3.1 Describing the Behavior of Functions

Overview. We have been learning about how functions are constructed and how they are defined. In many instances, before we construct a formula for a function, we need to identify what behavior we are attempting to model. At other times, we have a formula and we need to know what behavior that predicts. We need specific language that we can use to describe behavior.

In this section, we will focus on three types of behavior: monotonicity, concavity, and end behavior. Monotonicity will describe where a function is increasing or decreasing. Concavity will describe where the slope ro rate of change of a function is increasing or decreasing. In a graph, concavity describes whether the curve is bending up or bending down. We also discuss simple end behavior including unbounded growth (tending to infinity) and horizontal asymptotes.

Our emphasis is in learning the language of behavior, describing graphs using this language, and creating graphs based on a description of a function. As our study of calculus develops, we will learn mathematical tools that will allow us to determine function behavior more precisely.

3.1.1 Functions Have Shapes

We often describe functions according to the shape of their graphs. The different possible shapes we see in graphs correspond to specific behaviors of the functions. We will focus on two aspects of a graph: monotonicity and concavity.

3.1.1.1 Monotonicity

The **monotonicity** of a function deals with whether the function is increasing or decreasing. We start with the mathematical definitions of increasing and decreasing functions. We will explore the ideas graphically in terms of maps and then graphs.

Definition 3.1.1 Monotonicity. A function f is **increasing** on a subset S of the domain (usually an interval) if for every $x_1, x_2 \in S$,

$$x_1 < x_2 \qquad \text{implies} \qquad f(x_1) < f(x_2).$$

A function f is **decreasing** on a subset S of the domain (usually an interval) if for every $x_1, x_2 \in S$,

$$x_1 < x_2 \qquad \text{implies} \qquad f(x_1) > f(x_2).$$

$\diamondsuit$

One way to think of monotonicity is that the function retains an ordering of the sets. An increasing function preserves the order, so that if two inputs are in a particular order, $x_1 < x_2$, then the resulting outputs have the same order, $f(x_1) < f(x_2)$. A decreasing function reverses the order, so that if inputs have an order, $x_1 < x_2$, then the outputs must have the opposite order $f(x_1) > f(x_2)$. A function that is not monotone (neither increasing or decreasing) does not maintain a sense of order uniformly over the set. Sometimes the outputs might have the same order as the inputs, and sometimes the outputs might have the opposite order.

Example 3.1.2 The function $f(x) = 2x + 1$ is a linear function with positive slope $m = 2$. We can show that f is an increasing function. Suppose $x_0 < x_1$. Multiplying both sides of an inequality by a positive number *preserves* the ordering, as does adding the same value to both sides:

$$\begin{aligned} x_0 &< x_1 \\ 2x_0 &< 2x_1 \\ 2x_0 + 1 &< 2x_1 + 1 \\ f(x_0) &< f(x_1) \end{aligned}$$

This is visualized in the following figure.

A deprecated JSXGraph interactive demonstration goes here in interactive output.

Figure 3.1.3 Dynamic illustration of the function $f(x) = 2x + 1$ as a map $x \mapsto y$ showing that f is increasing.

Thinking of the map dynamically, we see that as we increase the input, the output also increases. This is captured in the graph of the function in the (x, y) plane. The graph shows y-values increasing as viewed from left to right, which is corresponding to x-values increasing.

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Figure 3.1.4 The graph of the function $y = f(x) = 2x + 1$ in the (x, y) plane.

□

Example 3.1.5 The function $f(x) = -2x+3$ is a linear function with negative slope $m = -2$. We can show that f is an decreasing function. Suppose $x_0 < x_1$. Multiplying both sides of an inequality by a negative number *reverses* the ordering, while adding the same value to both sides preserves the order:

$$\begin{aligned} x_0 &< x_1 \\ -2x_0 &> -2x_1 \\ -2x_0 + 3 &> -2x_1 + 3 \\ f(x_0) &> f(x_1) \end{aligned}$$

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Figure 3.1.6 Dynamic illustration of the function $f(x) = -2x + 3$ as a map $x \mapsto y$ showing that f is decreasing.

The map shows that the order of outputs is always opposite to the order of the inputs. Thinking of the map dynamically, we see that as we increase the input, the output decreases. The graph of the function in the (x, y) plane captures the same information. Viewing the graph from left to right (as x increases), the y-values decrease.

A deprecated JSXGraph interactive demonstration goes here in interactive output.

