dominant genotype can be calculated knowing the proportion of all alleles that are dominant.

Because there are two distinct ways to exhibit the dominant genotype, the probability of exhibiting the dominant genotype will be the *sum* of the probabilities of being homozygous dominant and heterozygous. This is often described as the sum rule of probability, which states that the probability of some outcome that can be attained through multiple pathways is the sum of the probabilities of each of the possible pathways. To calculate the probability of each pathway, we use a *product* rule associated with sequential events. When a pathway requires that a sequence of random outcomes occur, the probability of that individual pathway is the product of the probabilities of the individual outcomes along the pathway.

We can create a diagram showing all of the pathways by creating a decision tree. An individual receives one allele from each parent. Our tree will consider which allele is received from each parent. Let us call p the proportion of alleles in the current generation with the dominant allele. The remaining alleles must be recessive, and we call $q = 1 - p$ the proportion of alleles that are recessive. The probability associated with receiving an allele from a parent will be equal to the proportion of that allele in the population.

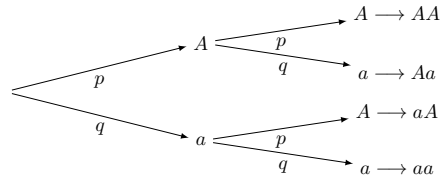


Figure 2.2.21 Tree showing inheritance of alleles from two parents.

There are three pathways that result in the dominant trait: AA , Aa , and aA . The probabilities associated with each pathway are p^2 , $pq = p(1-p)$, and $qp = (1-p)p$, respectively. Consequently, the probability that an offspring will have the dominant trait will be

$$f(p) = p^2 + p(1-p) + (1-p)p.$$

The structure of this unsimplified formula reveals a direct relation to the tree. A slightly simplified version,

$$f(p) = p^2 + 2pq = p^2 + 2p(1-p),$$

combines the two pathways resulting in a heterozygous genotype. \square

Example 2.2.22 Suppose a population of plants reproduces annually and is subject to density dependence. Density dependence typically results from the effects of competition and crowding with other individuals. The number of seeds each plant can produce is likely to depend on the population density. In addition, the probability that individual seeds will germinate and grow to maturity in the subsequent generation also depends on the population density. If we could characterize these dependencies as functions, then we could create a function that would predict the population size in a subsequent generation.

Let P_0 represent the population size of the current generation. The subscript 0 refers to the number of generations in the future. We wish to create a function $f : P_0 \mapsto P_1$, where P_1 is the population one generation in the future. Suppose that S_0 measures the average number of seeds produced by each plant in the current generation. The function $s : P_0 \mapsto S_0$ characterizes the dependence of seed production on the population size so that $s(P_0)$ gives the average number of seeds per plant in a population of size P_0 . Now suppose that $\sigma(P_0)$ is another function that gives the success probability for an individual seed to survive to maturity coming from a population of size P_0 .

We can use these elements to construct our function $f(P_0)$. The total number of seeds produced will be the current population size P_0 times the average number of seeds produced per plant. This means that $P_0s(P_0)$ gives the total number of seeds produced. Not all seeds survive to maturity, so we multiply this by the success probability to give

$$f(P_0) = P_0 s(P_0) \sigma(P_0).$$

Thus, the function used to project the subsequent generation's population size is constructed as a product of terms. If there were other ways that seeds could mature to new plants, we would add similar models for those other terms. \square

Example 2.2.23 At the beginning of 2018, the US national debt was 20.493 trillion dollars. At the end of the year, the debt had risen to 21.974 trillion dollars. At the beginning of 2018, the US population was 326.2 million. A year later, the population was 328.2 million.

Develop a model for the per capita debt as a function of time, where per

capita debt is calculated as the ratio of the total debt to the total population size.

Solution. The per capita debt will be the total debt D (trillions of dollars) divided by the total population P (millions of individuals). To create a model, we need to make some modeling choices for $t \mapsto D$ and $t \mapsto P$, where t measures the year.

