3.3 Accumulation of Change

Overview. Many students learn a basic rule relating distance d, speed r and time t: $d = rt$ or "distance equals rate times time." This statement is really only true when the rate is unchanging. If the speed is constant at a rate of r_1 for a time t_1 and then instantly changes to a new constant speed at rate r_2 for another time t_2, then the total distance d traveled over the total time is

$$d = r_1 t_1 + r_2 t_2.$$

This generalizes to any number of intervals of constant rate, that the total change in position (displacement) equals the sum of the products of the rate times the increment of time at that rate.

The **definite integral** is the mathematical generalization of the idea that we just described. Given any rate of change $r(x)$ for a quantity Q with respect to an independent variable x, the definite integral's purpose is to compute the increment of change in Q when the independent variable changes from one value, $x = a$, to another value, $x = b$. We write

$$Q(b) - Q(a) = \int_a^b r(x)dx \qquad \Leftrightarrow \qquad Q(b) = Q(a) + \int_a^b r(x)dx.$$

This section introduces the idea of the definite integral for special functions for which we can compute the increment of change without knowing any additional calculus rules. We start with simple functions, which means the functions are constant on intervals. These functions motivate the basic properties which we then apply graphically and numerically. We will learn the rules later.

3.3.1 Rate of Change

Suppose that we have two variables that are related by a function. In mathematics, we often think of the prototypical variables x and y with some function $f : x \mapsto y$. But in physical situations, we are often considering changes in time so that we use the independent variable t for time. The official definition for **rate of change** is as the **derivative**. In the present context, we will not need to know how to compute derivatives. We only need to consider that there is a function that physically measures a rate of change.

For example, a speedometer measures speed which is a rate of change of distance with respect to time. As another example, we can physically measure the rate at which water flows through a pipe which represents a rate of change of a reservoir (e.g., a tub or a pool) that is being filled or drained. An electrical analog of water flow is electrical current which measures the rate of change of electrical charge along an electrical path. In biology, the rate of change of a population is physically measured through birth, death and migration rates.

When any of these rates are constant over an interval $t \in [a, b]$, the net change in the quantity of interest Q is equals the rate times the increment of time. The following definition makes this clear.

Definition 3.3.1 Constant Rate of Change. Given a quantity Q that is a function of independent variable t, say $t \mapsto Q(t)$, we say that Q has a constant rate of change r on an interval $[a, b]$ if for any t_1, t_2 satisfying $a \leq t_1 < t_2 \leq b$,

$$Q(t_2) - Q(t_1) = r \cdot (t_2 - t_1),$$

which is often written $\Delta Q = r \cdot \Delta t$. $\Diamond$

A quantity that has a constant rate of change satisfies a linear equation on the given interval and the rate of change corresponds to the slope of that line. In particular, if c is any value for t in the interval, $c \in [a, b]$, then the accumulation $Q(t)$ is a linear function of t,

$$Q(t) = Q(c) + r(t - c),$$

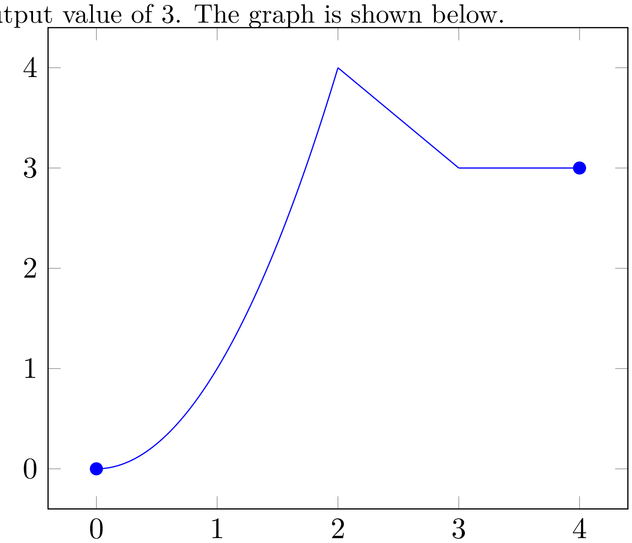
using the point–slope equation of a line. The value $Q(c)$ represents the initial value while $r(t - c)$ represents the increment of change in Q when the independent variable goes from c to the value t.