Figure 3.1.7 The graph of the function $y = f(x) = -2x + 3$ in the (x, y) plane.

□

Some functions are not monotone because the map does not retain the ordering of the sets. Dynamically, this is because the output will sometimes increase and sometimes decrease as the input is increased.

Example 3.1.8 The function $f(x) = x^2$ is not a linear function and is not monotone. We can show this by illustrating that the function is inconsistent in ordering the output values relative to the input values. Consider $x_0 = -2$ and $x_1 = -1$. We have $f(x_0) = 4 > f(x_1) = 1$, so for these inputs the order is reversed. However, for $x_0 = 1$ and $x_1 = 2$, we have $f(x_0) = 1 > f(x_1) = 4$ and the order is preserved. This function is not increasing or decreasing, but is a combination.

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Figure 3.1.9 Dynamic illustration of the function $f(x) = x^2$ as a map $x \mapsto y$ showing that f is not monotone.

We can see graphically that f is decreasing on $(-\infty, 0]$ because for any two inputs in this interval, the order of the outputs is reversed. We can also see that f is increasing on $[0, \infty)$ because for any two inputs in that interval, the order of the outputs is preserved. This point where monotonicity switches corresponds to the vertex of the parabola $y = x^2$.

A deprecated JSXGraph interactive demonstration goes here in interactive output.

Figure 3.1.10 Dynamic illustration of the function $f(x) = x^2$ as a map $x \mapsto y$ showing that f is not monotone.

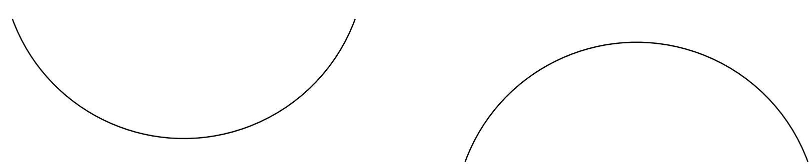
□

One of our goals in calculus will be to develop a method to determine the intervals on which a function is increasing or decreasing. When we motivated monotonicity with linear functions, we saw that a positive slope implied an increasing function and a negative slope implied a decreasing function. Calculus will develop a more general sense of the slope of a function using the derivative such that we will describe monotonicity based on the signs of the derivative.

Note 3.1.11 When listing intervals on which a function is increasing or decreasing, it is important *not* to use a union of the intervals. The reason is that we are saying that the function is increasing on *each* of the intervals individually and not on the set formed by the union. If listing multiple intervals, simply form a comma-separated list.

3.1.1.2 Concavity

Concavity describes how the graph of a function in the (x, y) plane bends. If the graph bends upward, we say the function is **concave up**. If the graph bends downward, we say the function is **concave down**.



(a) Concave up **(b)** Concave down

Figure 3.1.12 Comparison of concave up and concave down graphs

As with monotonicity, these attributes of functions apply over intervals rather than at individual points. When a graph changes concavity at a point, for example switching from bending up to bending down, the function has an **inflection point**. A technical definition of concavity that depends on the concept of a derivative will be provided later.

However, we can capture the essential idea by thinking about how the slope is changing between points. A function that has an increasing slope or rate of change over an interval is concave up on the interval. A function that has decreasing slope or rate of change is concave down.

Definition 3.1.13 Concavity. A function f is **concave up** on a subset S of the domain (usually an interval) if for every $x_1, x_2, x_3 \in S$, the slope or rate of change is increasing,

$$x_1 < x_2 < x_3 \qquad \text{implies} \qquad \frac{f(x_2)-f(x_1)}{x_2-x_1} < \frac{f(x_3)-f(x_2)}{x_3-x_2}.$$

A function f is **concave down** on a subset S of the domain (usually an interval) if for every $x_1, x_2, x_3 \in S$, the slope or rate of change is decreasing,

$$x_1 < x_2 < x_3 \qquad \text{implies} \qquad \frac{f(x_2)-f(x_1)}{x_2-x_1} > \frac{f(x_3)-f(x_2)}{x_3-x_2}.$$

♢

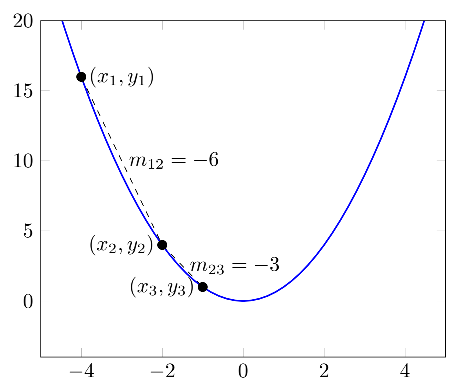
This definition is not very easy to use directly. When we learn more about derivatives to describe the slope at individual points, we will have a much better method known as the second derivative test for concavity. However, the following examples will illustrate what is happening.

