3.2 Rate of Accumulation and the Derivative

3.2.1 Overview

For functions which are defined as the accumulation of a given rate,

$$f(x) = f_0 + \int_{x_0}^{x} f'(z)\,dz, \tag{3.2.1}$$

we can describe their monotonicity and concavity using the signs and monotonicity of their corresponding rates of accumulation $f'(x)$. Of course, most functions are not written using an accumulation formula representation. For polynomials, where we know simple accumulation formulas, we know how to calculation the corresponding rate of accumulation $f'(x)$. What about other functions?

This raises a central question of calculus: Which functions *can* be expressed as an accumulation? And if a function can be expressed as an accumulation, how do we find the formula for the rate of accumulation that will be the integrand in that representation?

This question will be partially resolved through the definition of the derivative. The derivative will provide a new interpretation of the concept of rate of change that is not directly connected to accumulation and definite integrals. At that point, we will have two potential concepts of the rate—the rate of accumulation and the derivative. The connection between these two rates as representing the same thing will ultimately be established through the Fundamental Theorem of Calculus. Anticipating this eventual equality, we will adopt the name **derivative** as being equivalent to the rate of accumulation.

In this section, we will use known elementary accumulation functions and their corresponding rates to compute the rate of accumulation for simple polynomials. The process of finding a rate of change inherits the linearity properties of integration. Using elementary formulas and linearity, we will learn to identify its rate of change or derivative of any polynomial. Once we have a rate of change, we can express the polynomial as an accumulation function. We can also classify the monotonicity and concavity of the polynomial.

3.2.2 The Rate of Accumulation

When a function is defined as an accumulation function in terms of a definite integral (3.2.1), it is easy enough to determine the rate of accumulation or derivative by identifying the function in the integrand. We just need to express the function as a constant (the initial value) plus an integral from the initial point. The function inside the integral, called the **integrand**, will be the rate of accumulation.

Example 3.2.1 If $f(x)$ is defined as $f(x) = 3 + \int_1^x z^2 - 5z\,dz$, then the integrand $z^2 - 5z$ must be $f'(z)$. Changing variables means $f'(x) = x^2 - 5x$ is the rate of accumulation or derivative. □

Example 3.2.2 Find $G'(x)$ for $G(x) = 4 - 3\int_1^x \frac{1}{z}\,dz$.

Solution. Because $G(x)$ is not yet written as a constant *plus* an integral, we need to use properties of integrals to put it in the standard form of an accumulation function. The (((Unresolved xref, reference "thm-definite-integral-constant-multiple"; check spelling or use "provisional" attribute)))constant multiple rule allows us to treat -3 as a constant multiplied inside the integral,

$$G(x) = 4 + \int_1^x \frac{-3}{z}\,dz.$$

Now that $G(x)$ is an accumulation function, we can find the rate to be

$$G'(x) = \frac{-3}{x}.$$

□

The elementary accumulation formulas for simple powers 3.4.3 can be interpreted as complementary rules used to find the rate of accumulation or derivative for simple powers. As an example, consider the known accumulation formula

$$\int_0^x z^2\,dz = \frac{1}{3}x^3.$$

If we multiply both sides of the equation by 3 to clear the fraction and move the constant inside the integral, we have an equivalent statement

$$x^3 = \int_0^x 3z^2\,dz.$$

That is, we have just found that for the function $f(x) = x^3$ the derivative is the rate of accumulation $f'(x) = 3x^2$. Every accumulation formula that we know provides a corresponding rate of accumulation for simple powers.

Theorem 3.2.3 The Power Rule for the Rate of Accumulation. *The elementary accumulation formulas lead to the following elementary rates of accumulation for powers of the independent variable.*

- *If* $f(x) = x$, *then* $f'(x) = 1$.
- *If* $f(x) = x^2$, *then* $f'(x) = 2x$.
- *If* $f(x) = x^3$, *then* $f'(x) = 3x^2$.
- *If* $f(x) = x^4$, *then* $f'(x) = 4x^3$.

Proof. Each formula follows by applying the same technique described above for $f(x) = x^3$ on the corresponding accumulation formula. ■

You might have noticed a pattern in these formulas, that the rate of accumulation involves a power that has been reduced by one from the accumulation function and that the power for the accumulation has become a constant multiple. Without creating more accumulation formulas, we can not prove that this pattern will always be true. In mathematics, we call a pattern that we believe might be true a **conjecture**.

Conjecture 3.2.4 Power Rule for Rate of Accumulation. *For any constant power* n, *the function* $f(x) = x^n$ *has a rate of accumulation* $f'(x) = nx^{n-1}$.