In preparation for extending the idea of rate of change, we need to recall the concept of piecewise functions 5.2.3. A piecewise function considers its domain as consisting of a collection of disjoint (non-overlapping) intervals. On each such interval, the function has a separate formula or rule of calculation.

Example 3.3.2 The function f is defined by the equation

$$f(x) = \begin{cases} x^2, & 0 \le x < 2, \\ 6 - x, & 2 \le x \le 3 \\ 3, & 3 < x \le 4 \end{cases}.$$

The domain of f is the union of disjoint intervals $[0, 2)$, $[2, 3]$ and $(3, 4]$ which corresponds to $[0, 4]$. The notation states that for input values x between 0 and 2, including 0, the function will square the input to give the output. Between 2 and 3, inclusively, the function will subtract the input from 6 for the output. For input values greater than 3 but less than or equal to 4, the function has a constant output value of 3. The graph is shown below.



□

Using piecewise functions, we can define something called a **simple function**. Such a function is piecewise constant, meaning that the domain is formed as a union of disjoint intervals and the function has a constant value on each interval. To define these intervals, we first introduce the idea of a **partition** which will be used to define the end points of these subintervals.

Definition 3.3.3 Partition. A **partition** of size n of an interval $[a, b]$ is an increasing, finite sequence of numbers $P = (x_0, x_1, \ldots, x_n)$ such that $x_0 = a$, $x_n = b$ and $x_j < x_{j+1}$. The **increments** of the partition correspond to the widths of subintervals, with

$$\nabla x_j = x_j - x_{j-1}$$

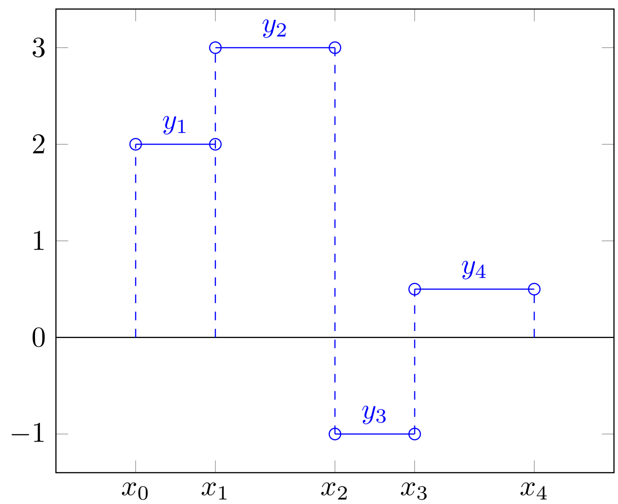
being the width of the subinterval $[x_{j-1}, x_j]$ for $j = 1, \ldots, n$. ◊

Definition 3.3.4 Simple Function. Given a partition P of size n of an interval $[a, b]$, a function f is a **simple function** on the partition P with values $(y_1, \ldots, y_n)$ if

$$f(x) = \begin{cases} y_1, & x_0 < x < x_1, \\ y_2, & x_1 < x < x_2, \\ \vdots & \\ y_n, & x_{n-1} < x < x_n. \end{cases}$$

◊

The figure below illustrates a simple function defined with a partition of size $n = 4$. Open circles are used on the edges of the segments because we did not define the value at the actual partition points, only on the intervals between those points. That is because when a rate changes instantaneously between two values, the rate can not be properly defined at the instant itself.