The simplest model might be to use linear functions for both. For a change in time $\Delta t = 1$ (year), we can see that

$$\begin{aligned}\Delta D &= 21.974 - 20.493 = 1.481 \\ \Delta P &= 328.2 - 326.2 = 2.0\end{aligned}$$

which are also slopes (dividing by $\Delta t = 1$ year). Consequently, our linear models for D and P are given by

$$\begin{aligned}D &= 20.493 + 1.481(t - 2018) \\ P &= 326.2 + 2.0(t - 2018)\end{aligned}$$

The per capita debt according to this model will be approximated by

$$f(t) = \frac{D}{P} = \frac{20.493 + 1.481(t - 2018)}{326.2 + 2.0(t - 2018)}.$$

We expect that populations and debt grow exponentially. Consequently, an exponential model for our functions might be more appropriate. Using exponential models, $D = Ab^t$ and $P = Ba^t$, we use our data to find equations for the model parameters.

$$\begin{aligned}t = 2018 &\Rightarrow 20.493 = Ab^{2018} \\ &\Rightarrow 326.2 = Ba^{2018} \\ t = 2019 &\Rightarrow 21.974 = Ab^{2019} \\ &\Rightarrow 328.2 = Ba^{2019}\end{aligned}$$

We might use the 2018 equations to solve for A and B ,

$$\begin{aligned}A &= \frac{20.493}{b^{2018}} \\ B &= \frac{326.2}{a^{2018}}\end{aligned}$$

Then we substitute our results into the 2019 equations:

$$\begin{aligned}21.974 &= \frac{20.493}{b^{2018}} b^{2019} = 20.493b \\ b &= \frac{21.974}{20.493} \\ 328.2 &= \frac{326.2}{a^{2018}} a^{2019} = 326.2a \\ a &= \frac{328.2}{326.2}\end{aligned}$$

Our models can now be written down:

$$\begin{aligned}D &= Ab^t = \frac{20.493}{b^{2018}} b^t = 20.493 b^{t-2018} \\ &= 20.493 \left(\frac{21.974}{20.493} \right)^{(t-2018)}\end{aligned}$$

$$\begin{aligned}
 P &= B a^t = \frac{326.2}{a^{2018}} a^t = 326.2 a^{t-2018} \\
 &= 326.2 \left(\frac{328.2}{326.2} \right)^{(t-2018)}
 \end{aligned}$$

The function for the per capita debt is then calculated as a ratio,

$$\begin{aligned}
 f(t) &= \frac{D}{P} = \frac{20.493 b^{t-2018}}{326.2 a^{t-2018}} \\
 &= \frac{20.493}{326.2} \left(\frac{21.974(326.2)}{20.493(328.2)} \right)^{t-2018} \\
 &\approx 6.2823 \times 10^{-2} (1.0657)^{t-2018}
 \end{aligned}$$

Because we modeled the units of the debt as trillions of dollars and of the population as millions of individuals, the units for the per capita debt is in trillions of dollars per millions of individuals. To make sense of the units, it would help to go back to simple units of dollars and individuals. We would need to multiply D by 10^{12} to account for each debt unit representing a trillion dollars. Similarly, we multiply P by 10^6 to account for each population unit representing a million individuals. The per capita debt is the ratio, so we multiply the numerator by 10^{12} and the denominator by 10^6 , with a net effect of multiplying by 10^6 .

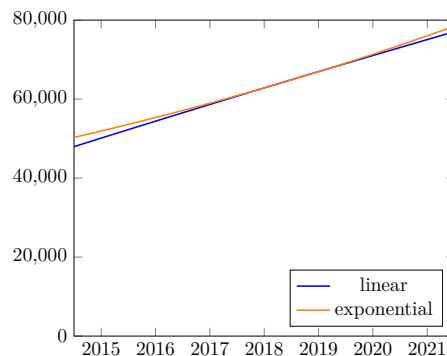


Figure 2.2.24 Models of US Per Capita Debt around 2018 in dollars per person.