Example 3.1.14 The function $f(x) = x^2$ is concave up on $(-\infty, \infty)$ (the entire domain). We will not *prove* that this is true because this is too difficult without derivatives. But we can illustrate the idea.

Consider the graph $y = f(x) = x^2$ and the particular values $x_1 = -4$, $x_2 = -2$, and $x_3 = -1$. We will calculate the slope or rate of change between $(x_1, y_1) = (-4, 16)$ and $(x_2, y_2) = (-2, 4)$ and between (x_2, y_2) and $(x_3, y_3) = (-1, 1)$.

$$\begin{aligned} m_{12} &= \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-16}{-2 - -4} = \frac{-12}{2} = -6 \\ m_{23} &= \frac{y_3 - y_2}{x_3 - x_2} = \frac{1-4}{-1 - -2} = \frac{-3}{1} = -3 \end{aligned}$$

We can see that the slope or rate of change is increasing, $m_{12} < m_{23}$. These slopes are illustrated in the following figure.



This is not a proof of concavity because we only illustrated the order for three specific points. Use the following dynamic figure to convince yourself that for *any* three points we might choose, the slopes increase from left to right.

A deprecated JSXGraph interactive demonstration goes here in interactive output.

Figure 3.1.15

The reason that f has an inflection point at $x = 0$ is that point is where f has the steepest negative slope. To the left, $x < 0$, the slope decreases; to the right, $x > 0$, the slope increases. □

Example 3.1.16 The function $f(x) = x^3 - 3x$ changes concavity at $x = 0$. f is concave down on $(-\infty, 0]$ and concave up on $[0, \infty)$. When three points are chosen with $x \in (-\infty, 0]$, the slope is decreasing. When the three points are chosen with $x \in [0, \infty)$, the slope is increasing. This can be verified in the following dynamic figure. However, the three points must all be in either $(-\infty, 0]$ or in $[0, \infty)$.

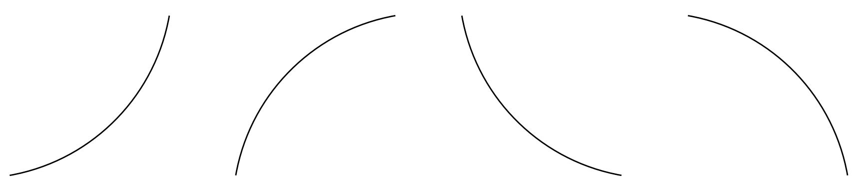
A deprecated JSXGraph interactive demonstration goes here in interactive output.

Figure 3.1.17

□

3.1.1.3 Combining Monotonicity and Concavity

The shape of a graph of a function is often defined in terms of the monotonicity and concavity combined. There are four basic shapes that correspond to the four quadrants of a circle, illustrated in the figure below. A curve that has a positive and increasing slope is increasing and concave up. A curve that has a positive but decreasing slope is increasing and concave down. A curve that has a negative but increasing (becoming less negative) slope is decreasing and concave up. A curve that has a negative and decreasing (becoming more negative) slope is decreasing and concave down.

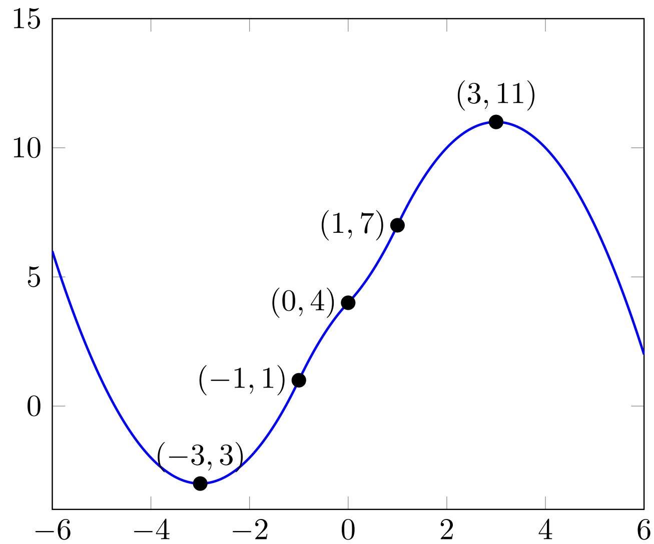


(a) Increasing, Concave up **(b)** Increasing, Concave Down **(c)** Decreasing, Concave up **(d)** Decreasing, Concave Down

Figure 3.1.18 Basic shapes defined by monotonicity and concavity.