We can use a simple function to represent a special case of a varying rate of change, namely a rate of change that is constant on subintervals but which changes instantly (not physically possible in most situations) at the points of a partition. Given a simple rate function, $r(x)$, on a partition P of size n of the interval $[a, b]$ with values $(r_1, r_2, \ldots, r_n)$, we can define an **accumulation function** that is piecewise linear on the same partition having initial value $(a, f(a))$:

$$f(x) = \begin{cases} f(a) + r_1(x - x_0), & x_0 \le x < x_1, \\ f(a) + r_1 \nabla x_1 + r_2(x - x_1), & x_1 \le x < x_2, \\ f(a) + \displaystyle\sum_{k=1}^{2} r_k \nabla x_k + r_3(x - x_2), & x_2 \le x < x_3, \\ f(a) + \displaystyle\sum_{k=1}^{3} r_k \nabla x_k + r_4(x - x_3), & x_3 \le x < x_4, \\ \vdots & \\ f(a) + \displaystyle\sum_{k=1}^{n-1} r_k \nabla x_k + r_n(x - x_{n-1}), & x_{n-1} \le x \le x_n. \end{cases}$$

The purpose of the summation is to represent the accumulation of change on all previous subintervals of the partition in order to make the accumulation function $f(x)$ continuous on the full interval $[a, b]$.

Definition 3.3.5 Definite Integral of Simple Function. Given a simple rate function, $r(x)$, on a partition P of size n of the interval $[a, b]$ with values $(r_1, r_2, \ldots, r_n)$, the total accumulated change associated with this rate is the definite integral represented by is given by

$$\int_a^b r(x)\,dx = \sum_{k=1}^{n} r_k \,\nabla x_k.$$

◊

It is common that the increments ∇x_k be instead written Δx_k. However, this is a notational abuse because Δx_k technically represents the forward difference $\Delta x_k = x_{k+1} - x_k$ with $k = 0, \ldots, n-1$. Ignoring this complaint, the total accumulation of change is often written

$$f(b) - f(a) = \sum_{k=1}^{n} r_k \,\Delta x_k.$$

(The complaint can formally be resolved by shifting the index values from $k = 1, \ldots, n$ to $k = 0, \ldots, n-1$.)

Example 3.3.6 A storage reservoir starts with 100 gallons of water. Over the next 20 minutes, water is added to the reservoir at a rate of 5 gal/min. Then water is pumped out at a rate of 12 gal/min for 10 minutes. For the next 30 minutes, water is added at a rate of 3 gal/min. Find a piecewise linear function describing the amount of water in the reservoir as a function of time (in minutes).

Solution. The rate function is a simple function using the partition P that starts at $x_0 = 0$ and has increments of $\nabla x_1 = 20$, $\nabla x_2 = 10$ and $\nabla x_3 = 30$. That is, the partition is given by $P = \{0, 20, 30, 60\}$. The rate of change of water is constant on the subintervals defined by this partition:

$$R(t) = \begin{cases} 5, & 0 < t < 20, \\ -12, & 20 < t < 30, \\ 3, & 30 < t < 60. \end{cases}$$

The amount of water in the reservoir is also a function of time, say $W(t)$, and is defined as an accumulation using the rate of change $R(t)$ found above. Because the reservoir begins with $W(0) = 100$, our initial value, we can write $W(t)$ as a piecewise linear function that accumulates the change in water over each of the subintervals. Consider first the total accumulation of change in water on each of the subintervals, which is equal to the rate of change times the increment of time for that subinterval.

$$\begin{aligned} W(20) - W(0) &= 5(20) = 100 \\ W(30) - W(20) &= -12(10) = -120 \\ W(60) - W(30) &= 3(30) = 90 \end{aligned}$$

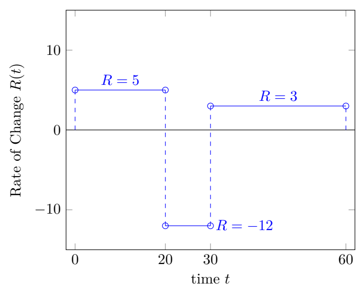
Notice that the total change in water volume over the entire interval $[0, 60]$ is the sum of these increment,