□

2.2.4 Piecewise-Defined Functions

It is often the case that we use different models for different parts of the domain. When we introduced restricted domains, we defined functions by stating an inequality that specified the domain. For example, the equation

$$f(x) = x^2, \quad x \geq 0$$

defines a function with a domain $[0, \infty)$ based on the restriction $x \geq 0$. If we wanted a different rule for $x < 0$, say

$$f(x) = -x, \quad x < 0,$$

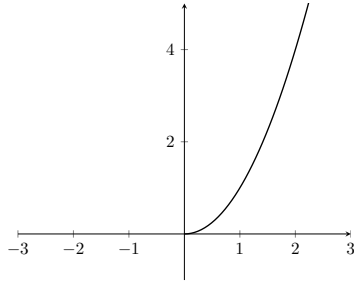
then the function now has domain $(-\infty, 0)$.

Functions that do this are called **piecewise-defined functions**. A piecewise-defined function allows us to specify rules on different parts of the domain. The

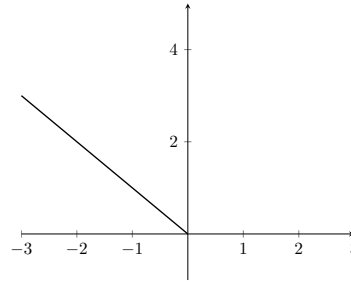
notation is similar to restricted domains, but we group all of the rules with a curly brace. The function

$$f(x) = \begin{cases} x^2, & x \geq 0, \\ -x, & x < 0 \end{cases}$$

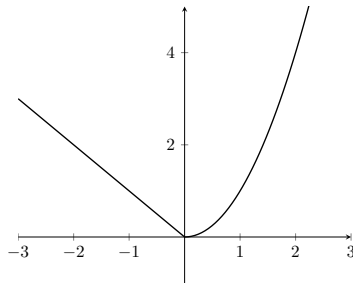
is defined for all values x so that the domain is $(-\infty, \infty)$.



(a) $f(x) = x^2, x \geq 0$



(b) $f(x) = -x, x < 0$



(c) $f(x) = \begin{cases} x^2, & x \geq 0, \\ -x, & x < 0 \end{cases}$

Figure 2.2.25 Comparison of functions with restricted domains and a piecewise-defined function.

Piecewise functions appear when there is a sudden change in behavior. The income tax structure in the United States is called a graduated tax because the tax rate increases as the amount of taxable income increases.

Example 2.2.26 For 2019, the first three IRS tax brackets for single individuals are as follows:

1. If taxable income is not over \$9700, then the tax is 10% of the taxable income.
2. If taxable income is over \$9700 but not over \$39475, then the tax is \$970 plus 12% of the excess over \$9700.
3. If taxable income is over \$39475 but not over \$84200, then the tax is \$4543 plus 22% of the excess over \$39475.

Create a piecewise function that calculates the tax given the taxable income.

Solution. The taxable income I is the independent variable. The “if” statements describing the taxable income levels describe the intervals of the domain. The first tax bracket is for $0 \leq I \leq 9700$, the second bracket is for $9700 < I \leq 39475$, and the third bracket is for $39475 < I \leq 84200$. Notice how the phrase “not over” is interpreted as including the stated value. The description of the tax amount uses percentages, which we will need to translate as a decimal multiplication. In addition, the phrase “excess over” will be

interpreted as subtraction. Putting the pieces together, we create the function

$$f(I) = \begin{cases} 0.10I, & 0 \leq I \leq 9700, \\ 970 + 0.12(I - 9700), & 9700 < I \leq 39475, \\ 4543 + 0.22(I - 39475), & 39475 < I \leq 84200. \end{cases}$$

The function could be extended further if we had additional information for the remaining tax brackets. \square

The absolute value function is a particularly important mathematical function defined piecewise. For values that are negative, the absolute value returns the opposite (positive) value. For zero or for values that are already positive, the absolute value returns the original value.

Definition 2.2.27 The absolute value function is defined as

$$\text{abs}(x) = |x| = \begin{cases} -x, & x < 0, \\ x, & x \geq 0. \end{cases}$$

\diamond

2.2.5 Summary

- Functions defined by formulas are typically constructed from elementary functions: constant functions, the identity function, power functions, exponential functions, logarithms, and trigonometric functions.
- Combinations of expressions can be arithmetic (sum, difference, product, or quotient) or the composition of functions.
- An expression is classified by the *last* operation used to construct that expression.
- Binary operations involving a constant operand are special cases. They can be constructed using only constant sums, constant multiples, and reciprocals.
- A parametrized family of functions is a set of functions that have the same structure with different constants. The constants that can change are called parameters.
- Common parametrized families of functions are linear, exponential, and power functions.