We can describe the shape of a graph by stating intervals on which the function satisfies each of the possible behaviors. The intervals are separated by points where the graph reaches either a maximum or minimum value (changes in monotonicity) or where the slope of the graph reaches an extreme and begins to bend the other direction (changes in concavity or points of inflection).

Example 3.1.19 The graph of a function $y = f(x)$ is shown below, with labeled extreme points and inflection points. Describe the shape of the graph by giving intervals of monotonicity and concavity.



Solution. Intervals for monotonicity are based on the function increasing or decreasing. The end-points of these intervals are the extreme points for the function. When the graph extends beyond the frame of the figure, we assume the function behavior continues as shown. Intervals always are read from left to right. The end-point of an interval is included (closed) if the behavior extends up to and including that point.

The function f is decreasing on $(-\infty, -3]$, increasing on $[-3, 3]$, and decreasing on $[3, \infty)$. Notice that the extremes at $x = -3$ and $x = 3$ are included in two intervals. The continuous function is decreasing on $(-\infty, -3)$ as an open interval. Because f decreases up to and including $x = -3$, we include the end-point.

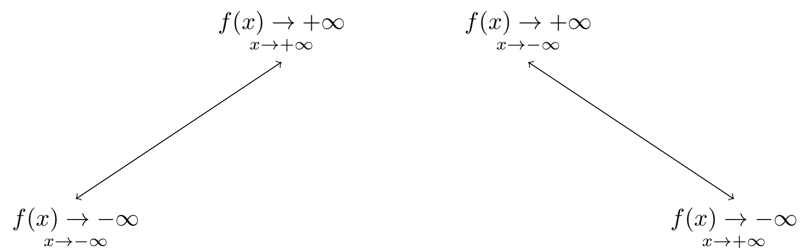
Intervals for concavity are based on where the slope is increasing or decreasing. Intervals on which the graph bends upward, f is concave up. Intervals on which the graph bends downward, f is concave down. Notice our graph has inflection points (where the concavity changes) at $x = -1$, $x = 0$, and $x = 1$. At these points, the graph starts to bend in the opposite direction.

The function f is concave up on $(-\infty, -1]$, concave down on $[-1, 0]$, concave up on $[0, 1]$, and concave down on $[1, \infty)$. We include the inflection points as the end points of the intervals (closed) because the slope is increasing or decreasing up to and including those points. □

3.1.2 End Behavior

End-behavior of a function describes what happens to a function as the size of the input grows. Consider the possibilities of a linear function, $y = f(x) = mx + b$. So long as the slope is non-zero, the function is *unbounded*, meaning that the graph eventually goes above every level and eventually goes below every level (on opposite sides of the graph).

If the slope is positive, $m > 0$, then the function is increasing. We say $f(x) \to +\infty$ as $x \to +\infty$, which we read as "the value of $f(x)$ tends to positive infinity as the value of x goes to positive infinity". This is because the y-values will eventually rise *above* any level on the right side of the graph (for sufficiently large positive values x). We also say $f(x) \to -\infty$ as $x \to -\infty$ because the y-values are *below* any specified value on the left side of the graph (for sufficiently large negative values x). When the slope is negative, $m < 0$, the unbounded behavior is reversed.



(a) Positive slope ($m > 0$) **(b)** Negative slope ($m < 0$)

Figure 3.1.20 Unbounded behavior of linear functions with positive and negative slopes.

For consistency in notation to describe the tendency of a function (as opposed to the value of a function), we use limits to describe unbounded behavior.