This conjecture happens to be true, but we will need to wait to develop the process of differentiation to prove it.

Example 3.2.5 For $f(x) = x^{10}$, what does the conjecture about the power rule predict will be $f'(x)$?

Solution. The function f is an elementary power function with $n = 10$. The power rule shown in the conjecture states that

$$f'(x) = 10x^9.$$

The reason this is a conjecture right now is that we would currently need to write x^{10} as an accumulation function

$$x^10 = \int_0^x 10z^9,$$

but we don't currently have a rule for a definite integral with a power that high. That would require knowing the limit of a Riemann sum

$$\int_0^x 10z^9 = \lim_{n\to\infty} \sum_{k=1}^{n} 10(\frac{kx}{n})^9 \frac{x}{n}.$$

Without knowing a sum accumulation for $\sum_{k=1}^{n} k^9$, we can't justify this integral.

□

We will need one more simple rate of accumulation to deal with constant terms. A constant function sees no change, so the accumulation must always be zero, and that will come from a rate of accumulation that is zero.

Theorem 3.2.6 Rate of Accumulation for Constants. *If $f(x) = c$ for some constant c (a constant function), then $f'(x) = 0$.*

Proof. Because $\int_a^b 0\,dx = 0$, we can write $f(x) = c + \int_a^x 0\,dz$ for any value a. Thus, the rate of accumulation $f'(x) = 0$. ■

The linearity properties of definite integrals imply that rates of accumulation also satisfy the same linearity properties.

Theorem 3.2.7 Linearity of Rates of Accumulation. *If $f(x)$ has a rate of accumulation $f'(x)$ and $g(x)$ has a rate of accumulation $g'(x)$, then the function $h(x) = c_1 f(x) + c_2 g(x)$ with constants c_1 and c_2 has a rate of accumulation $h'(x) = c_1 f'(x) + c_2 g'(x)$.*

Proof. Using a common initial point at $x = a$, we can write

$$\begin{aligned} f(x) &= f(a) + \int_a^x f'(z)\,dz, \\ g(x) &= g(a) + \int_a^x g'(z)\,dz. \end{aligned}$$

Because $h(x) = c_1 f(x) + c_2 g(x)$, we can use the linearity properties of definite integrals to rewrite

$$\begin{aligned} h(x) &= c_1\Big(f(a) + \int_a^x f'(z)\,dz\Big) + c_2\Big(g(a) + \int_a^x g'(z)\,dz\Big) \\ &= c_1 f(a) + c_2 g(a) + c_1 \int_a^x f'(z)\,dz + c_2 \int_a^x g'(z)\,dz \\ &= (c_1 f(a) + c_2 g(a)) + \int_a^x (c_1 f'(z) + c_2 g'(z))\,dz \end{aligned}$$

Since $h(a) = c_1 f(a) + c_2 g(a)$, we can see that $h(x)$ has been written in the form of an accumulation with a rate of accumulation $h'(x) = c_1 f'(x) + c_2 g'(x)$. ■

A polynomial is a linear combination of simple powers. Consequently, the derivative of a polynomial will be the same linear combination of the derivatives of those powers.

Example 3.2.8 Find $f'(x)$ for $f(x) = x^2 - 6x + 5$.

Solution. We look at $f(x)$ as a sum of three terms:

$$f(x) = \underbrace{x^2}_{f_1(x)} + \underbrace{-6x}_{f_2(x)} + \underbrace{5}_{f_3(x)}.$$

First, $f_1(x) = x^2$ is an elementary power for which we know $f_1'(x) = 2x$. Next, $f_2(x) = -6x$ is a constant -6 times x, so $f_2'(x)$ is the same constant times 1, $f_2'(x) = -6 \cdot 1 = -6$. Finally, $f_3(x) = 5$ is a constant function so that $f_3'(x) = 0$. The linearity for rates of accumulation implies

$$f'(x) = \underbrace{2x}_{f_1'(x)} + \underbrace{-6}_{f_2'(x)} + \underbrace{0}_{f_3'(x)} = 2x - 6.$$

□

3.2.3 Using the Rate of Accumulation

Now that we know how to find the rate of accumulation for simple polynomials, we can express a polynomial as an accumulation with that rate. Although we know the rate of accumulation, we also need to be careful that the initial value matches the function of interest.

Example 3.2.9 Express $f(x) = x^3 - 6x^2 - 4$ as an accumulation from $x = 1$.