Parametric Formula	Description
$f(x) = mx + b$	linear, slope-intercept
$f(x) = Ax^p$	power
$f(x) = Ab^x$	exponential, general base b
$f(x) = Ae^{kx}$	exponential, natural base e

- A polynomial is a linear combination of simple powers $(1, x, x^2, \dots, x^n)$, or, in other words, a sum of constant multiples of these powers,

$$f(x) = a_n x^n + \dots + a_2 x^2 + a_1 x + a_0.$$

The constant multiples (a_0, a_1, \dots, a_n) are called the coefficients. The term $a_n x^n$ is called the leading term.

- A piecewise-defined function uses different rules for different portions of the domain.

2.2.6 Exercises

1. Classify each elementary function.

(a) $f(x) = \pi$

(b) $g(x) = x$

(c) $h(x) = x^\pi$

(d) $j(x) = \pi^x$

(e) $k(x) = \sin(x)$

2. Classify each function according to the last operation. Then classify the component expressions. Make note if the operation involves a constant expression.

(a) $f(x) = 4x^4$

(b) $g(x) = 2^{3x} + 5$

(c) $h(x) = 3^{5x-1}$

(d) $j(x) = 3\sqrt{x} + \frac{1}{x^2}$

(e) $k(x) = 4x^2e^{3x}$

(f) $m(x) = \frac{x^2(3x-1)}{(x^2+1)^4}$

3. For each polynomial, determine the degree and list the coefficients.

(a) $f(x) = 3x^2 + 5x - 1$

(b) $f(x) = x^3 - 2x + 8$

(c) $f(x) = x^4 - 1$

(d) $f(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$

Find the equation of the function $x \mapsto y$, if possible, for each of the following parametric models satisfying the states $(x, y) = (0, 3)$ and $(x, y) = (5, 9)$.

4. linear function

5. power function

6. exponential function

7. quadratic function of the form $y = a + bx^2$

8. quadratic function of the form $y = ax + bx^2$

Find the equation of the function $x \mapsto y$, if possible, for each of the following parametric models satisfying the states $(x, y) = (1, 3)$ and $(x, y) = (4, 6)$.

9. linear function

10. power function

11. exponential function

12. quadratic function of the form $y = a + bx^2$

13. quadratic function of the form $y = ax + bx^2$

14. a function the form $y = \frac{ax}{x+b}$

Use the description of each relation to create a corresponding piecewise-defined function.

15. The marginal tax rate is the percentage rate applied to the amount of taxable income that falls in the tax bracket. Based on the example, we see the marginal tax rate is 10% for income no greater than \$9700, 12% for income greater than \$9700 and no greater than \$39475, and 22% for income greater than \$39475 and no greater than \$84200. Define the function that takes the taxable income and returns the marginal tax rate for these three brackets.
16. Many bulk supplies are sold at a discount when enough items are purchased at once. An online gem store sells packages with two amethyst beads. If you purchase fewer than 15 packages, each package costs \$10.89. If you purchase at least 15 packages but fewer than 50, each package costs \$8.57. If you purchase at least 50 packages but fewer than 100, each package costs \$6.42. If you purchase at least 100 packages, each package costs \$5.87. Define the function that takes the number of packages ordered and returns the per package cost. Be clear about the domain.
17. For the gem example in [Exercise 2.2.6.16](#), define the function that takes the number of packages ordered and returns the total cost of the order.
18. An electronic scooter can be unlocked for \$1.00 and then you are charged \$0.15 per minute of use. Partial minutes are rounded up to the next minute, so a rental of two minutes and fifteen seconds would be charged for three minutes or \$1.45 total. Define a piecewise function that gives the cost for rental times up to five minutes. What is the domain?
19. A car has a gas tank that holds 12 gallons and drives 35 miles per gallon. The owner starts with a full tank of gas, drives 300 miles, refills the tank, and then drives another 200 miles. Define a piecewise function that gives the amount of gas in the tank as a function of total distance traveled. What is the domain? Are there any ambiguities?