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= -\infty && \text{means } f(x) \to -\infty \text{ as } x \to -\infty \\
\lim_{x\to-\infty} f(x) &= \infty && \text{means } f(x) \to +\infty \text{ as } x \to -\infty \\
\lim_{x\to\infty} f(x) &= -\infty && \text{means } f(x) \to -\infty \text{ as } x \to +\infty \\
\lim_{x\to\infty} f(x) &= \infty && \text{means } f(x) \to +\infty \text{ as } x \to +\infty
\end{aligned}$$

When the graph of a function f behaves more and more like a constant function (horizontal line) for larger and larger values of the independent variable, we say f has a **horizontal asymptote**. A horizontal asymptote $y = L$ on the right side (large, positive values for x) uses the limit statement

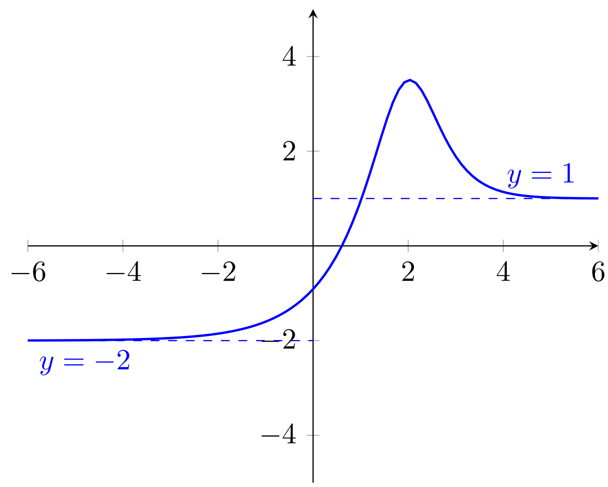
$$\lim_{x\to\infty} f(x) = L,$$

which means that the value of $f(x)$ approaches the constant value L as $x \to +\infty$. When f has a horizontal asymptote $y = L$ on the left side (large, negative values of x), we use the limit statement

$$\lim_{x\to-\infty} f(x) = L.$$

Example 3.1.21 The graph of a function $y = f(x)$ is shown below. This function has two horizontal asymptotes: $y = -2$ as $x \to -\infty$ and $y = 1$ as $x \to +\infty$. We write

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= -2, \\
\lim_{x\to+\infty} f(x) &= 1.
\end{aligned}$$



□

A function can also have unbounded behavior near a particular input value, say at $x = a$. Using limit notation, this means that at least one of the following must be true.

$$\lim_{x\to a^-} f(x) = +\infty$$
$$\lim_{x\to a^-} f(x) = -\infty$$
$$\lim_{x\to a^+} f(x) = +\infty$$
$$\lim_{x\to a^+} f(x) = -\infty$$

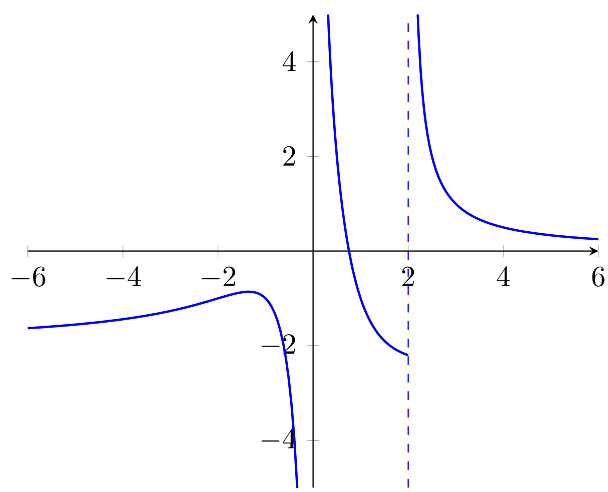
The graph has a **vertical asymptote** at $x = a$, meaning that the graph of the function approaches closer and closer to this vertical line.

Example 3.1.22 The graph of a function $y = f(x)$ is shown below with two vertical asymptotes. The vertical asymptote at $x = 0$ corresponds to left- and right-limits

$$\lim_{x\to 0^-} f(x) = -\infty,$$
$$\lim_{x\to 0^+} f(x) = +\infty,$$

The vertical asymptote at $x = 2$ only corresponds to the right-limit

$$\lim_{x\to 2^+} f(x) = +\infty.$$



It is hard to tell from a graph alone where a vertical asymptote occurs. Using only the limited graph window, it is not obvious that the vertical asymptote is at exactly $x = 0$ since the graph is still fairly far away from that vertical line from this perspective. □

Note 3.1.23 A common false impression about horizontal asymptotes is that the graph of a function can not cross the asymptote. A function can not cross a vertical asymptote, but that is only because a function can not intersect a vertical line at more than one point. An asymptote only requires that the graph behaves more and more like the line.

