4.3 Accumulation Sequences

Overview. One of the most important mathematical ideas in calculus is that of an accumulation of change for physical quantities. As we have been learning about sequences, we have talked about how we can define sequences using explicit formulas and using recursive definitions. More recently, we have looked at how the increments of a sequence can help us understand the behavior of a sequence. For some sequences, we learned that patterns in the increments could be used to find additional terms in a sequence.

We are now ready to think about this more generally. Given any sequence of values, we wish to find that sequence for which the given sequence matches the increments. We call the sequence that we are finding the **accumulation sequence** of the given sequence.

In this section, we formally define and discuss the theory of accumulation sequences. Summation notation is introduced. We establish conditions that guarantee two sequences are equivalent. Then we illustrate applying these conditions to demonstrate that the explicit and recursive definitions for arithmetic and geometric sequences are equivalent.

4.3.1 Accumulation of Change

There are many examples of quantities where we track changes to the quantity rather than repeated measure the quantity itself. Consider a bank balance. We do not count our money every month. Instead, we add up all of our deposits and withdrawals and use them to adjust our record for the balance. Similarly, consider a population under study. It could be very costly to count all of the individuals every month. If instead we could track how many births and deaths occurred during the month, we could calculate a new population count by adding births and subtracting deaths.

Example 4.3.1 At the start of the year, you had $1500 in an account. Suppose that the sequence

$$W = (W_m)_{m=1}^{12} = (240, 300, 270, 450, 250, 310, 360, 270, 320, 300, 350, 480)$$

represents the total of monthly withdrawals from the account, and the sequence

$$D = (D_m)_{m=1}^{12} = (280, 280, 280, 280, 280, 280, 280, 280, 280, 280, 280, 280)$$

represents the total of monthly deposits into the account. Find the sequence of monthly balances in the account.

Solution.   Let B represent the monthly balance. Before any months pass, we have a balance of 1500 dollars. This gives an initial value $B_0 = 1500$. We wish to define the sequence $B = (B_m)_{m=0}^{12}$.

After one month, our account has had $240 withdrawn and $280 deposited. The balance after the end of the month is thus given by

$$B_1 = B_0 - W_1 + D_1 = 1500 - 240 + 280 = 1540.$$

Once we have the balance after one month, we can repeat this process for the other eleven months.

m (month)	B_m (balance in dollars)
0	1500
1	$1500 - 240 + 280 = 1540$
2	$1540 - 300 + 280 = 1520$
3	$1520 - 270 + 280 = 1530$
4	$1530 - 450 + 280 = 1360$
5	$1360 - 250 + 280 = 1390$
6	$1390 - 310 + 280 = 1360$
7	$1360 - 360 + 280 = 1280$
8	$1280 - 270 + 280 = 1290$
9	$1290 - 320 + 280 = 1250$
10	$1250 - 300 + 280 = 1230$
11	$1250 - 350 + 280 = 1160$
12	$1160 - 480 + 280 = 960$

$\square$

When we create a sequence of values based on knowing the increments, we are creating what we call an **accumulation sequence**.

Definition 4.3.2 Given a sequence $x = (x_k)_{k=m}^n$, we say u is an **accumulation sequence** of x if $u = (u_k)_{k=m-1}^n$ with $\nabla u_k = x_k$. $\diamond$

4.3.2 Equivalent Sequences

A given sequence of increments has infinitely many different accumulation sequences which differ in their initial value. However, for a given initial value and sequence of increments, the resulting accumulation sequence is unique. That is, any two sequences that have the same initial value and increments sequences that are equal for all values, then the sequences themselves are equal for all values.

Theorem 4.3.3 Uniqueness Conditions for Accumulation Sequences.
Given two sequences u and w. If $u_m = w_m$ and $\nabla u_k = \nabla w_k$ for all $k > m$, then $u_k = w_k$ for all $k \geq m$.
Proof. In mathematics, to prove that every statement from a sequence of statements is true, we often use an approach called the **Principle of Mathematical Induction**. This requires demonstrating that the first statement in the sequence is true, and then showing that anytime one of the statements is true, the subsequent statement must also be true.