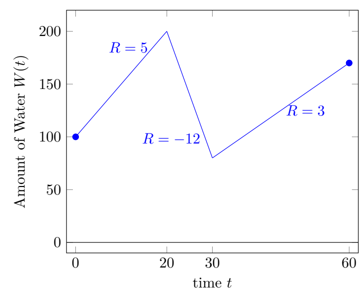
$$W(60) - W(0) = 100 + -120 + 90 = 70.$$

The accumulation function $W(t)$, which has an initial value $W(0) = 100$, is therefore defined by

$$W(t) = \begin{cases} 100 + 5(t-0) = 100 + 5t, & 0 \le t < 20, \\ 100 + 100 - 12(t-20) = 200 - 12(t-20), & 20 \le t < 30, \\ 200 - 120 + 3(t-30) = 80 + 3(t-30), & 30 \le t \le 60. \end{cases}$$

The graphs of the rate function $R(t)$ and the water level $W(t)$ are shown below. Notice that although the rate function is not defined at the partition points, the water level function $W(t)$ is defined and continuous at those points. It is continuous because the accumulations are designed to start on the next interval exactly where it stops from the previous interval with no sudden jumps.





□

There is an important geometric interpretation of accumulation in terms of area on the graph. Recall that the area of a rectangle is defined as the product of the height and the width. Mathematically, this is the same operation as when we calculate an increment as the product of a rate and the increment of the independent variable, except that a rate can be negative. Consequently, we introduce the idea of **signed area**.

Definition 3.3.7 Signed Area (Informal). Suppose we have the graph of a function $y = f(x)$ that is continuous on an interval (a, b) and is either entirely above the axis, $f(x) > 0$ for all $x \in (a, b)$, or entirely below the axis, $f(x) < 0$ for all $x \in (a, b)$. Then we can define the **signed area** of the graph by considering the area A (area itself is always positive) of the region between the curve $y = f(x)$ and the axis $y = 0$ and between the vertical lines $x = a$ and $x = b$. If $f(x) > 0$ (above the axis), then we say that we have **positive area** A; if $f(x) < 0$ (below the axis), then we say that we have **negative area** $-A$.

If the graph $y = f(x)$ on an interval (a, b) has a finite number of discontinuities or crosses the axis so that sometimes the graph is above the axis and sometimes below, then we can consider a partition of $[a, b]$ using the x-values of the discontinuities and zeros of f. Then on every subinterval from this partition, the earlier definition applies and we have a signed area for each subinterval. The signed area for the entire graph is the sum of the signed areas of the subintervals, adding areas that are above the axis and subtracting areas that are below the axis. ♢

Given any simple rate function $f(x)$, the signed area of the graph $y = f(x)$ on the interval $[a, b]$ consists of the sum of signed areas of rectangles. This exactly matches the definite integral

$$\int_a^b f(x)\,dx = \sum_{k=1}^{n} r_k \,\nabla x_k.$$

Therefore, we adopt the definite integral as our formal definition of signed area.

Definition 3.3.8 Signed Area and Accumulated Change (Formal). Suppose we have a function $y = f(x)$ that is bounded and piecewise continuous on an interval (a, b) $(a < b)$. The signed area of f on the interval (a, b) is defined as the definite integral

$$\int_a^b f(x)\,dx.$$

If $f(x)$ gives the rate of change of a quantity Q with respect to the independent variable x, then the definite integral also gives the increment of change in Q:

$$Q(b) - Q(a) = \int_a^b f(x)\,dx.$$

The function $f(x)$ is called the **integrand** and the variable x is called the **variable of integration**. The values a and b are called the **limits of integration**. ♢

The notation of the definite integral uses an elongated "S" called the integral symbol $\int$ that should remind you of the idea of summing increments of signed area. The limits of integration a (lower) and b (upper) represent the starting and ending points of integration, respectively. The increments of signed area are represented by the formula $f(x)\,dx$ which represents a strip of signed area with signed height $f(x)$ (generalizing the constant height of a simple rate function) and infinitesimally small width dx (generalizing the increments of a partition ∇x).

