4.1 Introduction to Sequences

Overview. Sequences are often introduced to us as examples for finding basic numerical patterns. We are shown the start to a list of numbers and asked if we can identify the next few numbers in the list or are asked to identify the rule being used to generate the sequence.

$$1, 5, 9, 13, \ldots$$
$$2, 6, 18, 54, \ldots$$

Do you see the patterns?

You probably recognized that in the first sequence, the next number would be 17 because the pattern involved adding 4 to the previous number. In the second sequence, you probably saw that we were multiplying by the value 3, so that the next number would have been 162. Not all sequences follow patterns. However, we use examples such as these to motivate the mathematical definition of a sequence.

We study sequences because they illustrate a number of ideas we will use in calculus. We eventually want to describe functions as dynamic models. Dynamic models for sequences are easier to illustrate than for general functions.

This section introduces the basic terminology for sequences. It explains how a sequence is a special type of function, where the domain is a set of integers. We will learn about explicit formulas for a sequence and recursive formulas for a sequence, using arithmetic and geometric sequences as our original motivation.

Later in this chapter, we will explore the dynamic ideas that will motivate calculus. Sequences that converge to a single value will be used to introduce the concept of limits. Recursive formulas for sequences will be used to introduce the ideas of accumulation which ultimately motivates the concept of integration. The dynamic behavior of a sequence will be analyzed in terms of its increment sequence which will motivate the calculus concept of the derivative.

4.1.1 Basic Terminology and Notation

A **sequence** is an ordered collection of numbers. The idea of being ordered is that we can say what the first number is, what the second number is, and so forth. To emphasize that the number have assigned positions, a sequence can be written as an **ordered list** using parentheses. The entire sequence can be assigned a symbol, just like a variable, so that a sequence assigned a symbol x and given by the values 1, 5, 9, 13, etc., would be written

$$x = (1, 5, 9, 13, \ldots).$$

Because the sequence has a specific order, we use an **index** as a way of counting through the sequence. For a given sequence, the term with index 1 is the first number of the sequence, the term with index 2 is the second number, the term with index 3 is the third number, and so forth. We use subscripts on a sequence to refer to an indexed value. So x_1 is the first value of sequence x and x_5 refers to the value of the sequence x with index 5.

Example 4.1.1 For the sequence $x = (1, 5, 9, 13, \ldots)$ and assuming the pattern continues, find each of the following values: x_1, x_3, and x_5.

Solution. The ordering of the list of values in the sequence can be made explicit with a table.

ordinal position	index	sequence value
first	1	1
second	2	5
third	3	9
fourth	4	13
fifth	5	17

Of course, you probably thought through the ordering in your head rather than make a table. From this ordering, we know that $x_1 = 1$, $x_3 = 9$, and $x_5 = 17$. □

The table in the solution for the previous example illustrates an explicit association between the index and the sequence value. We could reorganize this table to create a **mapping** between values.

$$\begin{array}{ccccccc} n & 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & & & & & & \\ x_n & 1 & 5 & 9 & 13 & 17 & 21 \end{array}$$

The mapping can be illustrated using two number lines, one for the index and the other for the sequence values, with arrows drawn from the index to the corresponding sequence value. We write $n \mapsto x_n$ to indicate that we have a mapping that goes from a value of n to a value x_n.

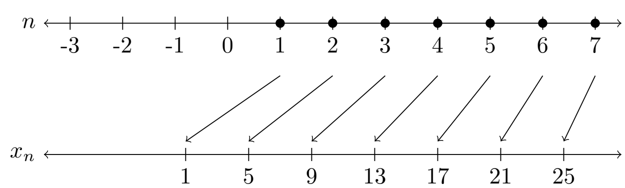


Figure 4.1.2 Illustration of the example sequence as a map $n \mapsto x_n$.

Another name for a mapping is a **function**. Sequences are functions whose domains correspond to an interval of the integers. The **domain** for a sequence is the set of possible values for the index. An interval of integers corresponds to a subset of integers with no gaps. The interval could be finite, as in $\{4, \ldots, 10\}$, or it could be infinite, as in $\{4, \ldots, \infty\}$. The usual domain for sequences is the set of natural numbers $D = \mathbb{N} = \{1, 2, 3, \ldots, \infty\}$. We often also want to include an initial value corresponding to an index value $n = 0$, in which case our domain is the extended natural numbers $D = \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

Definition 4.1.3 Sequence. A **sequence** x is a function with a domain D that is an interval of integers and values that are real numbers. We can write this in symbols using mapping notation,

$$x : n \in D \mapsto x_n \in \mathbb{R}.$$

◊

The mapping notation used in the definition of a sequence is a symbolic representation of the statement that a sequence is a function or a map. In particular, it says there is a map ($\mapsto$) named x that takes a value n from the set D ($n \in D$) and returns a value x_n from the set of real numbers $\mathbb{R}$ ($x_n \in \mathbb{R}$).