When a function physically relates two variables, $x \mapsto y$, a horizontal asymptote indicates that for sufficiently large values of the independent variable, the dependent variable is essentially a constant. A common description in physical settings for this constant is a **saturation value**. We think of the quantity measured by the independent variable as a control variable. The dependent variable can be thought of as a response. As the control variable is increased, the response will pass through some of its range of values. However, there will come a point where even though you continue to increase the control variable, the response is no longer able to change very much at all. That is, the response has saturated.

Example 3.1.24 An enzyme is a protein that helps catalyze a chemical reaction. The rate or velocity of reaction V depends on the concentration of the reactant C. Commonly, the function $C \mapsto V$ is increasing, concave down, and has a horizontal asymptote, known as Michaelis–Menten reaction kinetics. The physical domain is $C \in [0, \infty)$. Because the relation is increasing, we know that adding more reactants will raise the reaction rate. Because the relation is concave down, we know that the degree to which the rate increases slows down as more reactants are added. The horizontal asymptote means that this increase in the reaction rate saturates to some maximum rate $V_{\max}$,

$$\lim_{C\to\infty} V = V_{\max}.$$

The reactant concentration where the reaction rate is halfway to the maximum value is called the half-saturation value, and is usually represented with a constant K.

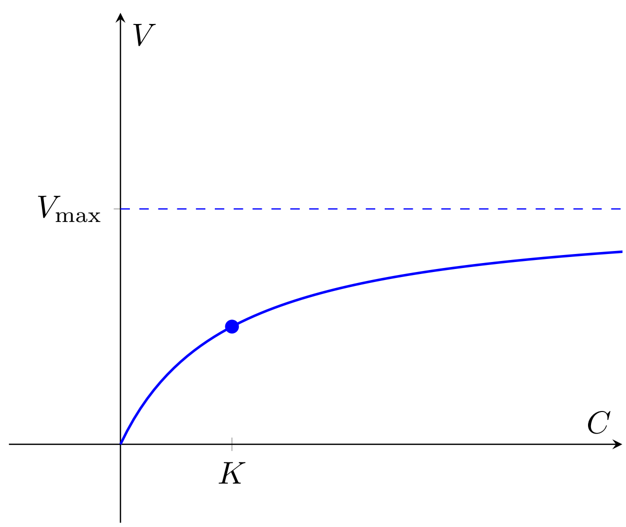


Figure 3.1.25 Michaelis–Menten reaction kinetics with saturating rate $V_{\max}$ and half-saturation constant K.

□

Example 3.1.26 Imagine a crop of plants growing in a field. The total biomass harvested B depends on the number of seeds S that are sown. If very few seeds are sown, the biomass harvested will be small. For more seeds sown, we expect the biomass would increase. However, if too many seeds are sown, then the crop will be overcrowded, resulting in a lower harvest. We expect that there might be an optimal number of seeds S^* for which the biomass is at a maximum.

Describe the behavior of the function $S \mapsto B$ and sketch a possible graph.

Solution. The function $S \mapsto B$ will have a physical domain of $S \in [0, \infty)$. Because B is a maximum at $S = S^*$, the function is increasing on $[0, S^*]$ and decreasing on $[S^*, \infty)$. The simplest assumption for concavity would be that the function starts concave down. However, a concave down and decreasing function will eventually approach $-\infty$, which is not physically possible for our physical scenario. Therefore, the function must change concavity at some inflection point after S^*, say at $S = S^\dagger$. Our function would be concave down on $[0, S^\dagger]$ and concave up on $[S^\dagger, \infty)$. Continuing to increase the number of seeds will result in ever smaller biomass due to overcrowding until it approaches some saturating biomass B_∞,

$$\lim_{S \to \infty} B = B_\infty.$$

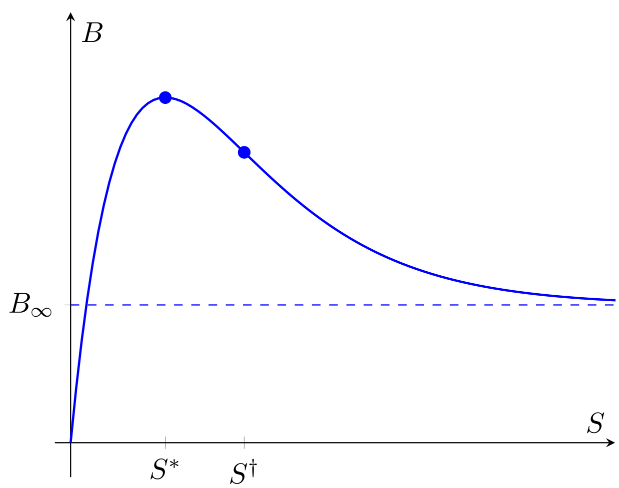


Figure 3.1.27 Possible graph of (S, B) with maximum at $S = S^*$ and inflection point at $S = S^\dagger$.