Although we have presented these ideas as definitions, they are really important consequences of the development of calculus. In particular, the statement that the increment of change $Q(b) - Q(a)$ is equal to the definite integral of the rate of change of Q is so most important that this result is called the (((Unresolved xref, reference "fundamental-theorem-calculus"; check spelling or use "provisional" attribute)))Fundamental Theorem of Calculus . One of the primary goals of learning calculus is to understand what this theorem means and why it is really true.

3.3.2 Interpretation of Definite Integrals as Signed Areas

We will learn integration methods later. For now, we will explore examples, including simple functions, where knowing the interpretation of a definite integral allows us to determine results using only the ideas of signed area. When exact area calculations can not be found, then approximations of signed area can allow us to estimate the value of the definite integral. We start by revisiting our earlier examples using the context of definite integrals.

Example 3.3.9 Integral of Constant Rate. Consider the case of a constant function $R(t) = r$, representing a constant rate of change for some quantity Q with respect to time t. Earlier we noted that for constant rate of change, the increment of change in Q as t changes from $t = a$ to $t = b$ is equal to

$$Q(b) - Q(a) = r(b - a).$$

Using the idea of a definite integral to represent the accumulated increment of change, we can rewrite this (for constant rate) as

$$Q(b) - Q(a) = \int_a^b r\,dt.$$

If we rewrite our increment of change in Q so that it is solved for $Q(b)$, we find

$$Q(b) = Q(a) + \int_a^b r\,dt = Q(a) + r(b-a).$$

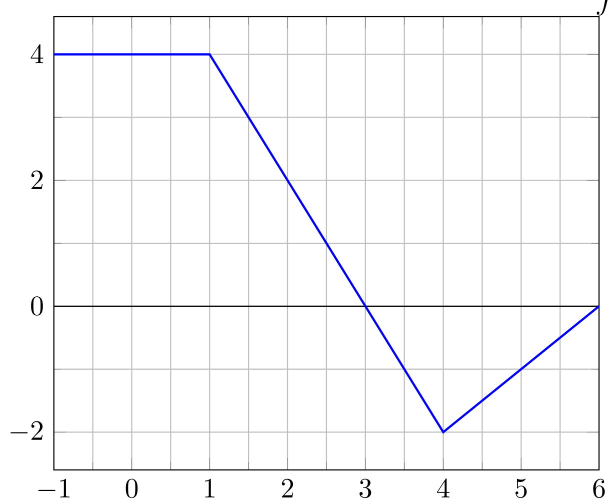
This is read as saying that $Q(b)$ equals the initial value $Q(a)$ plus the total increment of change in Q as t goes from a to b. Because this equation is true for any value of b, we can replace it with a variable and obtain

$$Q(x) = Q(a) + \int_a^x r\,dt = Q(a) + r(x-a).$$

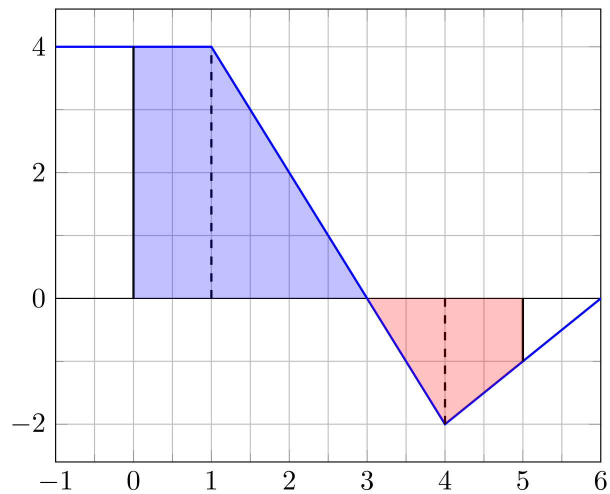
That is, we should recognize the point–slope equation of a line as a special case of an initial value plus an increment of change. □

Knowing the area of regions of a graph can allow us to compute some definite integrals.