This theorem is perfectly suited to apply mathematical induction. The sequence of statements we wish to prove is

$$u_k = w_k, \quad k = m, m+1, \ldots.$$

The first statement in the sequence, $u_m = w_m$ is true by assumption—one condition is that the sequences u and w have the same initial values. The inductive step is to go from an arbitrary statement in the sequence of statements to the next. So suppose $u_k = w_k$ for some index k in $\{m, m+1, \ldots\}$. We know that $\nabla u_{k+1} = \nabla w_{k+1}$ by the assumption that the sequences have equal increments. We now use substitution twice:

$$u_{k+1} = u_k + \nabla u_{k+1}$$
$$= w_k + \nabla w_{k+1}$$
$$= w_{k+1}.$$

This shows that $u_{k+1} = w_{k+1}$ whenever $u_k = w_k$. By mathematical induction, the entire sequence of statements must be true. $\blacksquare$

Example 4.3.4 Consider the explicitly defined sequence $x = (3k+4)_{k=1}^\infty$ and the sequence $y = (y_n)_{n=1}^\infty$ defined recursively with an initial value $y_1 = 7$ and iteration $y_n = y_{n-1} + 3$ for $n = 2, 3, \ldots$. Show that x and y represent the same sequence.

Solution. To apply Theorem 4.3.3, we need to show that the sequences have the same initial value and the same increments. We just show the two initial values and verify they are the same.

$$x_1 = 3(1) + 4 = 7$$
$$y_1 = 7$$

We next compare the increments. Using the explicit formula for x, we find

$$\begin{aligned}
\nabla x_k &= x_k - x_{k-1} \\
&= (3k+4) - (3(k-1)+4) \\
&= 3k + 4 - (3k - 3 + 4) \\
&= 3.
\end{aligned}$$

Using the recursive formula for y, we find

$$\nabla y_k = y_k - y_{k-1} = 3.$$

The sequences x and y have the same initial value and the same increments. Therefore, they have all the same values: $x_k = y_k$ for all $k = 1, 2, \ldots$. $\square$

Theorem Theorem 4.3.3 can be generalized from having two sequences with equal increments to two sequences sharing any recurrence relation involving the previous term. For example, a geometric sequence has a recurrence relation $x_n = \rho x_{n-1}$, so that the increment using the relation itself depends on the previous term, $\nabla x_n = (\rho - 1)x_{n-1}$.

Theorem 4.3.5 *Suppose u and w are two sequences with common initial values, $u_m = w_m$. If there is a sequence of projection functions f_k so that u and w satisfy the same relations,*

$$u_k = f_k(u_{k-1})$$

and

$$w_k = f_k(w_{k-1}),$$

then $u_k = w_k$ for all $k = m, m+1, \ldots$.

For a recursively defined sequence, the sequence of projection functions would all be the same function.

Example 4.3.6 Consider the explicitly defined sequence $x = (10 \cdot \frac{1}{2^k})_{k=1}^\infty$ and the sequence $y = (y_n)_{n=1}^\infty$ defined recursively with an initial value $y_1 = 5$ and iteration $y_n = \frac{1}{2}y_{n-1}$ for $n = 2, 3, \ldots$. Show that x and y represent the same sequence.

Solution. To apply Theorem 4.3.5, we need to show that the sequences have the same initial value and the satisfy the same recurrence relations. The initial values are:

$$x_1 = 10 \cdot \frac{1}{2^1} = 5,$$
$$y_1 = 5.$$

We next compare the recurrence relations. We know that y has projection

function $f : y_{n-1} \mapsto y_n = \frac{1}{2}y_{n-1}$. We need to show that x satisfies the same relation, $x_k = \frac{1}{2}x_{k-1}$. Using the explicit formula for x, we compute both sides of the recurrence equation and show they are equivalent.

$$x_k = 10 \cdot \frac{1}{2^k}$$

$$\frac{1}{2}x_{k-1} = \frac{1}{2} \cdot 10 \cdot \frac{1}{2^{(k-1)}}$$

$$= 10 \cdot \frac{1}{2} \cdot \frac{1}{2^{(k-1)}}$$

$$= 10 \cdot \frac{1}{2^{(k-1)+1}}$$

$$= 10 \cdot \frac{1}{2^k}$$

Comparing the formulas, we see that $x_k = \frac{1}{2}x_{k-1}$.