Defining a sequence as a function allows us more flexibility in what we include as sequences. The new definition allows us to have our first index value start at a value other than 1. It also allows us to use other variables for our index. The variable used for an index is most often a letter from the middle of the alphabet.

Example 4.1.4 Interpret the statement $u : k \in \{0, \ldots, 10\} \mapsto u_k \in \mathbb{R}$.

Solution. The map u defines a sequence with index values k going from $k = 0$ to $k = 10$. Because we do not have more information, we do not yet know the values of this sequence. □

Although mapping notation is useful to remind us that a sequence is a map or function, it can be a little cumbersome to use all the time. Mathematicians developed a more concise representation that reminds us of an ordered list. If we consider an interval of integers $\{n_i, \ldots, n_f\}$ (n_i is the *initial* value in the interval and n_f is the *final* value in the interval), then the mapping notation

$$x : n \in \{n_i, \ldots, n_f\} \mapsto x_n \in \mathbb{R}$$

is equivalent to the more compact sequence notation

$$x = (x_n)_{n=n_i}^{n_f}.$$

Example 4.1.5 Rewrite $u : k \in \{0, \ldots, 10\} \mapsto u_k \in \mathbb{R}$ using sequence notation.

Solution. We would write $u = (u_k)_{k=0}^{10}$. □

Sequence notation can be coupled with an ordered list of values to define a sequence that follows a pattern with an index that starts at a value other than 1.

Example 4.1.6 Interpret

$$w = (w_j)_{j=4}^{\infty} = (1, 2, 4, 8, 16, \ldots),$$

assuming the sequence follows a simple pattern.

Solution. This sequence notation tells us that w is a sequence, the index variable is j, and the interval of integers used for the index starts at $j = 4$ and continues through all integers greater than 4. The first few terms in a table showing the mapping are given below.

$$\begin{array}{ccccccc} j & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & & & & & & \\ w_j & 1 & 2 & 4 & 8 & 16 & 32 \end{array}$$

For this sequence, w_1, w_2, and w_3 are not defined because the values 1, 2, and 3 are not in the domain interval. □

Example 4.1.7 Interpret the sequence

$$u = (u_k)_{k=-1}^{\infty} = (8, 5, 2, -1, -4, \ldots),$$

assuming the sequence follows a simple pattern.

Solution. We have defined a sequence u with an index variable k. The first index value is $k = -1$. The sequence has the following values:

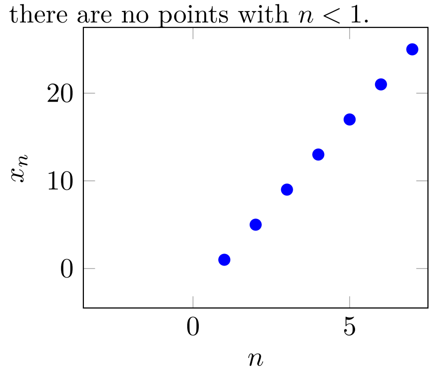
$$\begin{aligned} u_{-1} &= 8, \\ u_0 &= 5, \\ u_1 &= 2, \\ u_2 &= -1, \\ u_3 &= -4, \\ u_4 &= -7. \end{aligned}$$

□

4.1.2 Graphs of Sequences

We can create a graph anytime we can find a relation between two variables. For a sequence x, there is a natural choice for the two variables—the index n and the value x_n. The natural graph for a sequence x consists of the points (n, x_n). Because the index comes from a domain that is an interval of integers, the graph will be a collection of isolated points. This is why a sequence is called a discrete model.