Example 3.3.10 The graph of $y = f(x)$ is shown below. Find $\displaystyle\int_0^5 f(x)\,dx$.



Solution. If we consider vertical lines at $x = 0$ and $x = 5$ and then look at the regions between these lines, the graph and the x-axis, we can identify the areas that need to be calculated. Regions above the axis are shaded in blue and represent positive signed area while regions below the axis are shaded in red and represent negative signed area. In addition, dashed lines have been included to represent convenient places to split the region into simple geometric shapes.



Consider the geometric regions contained in the figure. There is a rectangle on the interval $[0,1]$ with a height $f(x)=4$. Since the area of the rectangle is 4 and the region is above the axis, we know

$$\int_0^1 f(x)dx = 4.$$

Next, we have a triangle on the interval $[1,3]$ with a horizontal base given by the increment $\Delta x = 3-1 = 2$ and a vertical height of 4. This region is also above the axis so that

$$\int_1^3 f(x)dx = \frac{1}{2}(2)(4) = 4.$$

Combining these regions, we know

$$\int_0^3 f(x)dx = 4+4 = 8.$$

The area of the region below the axis can be found in several ways. One way is to identify a triangle on interval $[3,4]$ and a trapezoid on $[4,5]$. The triangle has horizontal width $\Delta x = 4-3 = 1$ and height 2 for a total area of $\frac{1}{2}(1)(2) = 1$. Since the region is below the axis, we have

$$\int_3^4 f(x)dx = -1.$$

A trapezoid is a geometric shape consisting of two parallel sides and its area is the average length of the parallel sides times the perpendicular length between those sides. The area of the trapezoid on $[4,5]$ uses parallel lengths of 2 and 1 with a perpendicular distance $\Delta x = 5-4 = 1$ for a total area of $\frac{1}{2}(2+1)(1) = \frac{3}{2}$. As a negative signed area, we have

$$\int_4^5 f(x)dx = -\frac{3}{2}.$$

Combining these two shapes for total signed area on $[3,5]$, we have

$$\int_3^5 f(x)dx = -1 + -\frac{3}{2} = -\frac{5}{2}.$$

Another way to find this area would be to consider a larger triangle coming from the interval $[3, 6]$ and then subtract the area of the smaller triangle on the interval $[5, 6]$ that should not be included:

$$\int_3^5 f(x)dx = -\big(\frac{1}{2}(6-3)(2) - \frac{1}{2}(6-5)(1)\big) = -\frac{5}{2}.$$

Finding the total signed area for the graph, we find the definite integral desired,

$$\int_0^5 f(x)dx = \int_0^3 f(x)dx + \int_3^5 f(x)dx = 8 + -\frac{5}{2} = \frac{11}{2}.$$

$\square$

3.3.3 Finding Definite Integrals with Technology

When we do not have easy tricks to compute a definite integral, we can get high accuracy estimates using technology. There are free websites that can compute integrals such as WolframAlpha or SageMath described previously. Most graphing calculators have the ability to compute a definite integral and therefore the ability to compute accumulated change or signed areas. In general, you will apply the following steps on a calculator.

1. Identify the integrand function (i.e., the rate of change or the function defining signed area) and the limits of integration.
2. Use the graphing feature of your calculator so that the function is graphed and the interval of interest is showing. You may need to change the window of you graph.
3. Use the menu system to find the integral. You will need to select your function and input the end points of the interval of interest.

Example 3.3.11 We wish to compute $\int_2^5 (2^x - 8)\, dx$. First, steps are given for evaluating this using a TI-83/84 graphing calculator. This is followed by a call to WolframAlpha.com and a SageMath script that compute the same integral.