The sequences x and y have the same initial value and the same sequence of recurrence relations. Therefore, they have all the same values: $x_k = y_k$ for all $k = 1, 2, \ldots$. $\qquad\square$

We end our discussion of showing two sequences are equivalent by establishing an explicit formula for sequences defined recursively by a *linear* projection function,

$$x_n = \alpha x_{n-1} + c,$$

with $\alpha \neq 1$. When $\alpha = 1$ we have an arithmetic sequence, which is a sequence we already know. When $c = 0$, we have a geometric sequence. The projection function $f : x_{n-1} \mapsto x_n$ is defined by the formula $f(x) = \alpha x + c$. The fixed point x^* is the solution to

$$\alpha x + c = x \qquad \Leftrightarrow \qquad x^* = \frac{c}{1 - \alpha},$$

defined only for $\alpha \neq 1$.

The linear projection function can be rewritten in terms of the fixed point using slope α and point (x^*, x^*) as

$$f(x) = x^* + \alpha(x - x^*).$$

This means that the recurrence relation can be written

$$x_n = x^* + \alpha(x_{n-1} - x^*) \qquad \Leftrightarrow \qquad x_n - x^* = \alpha(x_{n-1} - x^*).$$

Consequently, $x_n - x^*$ is a geometric sequence with ratio α. This allows us to find an explicit formula for x_n.

Theorem 4.3.7 Explicit Formula for Linear Recursive Sequences.
Suppose x_n is defined recursively by the equation

$$x_n = \alpha x_{n-1} + c$$

with $\alpha \neq 1$. Then x_n is defined explicitly by a shifted geometric sequence

$$x_n = x^* + (x_0 - x^*) \cdot \alpha^n = x^* + (x_1 - x^*) \cdot \alpha^{n-1},$$

where $x^* = \dfrac{c}{1 - \alpha}$ is the equilibrium of the sequence.

4.3.3 Summation Notation

In mathematics, the idea of adding terms from a sequence appears so frequently that a special notation, called summation notation or sigma notation for the Greek letter sigma Σ, was created to represent the sum.

Definition 4.3.8 Summation Notation. Given any sequence x and integers $m \leq n$, the sum of terms x_k with index k satisfying $m \leq k \leq n$ is written

$$\sum_{k=m}^{n} x_k = x_m + x_{m+1} + \cdots + x_n.$$

The starting index m is called the **lower limit** of the sum while the ending index n is called the **upper limit**.   ◇

The sequence of terms being added is often given as an explicit function of the index. In that case, the explicit formula is used in place of the sequence name in the summation.

Example 4.3.9 Evaluate the following sums.

1. $\displaystyle\sum_{k=3}^{7}[2k + 3]$

2. $\displaystyle\sum_{k=1}^{4} \frac{1}{k^2 + k}$

Solution.

1. The sum $\displaystyle\sum_{k=3}^{7}[2k + 3]$ involves the increment sequence $a_k = 2k + 3$ and is the sum of terms with index from 3 to 7:

   $$a_3 = 2(3) + 3 = 9, \ a_4 = 2(4) + 3 = 11, \ a_5 = 13, \ a_6 = 15, \ a_7 = 17.$$

   Consequently, we can find the sum

   $$\sum_{k=3}^{7}[2k + 3] = 9 + 11 + 13 + 15 + 17 = 65.$$

2. The sum $\displaystyle\sum_{k=1}^{4} \frac{1}{k^2 + k}$ involves an increment sequence $b_k = \dfrac{1}{k^2 + k}$. The index values involved go from 1 to 4 so that we find

   $$\begin{aligned}
   \sum_{k=1}^{4} \frac{1}{k^2 + k} &= \frac{1}{1^1 + 1} + \frac{1}{2^2 + 2} + \frac{1}{3^2 + 3} + \frac{1}{4^2 + 4}\\
   &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20}\\
   &= \frac{30}{60} + \frac{10}{60} + \frac{5}{60} + \frac{3}{60}\\
   &= \frac{48}{60} = \frac{4}{5}.
   \end{aligned}$$