Solution. Start by finding the rate of accumulation.

$$f'(x) = 3x^2 - 6(2x) + 0 = 3x^2 - 12x$$

The rate of accumulation becomes the integrand of the accumulation using $f'(z)$. Because the integral will start at $x = 1$, the initial value will be

$$f(1) = 1^3 - 6 \cdot 1^2 - 4 = -9.$$

We can now write $f(x)$ as an accumulation from $x = 1$:

$$\begin{aligned} f(x) &= f(1) + \int_1^x f'(z)\,dz \\ &= -9 + \int_1^x 3z^2 - 12z\,dz. \end{aligned}$$

□

The sign of the rate of accumulation determines whether an accumulation is increasing or decreasing. Furthermore, the monotonicity of the rate of accumulation determines the concavity of the accumulation. If we determine the rate of accumulation for a polynomial $f(x)$, then we can use sign analysis on the resulting formula $f'(x)$ to characterize monotonicity of $f(x)$.

Because the rate of accumulation $f'(x)$ will itself be a new polynomial, we can describe its monotonicity using the rate of accumulation for that rate of accumulation. The rate of accumulation of the rate of accumulation is called the **second derivative** and is named $f''(x)$. Because the monotonicity of the rate $f'(x)$ determines concavity of f, we can use sign analysis of $f''(x)$ to characterize the concavity f.

Example 3.2.10 We found that $f(x) = x^3 - 6x^2 - 4$ has corresponding rate $f'(x) = 3x^2 - 12x$. Describe the monotonicity and concavity of $f(x)$.

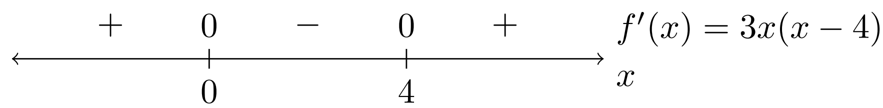
Solution. Sign analysis of the rate of accumulation $f'(x)$ will be used to describe the monotonicity of $f(x)$. We factor

$$f'(x) = 3x^2 - 12x = 3x(x-4).$$

The zeros (x-intercepts) occur at $x = 0$ and $x = 4$ and $f'(x)$ is continuous. We need to test the sign of $f'(x)$ in the intervals $(-\infty, 0)$, $(0, 4)$, and $(4, \infty)$. Only the sign matters, so using the factored formula, we can count how many factors are positive and negative.

- $f'(-1) = 3(-1)(-5) = +$ (2 negative factors)
- $f'(1) = 3(1)(-3) = -$ (1 negative factors)
- $f'(5) = 3(5)(1) = +$ (0 negative factors)

When doing such a problem, we just use a sign analysis number line to record our results.

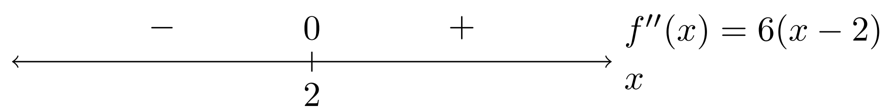


We interpret the signs of $f'(x)$ to deduce the monotonicity of $f(x)$. The function $f'(x)$ is positive on the intervals $(-\infty, 0)$ and $(4, \infty)$ and negative on the interval $(0, 4)$. Consequently, $f(x)$ must be increasing on the intervals $(-\infty, 0)$ and $(4, \infty)$ and decreasing on the interval $(0, 4)$. Because $f(x)$ is continuous (all polynomials are continuous), we can extend each of these intervals to include the end-points at $x = 0$ and $x = 4$.

Concavity of $f(x)$ depends on the monotonicity of $f'(x)$. Because $f'(x) = 3x^2 - 12x$ is itself a polynomial, we can find its rate of accumulation $f''(x)$ and perform sign analysis to deduce that monotonicity.

$$f''(x) = 6x - 12 = 6(x-2)$$

The only zero of $f''(x)$ is at $x = 2$. We test the intervals $(-\infty, 2)$ and $(2, \infty)$ using the signs of $f''(x)$, summarized by the following number line.



Because $f''(x) < 0$ on the interval $(-\infty, 2)$, we know that $f'(x)$ is decreasing on $(-\infty, 2)$. Consequently, $f(x)$ is concave down on $(-\infty, 2)$. Because $f''(x) > 0$ on the interval $(2, \infty)$, we know that $f'(x)$ is increasing on $(2, \infty)$. Consequently, $f(x)$ is concave up on $(2, \infty)$. Again, because $f(x)$ is continuous, intervals of concavity can be extended to include the end-point at $x = 2$.