Solution.

- Find the integral using a TI-83/84 graphing calculator.

    1. Identify the function. In this example, the formula $f(x) = 2^x - 8$ represents the integrand while the interval is based on the limits of integration $[2, 5]$.
    2. Graph the function over this interval. Each step is given as a separate line.

    ```
    Y=
    Y1= 2^x-8
    WINDOW
    Xmin= 2
    Xmax= 5
    GRAPH
    ZOOM: ZoomFit
    ```

    3. Compute the definite integral.

        CALC: $\int f(x)dx$

`Lower Limit?` 2

`Upper Limit?` 5

The calculator reports back $\int f(x)dx = 16.3954611$.

- Find the integral using WolframAlpha.

  In the prompt box, just type "`integrate 2^x-8 from 2 to 5`". An exact and approximate answer is given

  $$\int_2^5 (2^x - 8)dx = \frac{28}{\log(2)} - 24 \approx 16.3955.$$

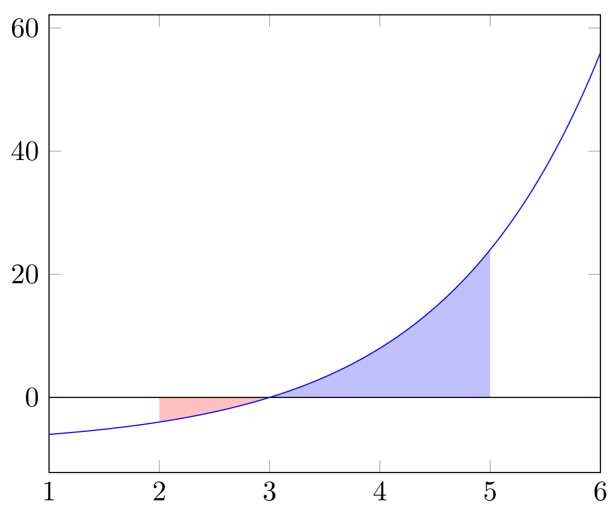
  Note that log on WolframAlpha refers to the natural logarithm ln.

- The SageMath script to compute the integral is just as similar. We need a separate command to show our result as a decimal approximation.

```
var("x")
value = integrate(2^x-8, [x,2,5])
show(value)
show(value.n())
```

□

The graph for the previous example, with the signed areas shaded, is shown below. Notice that the graph has two regions, one of which is negative (red) and one of which is positive (blue). We can find the point where the sign switches by solving $2^x - 8 = 0$ which is $x = 3$. In the next example, we will find the signed area of each interval separately and relate the values to the overall signed area.



Example 3.3.12 Compute $\displaystyle\int_2^3 (2^x - 8)dx$ and $\displaystyle\int_3^5 (2^x - 8)dx$.

Solution. As long as the interval of interest is graphed, we can compute the definite integral to get the signed area. Since we already have our integrand $f(x) = 2^x - 8$ in our calculator, we can just go to the compute steps.

- `CALC:` $\int f(x)dx$

  `Lower Limit?` 2

  `Upper Limit?` 3

The calculator reports back $\int f(x)dx = -2.22922$.

- `CALC:` $\int f(x)dx$

  `Lower Limit? 3`

  `Upper Limit? 5`

The calculator reports back $\int f(x)dx = 18.624681$.

We interpret these results. The first integral was

$$\int_2^3 (2^x - 8)dx \approx -2.22922$$

and means that the area of the first region from interval $(2, 3)$ is 2.22922. Because the graph is below the axis, the integral counts as negative area. The second integral was

$$\int_3^5 (2^x - 8)dx \approx 18.624681$$

which means that the second region on interval $(3, 5)$ has area 18.624681 and counts as positive area. The total signed area is the sum of these parts,

$$\begin{aligned}\int_2^5 (2^x - 8)dx &= \int_2^3 (2^x - 8)dx + \int_3^5 (2^x - 8)dx \\ &= -2.22922 + 18.624681 = 16.395461.\end{aligned}$$

Notice that there is a slight numerical discrepancy between our two methods. This is because numerical calculation of definite integrals involves an approximation. Approximations of necessity comes with some unavoidable errors. □

We can solve some applications about change by identifying an appropriate accumulation of a rate of change. For example, velocity is a rate of change of position. Consequently, the change in position (displacement) can be computed as an accumulation of change using velocity with a definite integral.