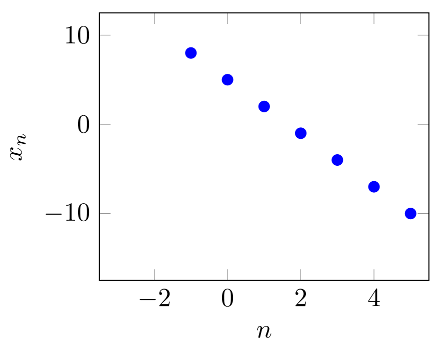
Example 4.1.8 The graph of the sequence $x = (1, 5, 9, 13, \ldots)$ consists of the points (n, x_n). The first few points of the graph—$(1, 1)$, $(2, 5)$, $(3, 9)$, and $(4, 13)$—are shown in the figure below. The sequence continues with addition points for $n > 4$, but there are no points with $n < 1$.



$\square$

Example 4.1.9 Create the graph for $u = (u_k)_{k=-1}^{\infty} = (8, 5, 2, -1, -4, \ldots)$.

Solution. The points in the graph use an index starting at $k = -1$. They include $(-1, 8)$, $(0, 5)$, $(1, 2)$, $(2, -1)$, and $(3, -4)$. The sequence continues to the right of these points.



$\square$

4.1.3 Explicit Sequence Representations

We sometimes have an **explicit representation** for a sequence, where the value of the sequence is a dependent variable in terms of the index as the independent variable. The expression defining the dependent variable could be used with each of the different notations.

Example 4.1.10 Each of the following notations define the same sequence.

$$x : n \in \{1, \ldots, \infty\} \mapsto x_n = \frac{n}{n+1}$$

$$x = \left(x_n = \frac{n}{n+1}\right)_{n=1}^{\infty}$$

$$x = \left(\frac{n}{n+1}\right)_{n=1}^{\infty}$$

We could simplify even further and write

$$x_n = \frac{n}{n+1}, n = 1, \ldots, \infty,$$

as the subscript notation x_n itself implies we have a sequence.

Writing the sequence value as a dependent variable provides a compact way of representing the entire sequence. To find a particular value of the sequence, we substitute the value for the independent variable into the expression.

$$\begin{aligned} x_1 &= x(1) = \frac{1}{1+1} = \frac{1}{2}, \\ x_2 &= x(2) = \frac{2}{2+1} = \frac{2}{3}, \\ x_{10} &= x(10) = \frac{10}{10+1} = \frac{10}{11}. \end{aligned}$$

□

In addition to substitution using actual integer values, we can also use substitution of expressions that have integer values. This includes using other variables that have integer values. To do this, we substitute the expression that appears in the subscript for every occurrence of the index in the expression.

Example 4.1.11 For the sequence defined by

$$x_n = \frac{n}{n+1}, n = 1, \ldots, \infty,$$

find the expressions defined by x_k, x_{n+1}, x_{2n}, and x_{n^2}.

Solution. With this interpretation, we can even do composition to find the sequence value at an index defined by a formula. Substituting the variable k for the index n in the dependent variable's expression, we find

$$x_k = \frac{k}{k+1}.$$

In a similar way, we substitute the expressions $n+1$, $2n$, and n^2 in the formula where n originally appeared to obtain

$$\begin{aligned} x_{n+1} &= \frac{(n+1)}{(n+1)+1} = \frac{n+1}{n+2}, \\ x_{2n} &= x(2n) = \frac{2n}{2n+1}, \\ x_{n^2} &= x(n^2) = \frac{n^2}{n^2+1}. \end{aligned}$$

□

Some sequences have patterns where we can easily find an explicit formula by recognizing how the numbers defining the sequence values relate to the index.

Example 4.1.12 Find an explicit formula for the sequence

$$x = (\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots),$$

and then find x_{12} and x_{2n}.

Solution. To find the explicit formula, we look for a pattern in the sequence and then try to find a relationship between the index and the pattern. Because the sequence domain was not specified, it is understood to be the natural numbers $\mathbb{N} = (1, \ldots, \infty)$. In this case, every sequence value is the reciprocal of a perfect square. If we look at this pattern with a table showing the index and the pattern, we find a relationship.

n	1	2	3	4
x_n	$\frac{1}{4} = \frac{1}{2^2}$	$\frac{1}{9} = \frac{1}{3^2}$	$\frac{1}{16} = \frac{1}{4^2}$	$\frac{1}{25} = \frac{1}{5^2}$

The pattern is that the number that is squared is always 1 greater than the index. So the explicit formula for this sequence is given by

$$x_n = \frac{1}{(n+1)^2}, \quad n \in \{1, 2, 3, 4, \ldots\}.$$

Using this explicit formula, we can find the desired values.