Notice how the graph of $y = f(x)$ reflects the monotonicity and concavity that we have determined.

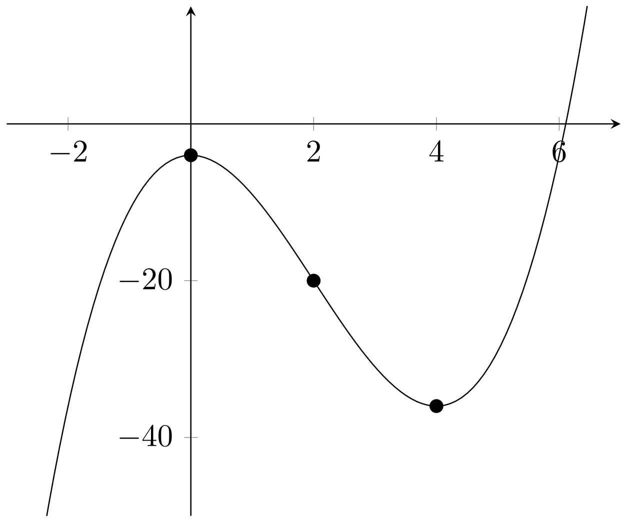


Figure 3.2.11 Graph of $y = f(x) = x^3 - 6x^2 - 4$, including turning points at $x = 0$ and $x = 4$ and an inflection point at $x = 2$.

□

3.2.4 Summary

- The rate of accumulation $f'(x)$ is the integrand function of an accumulation function $f(x)$,

$$f(x) = f(a) + \int_a^x f'(z)\,dz$$

- The rate of accumulation for a function is equivalent to what we will later define as the derivative of the function. Thus, we often just call the rate of accumulation $f'(x)$ the derivative of $f(x)$. Showing this equivalence will be the goal of the Fundamental Theorem of Calculus.

- Our known accumulation formulas have a complementary interpretation as derivatives:

  - $f(x) = x$ has derivative $f'(x) = 1$
  - $f(x) = x^2$ has derivative $f'(x) = 2x$
  - $f(x) = x^3$ has derivative $f'(x) = 3x^2$
  - $f(x) = x^4$ has derivative $f'(x) = 4x^3$

  A pattern in these derivatives suggest a conjecture (which we later prove) that any power $f(x) = x^n$ has derivative $f'(x) = nx^{n-1}$.

  In addition, any constant function $f(x) = c$ has derivative $f'(x) = 0$.

- Rates of accumulation (derivatives) satisfy the linear properties of a sum rule and a constant multiple rule.

- For simple polynomials $f(x)$, we compute $f'(x)$ as a related polynomial in order to answer questions about monotonicity. Because $f'(x)$ is a polynomial, it also has its own derivative $f''(x)$ (called the second derivative of $f(x)$). The sign of $f''(x)$ determines the monotonicity of $f'(x)$, which in turn determines the concavity of $f(x)$.

3.2.5 Exercises

For each function defined in terms of an integral, identify the rate of accumulation.

1. $f(x) = -5 + \int_1^x ze^{-z}\,dz$

2. $Q(x) = \int_0^x \frac{s}{s^2+1}\,ds$

3. $A(x) = 4\int_{-2}^x t + 3t^3\,dt$

4. $R(x) = 1 + 2\int_1^x \sin(z)\,dz - 3\int_1^x \cos(z)\,dz$

Find the derivative of each polynomial.

5. $p(x) = x^2 - 5x$

6. $q(x) = x^3 - 6x^2$

7. $r(x) = x^4 + 2x^3 - 5$

Write each polynomial as an accumulation function from the indicated starting point.

8. $p(x) = x^2 - 5x$ from $x = 1$

9. $p(x) = x^2 - 5x$ from $x = -2$

10. $q(x) = x^3 - 6x^2$ from $x = 2$

11. $q(x) = x^3 - 6x^2$ from $x = 0$

12. $r(x) = x^4 + 2x^3 - 5$ from $x = -1$

13. $r(x) = x^4 + 2x^3 - 5$ from $x = 2$

Determine the monotonicity and concavity of each polynomial.

14. $f(x) = 3x^2 - 48x + 5$

15. $g(x) = 100 + 80x - 2x^2$

16. $h(x) = 9x^2 - x^3$

17. $w(x) = x^3 - 9x^2 + 15x + 4$

18. $u(x) = x^4 - 6x^2 + 15$

19. $M(x) = 15 + 4x^3 - x^4$