Example 3.3.13 Suppose a hovercraft starts 40 meters away from the shore. If the velocity (meters per second) of the hovercraft is a function of time (seconds) $v(t) = t(t-3)(t-5)$ (a positive velocity is moving away from shore, increasing the distance). What is the position of the hovercraft after 3 seconds and again at 5 seconds?

Solution. Let $x(t)$ measure the position of the hovercraft (meters from shore) as a function time (seconds). By the principle of accumulation of change, the change in position over 3 seconds is equal to the definite integral of the rate of change,

$$x(3) - x(0) = \int_0^3 v(t)dt = \int_0^3 [t(t-3)(t-5)]dt.$$

Using technology, we find the accumulated change, such as asking WolframAlpha “integral of $t(t-3)(t-5)$ from 0 to 3”,

$$\int_0^3 [t(t-3)(t-5)]dt = \frac{63}{4} = 15.75.$$

Since the hovercraft started 40 meters from shore and has moved a net amount 15.75 meters (away from shore), the hovercraft is at a position 55.75 meters after 3 seconds.

We can compute the change in position over the next two seconds using

$$x(5) - x(3) = \int_3^5 v(t)dt = \int_3^5 [t(t-3)(t-5)]dt.$$

Using technology, we find the accumulated change, such as asking WolframAlpha "integral of $t(t-3)(t-5)$ from 3 to 5",

$$\int_3^5 [t(t-3)(t-5)]dt = \frac{-16}{3} \approx -5.333.$$

The hovercraft has moved 5.333 meters back toward the shore during the last two seconds. Since the craft was at 55.75 meters after 3 seconds, after 5 seconds it is at 50.417 meters away from the shore.

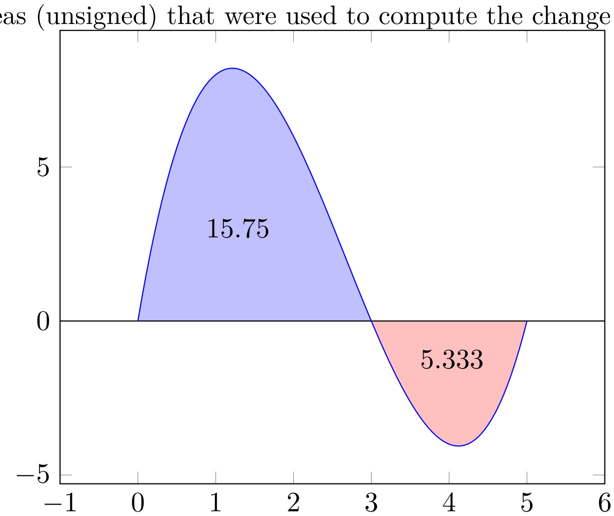
Alternatively, we could do a single integral finding the total change over all 5 seconds,

$$x(5) - x(0) = \int_0^5 v(t)dt = \int_0^5 [t(t-3)(t-5)]dt.$$

We find

$$\int_0^5 [t(t-3)(t-5)]dt = \frac{125}{12} \approx 10.417,$$

which implies that the hovercraft has moved from $x(0) = 40$ meters from shore to a new position of $x(5) = 50.417$ meters from shore, in agreement to our earlier calculations. The figure below shows a graph of the velocity function with the areas (unsigned) that were used to compute the change in position.



□

3.3.4 Summary

- Pending further edits.

3.3.5 Exercises

1. Pending.