Note: The asterisk and dagger are decorations so that the symbols S^* and $S^\dagger$ represent general constants. We don't know actual values for the maximum and inflection point, so we can't use numbers. The symbols are place-holders for values that would be determined experimentally. Similarly, the symbol B_∞ represents the value for the biomass harvested when the number of seeds sown saturates the system. □

3.1.3 Summary

- Describing the **monotonicity** of a function is determining intervals on which the function is increasing or decreasing.

- A function f is **increasing** on a set S if the function is order preserving: For all $x_1, x_2 \in S$, we must have

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2).$$

This corresponds to a graph that is rising left to right (positive slopes).

A function f is **decreasing** on a set S if the function is order reversing: For all $x_1, x_2 \in S$, we must have

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) > f(x_2).$$

This corresponds to a graph that is falling left to right (negative slopes).

- Describing the concavity of a function is determining intervals on which the function is concave up or concave down.
- A function f is **concave up** on a set S if the slope or rate of change is increasing on S: For all $x_1, x_2, x_3 \in S$, we must have

$$x_1 < x_2 < x_2 \quad \Rightarrow \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

The graph will be bending upward.

A function f is **concave down** on a set S if the slope or rate of change is decreasing on S: For all $x_1, x_2, x_3 \in S$, we must have

$$x_1 < x_2 < x_2 \quad \Rightarrow \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} > \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

The graph will be bending downward.

- A **point of inflection** is a point where a function is continuous and changes concavity.
- Lists of intervals of monotonicity and concavity should be separated by commas and not joined by unions.
- Limits as $x \to \pm\infty$ describe end behavior.
  - To say $f(x) \to +\infty$ means values of $f(x)$ eventually rise above *any* possible value.
  - To say $f(x) \to -\infty$ means values of $f(x)$ eventually fall below *any* possible value.
  - To say $f(x) \to L$ means values of $f(x)$ eventually approaches a horizontal asymptote $y = L$.

3.1.4 Exercises

Each of the following problems asks you to prove that the given function is either increasing or decreasing on a particular interval.

1. Prove that $f(x) = 5x - 12$ is an increasing function by showing that whenever $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

2. Prove that $f(x) = -3x - 2$ is a decreasing function by showing that whenever $x_1 < x_2$, we have $f(x_1) > f(x_2)$.

3. Prove that $f(x) = x^2$ is an increasing function on $[0, \infty)$ by showing that whenever $0 < x_1 < x_2$, we have $f(x_1) > f(x_2)$.
Hint: Show that $f(x_2) - f(x_1) > 0$ by factoring and determining the signs of the factors.

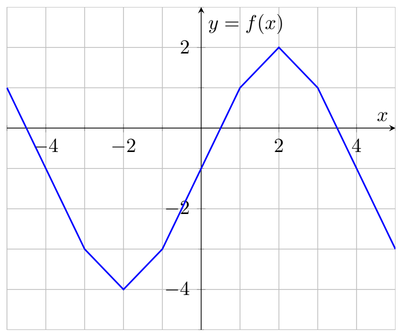
4. Prove that $f(x) = x^2$ is a decreasing function on $(-\infty, 0]$ by showing that whenever $x_1 < x_2 < 0$, we have $f(x_1) > f(x_2)$ or $f(x_2) - f(x_1) <$

0.

Hint: Show that $f(x_2) - f(x_1) < 0$ by factoring and determining the signs of the factors.

Consider each of the following graphs of functions. Use the graph to determine the intervals of monotonicity for that function.

5.