$$\begin{aligned} x_{12} = x(12) &= \frac{1}{(12+1)^2} = \frac{1}{169} \\ x_{2n} = x(2n) &= \frac{1}{(2n+1)^2} \end{aligned}$$

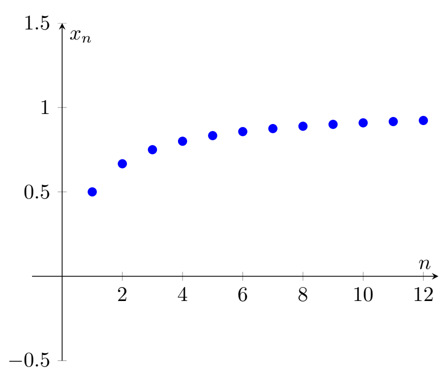
□

Knowing the explicit formula for a sequence, we can compute the values of the sequence to use in a graph.

Example 4.1.13 Graph the sequence $x_n = \dfrac{n}{n+1}$, defined for $n = 1, 2, 3, \ldots$.

Solution. This is the sequence discussed above. The plot will include the points

$$\{(n, x_n) : x_n = \frac{n}{n+1}, n = 1, 2, 3, \ldots\} = \{(1, \frac{1}{2}), (2, \frac{2}{3}), (3, \frac{3}{4}), (4, \frac{4}{5}), \ldots\}.$$



□

4.1.4 Summary

- Sequences are functions with domains that are intervals of integers. The independent variable (input) is called the **index**, and the dependent variable (output) is called the **value**. The value of a sequence x at index n is represented using subscripts for the index x_n.

- An **explicit representation** of a sequence x is when the function or map $n \mapsto x_n$ can be written with x_n as a dependent variable in terms of the index n.

- Sequence evaluation with an explicit formula involves substitution of the index variable by whatever expression appears in the subscript position.
- The standard graph of a sequence x uses the points (n, x_n) where n is chosen from the domain set of the sequence.

4.1.5 Exercises

In the following group of exercises, a sequence is defined. Identify the variable representing the sequence, the variable representing the index, and the domain or interval of integers the values of the index come from.

1. $u : i \in \{-3, -2, \ldots, 5\} \mapsto u_i \in \mathbb{R}$

2. $v : n \in \{5, \ldots, \infty\} \mapsto v_n \in \mathbb{R}$

3. $z = (z_k)_{k=2}^{\infty}$

4. $M = (M_t)_{t=-\infty}^{10}$

In the following group of exercises, a sequence with a pattern is given. Identify the values of the requested terms from the sequence. Create a graph that includes the first ten values from the sequence.

5. $x = (x_n)_{n=1}^{\infty} = (2, 4, 6, 8, \ldots)$
Find x_3, x_5, and x_7.

6. $y = (y_k)_{k=0}^{\infty} = (12, 9, 6, 3, \ldots)$
Find y_1, y_4, and y_6.

7. $w = (w_i)_{i=-2}^{\infty} = (24, 12, 6, 3, \ldots)$
Find w_0, w_2, and w_4.

8. $P = (P_t)_{t=0}^{\infty} = (100, 110, 125, 145, 170, \ldots)$
Find P_1, P_4, and P_6.

Find an explicit formula for each of the following sequences by identifying patterns relating the index and the expressions shown for the values.

9. $x = (x_n)_{n=0}^{\infty} = (1, 4, 9, 16, 25, \ldots)$

10. $y = (y_n)_{n=1}^{\infty} = (\frac{1}{4}, \frac{2}{9}, \frac{3}{16}, \frac{4}{25}, \ldots)$

11. $z = (z_n)_{n=0}^{\infty} = (0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots)$

In each of the following exercises, a sequence is defined explicitly. Evaluate the requested expressions.

12. $x_n = -3n + 20, \qquad n = 0, 1, 2, 3, \ldots$
Find x_0, x_1, and x_2.
Evaluate x_{k+2} and $x_k + 2$.

13. $y_k = \frac{2^{k-2}}{3^k}, \qquad k = 0, 1, 2, 3, \ldots$
Find y_0, y_1, and y_2.
Evaluate y_{n+1}, y_{k+1}, and $\dfrac{y_{k+1}}{y_k}$.