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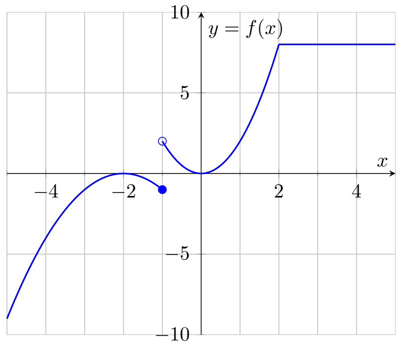

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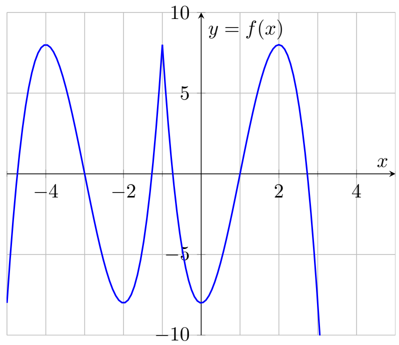


Each of the following problems asks you to illustrate the concavity of the given function.

8. Illustrate that $f(x) = \dfrac{1}{x}$ is concave up on $(0, \infty)$ by showing that the slope is increasing for the sequential points $x_1 = \frac{1}{2}$, $x_2 = 1$, and $x_3 = 2$.

9. Illustrate that $f(x) = \dfrac{1}{x}$ is concave down on $(-\infty, 0)$ by showing that the slope is decreasing for the sequential points $x_1 = -2$, $x_2 = -1$, and $x_3 = -\frac{1}{2}$.

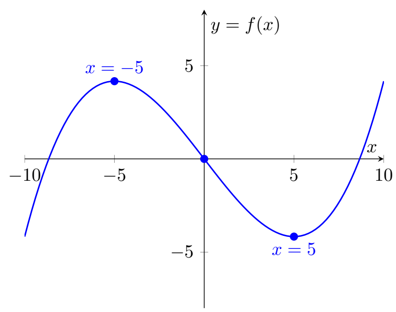
10. Illustrate that $f(x) = 2^x$ is concave up on $(-\infty, \infty)$ by showing that the slope is increasing for the sequential points $x_1 = -1$, $x_2 = 0$, and

$x_3 = 1$.

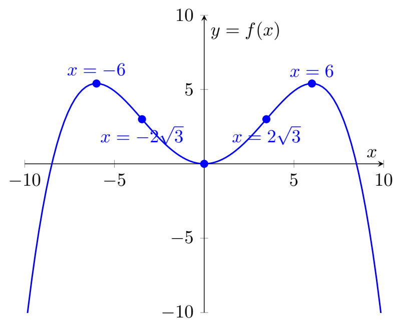
11. Illustrate that $f(x) = 2^{-x}$ is concave up on $(-\infty, \infty)$ by showing that the slope is increasing for the sequential points $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$.

Consider each of the following graphs of functions, which includes turning points and inflection points. Use the graph to determine the intervals of monotonicity and concavity for that function.

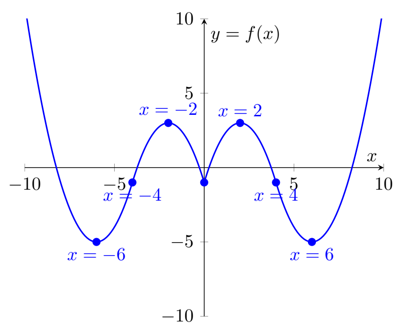
12.



13.

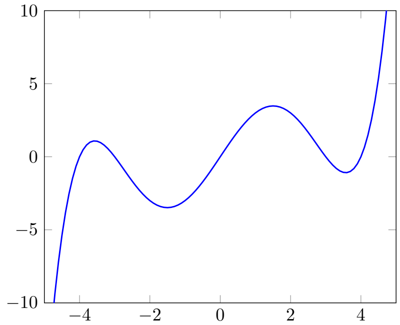


14.



Use the graphs to answer the questions about limits. Assume that the behavior of the graph shown in the window continues outside the window.

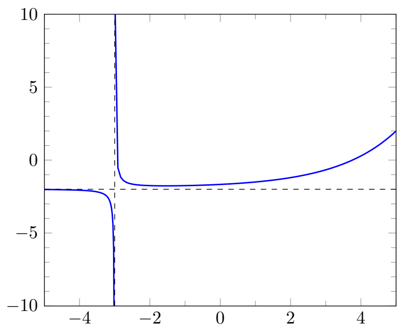
15.



(a) $\lim_{x\to-\infty} f(x)$

(b) $\lim_{x\to+\infty} f(x)$

16.



(a) $\lim_{x\to 3^-} f(x)$

(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to-\infty} f(x)$

(d) $\lim_{x\to+\infty} f(x)$