   □

An accumulation sequence is closely related to summation. The accumula-

tion sequence is a new sequence formed by starting with an initial value and then adding one increment at a time. Suppose $x = (x_k)_{k=1}^\infty$ and u is the corresponding accumulation sequence with initial value u_0. We can write each term of u as the initial value plus a partial sum of the increments.

$$u_1 = u_0 + x_1 = u_0 + \sum_{k=1}^{1} x_k$$

$$u_2 = u_0 + x_1 + x_2 = u_0 + \sum_{k=1}^{2} x_k$$

$$u_3 = u_0 + x_1 + x_2 + x_3 = u_0 + \sum_{k=1}^{3} x_k$$

$$\vdots$$

In general, we have

$$u_n = u_0 + \sum_{k=1}^{n} x_k.$$

Notice how the index for u appears as the upper limit of the summation and that the index of summation is a different variable. The index of summation can be any other unused variable, so that we might have instead written

$$u_n = u_0 + \sum_{i=1}^{n} x_i.$$

Also, notice that for consistency, we require

$$\sum_{k=1}^{0} x_k = 0,$$

regardless of the sequence x to indicate that no terms have been added in the summation. In general, we have the following representation.

Theorem 4.3.10 *If* $x = (x_k)_{k=m}^n$ *and* u *is the accumulation sequence with initial value* u_{m-1}, *then we can write*

$$u_k = u_{m-1} + \sum_{i=m}^{k} x_i,$$

for $i = m, \ldots, n$.

Example 4.3.11 Write the accumulation sequence $z = (z_n)_{n=0}^\infty$ with initial value $z_0 = 4$ and an increment sequence $a = (3, 5, 7, 9, 11, 13, \ldots)$ as a summation with an explicit formula for the increments.

Solution. The sequence z has initial value 4 which corresponds to index 0,

$$z_0 = 4.$$

For index values $n > 0$, the sequence is computed with an accumulation of values from the sequence a.

$$z_1 = 4 + \sum_{k=1}^{1} a_k = 4 + 3 = 7$$

$$z_2 = 4 + \sum_{k=1}^{2} a_k = 4 + 3 + 5 = 12$$

$$z_3 = 4 + \sum_{k=1}^{3} a_k = 4 + 3 + 5 + 7 = 19$$

$$z_4 = 4 + \sum_{k=1}^{4} a_k = 4 + 3 + 5 + 7 + 9 = 28$$

We need an explicit formula for the sequence $k \mapsto a_k$. We recognize that a is an arithmetic sequence with $a_1 = 3$ and constant increment $\nabla a_k = 2$. By Theorem 13.2.8, we know $a_k = a_1 + 2(k-1) = 2k + 1$. Using this explicit formula in the summation, we find

$$z_n = 4 + \sum_{k=1}^{n}(2k + 1).$$

$\square$

Example 4.3.12 Show that $\displaystyle\sum_{k=1}^{n}(2k - 1) = n^2$ for $n = 0, 1, \ldots$.

Solution.   There are two distinct sequences appearing in the equation— the sequence defined by the accumulation,

$$u_n = \sum_{k=1}^{n}(2k - 1),$$

and the sequence defined by an explicit formula. As u_n includes only a summation, we must have a zero initial value, $u_0 = 0$.

Because we know that $x_k = 2k - 1$ is the increment sequence for u, we only need to show that w has the same initial value and increment sequence. The initial value of w is

$$w_0 = 0^2 = 0,$$

in agreement with that of $u_0 = 0$. The increment is computed using the backward difference.

$$\begin{aligned}
\nabla w_n &= w_n - w_{n-1} \\
&= n^2 - (n-1)^2 \\
&= n^2 - (n^2 - 2n + 1) \\
&= n^2 - n^2 + 2n - 1 \\
&= 2n - 1
\end{aligned}$$

The explicit formula for the increment of w is the same as that for u. Consequently, we know that $u_n = w_n$ for all $n = 0, 1, 2, \ldots$. $\square$

4.3.4 Summary

- An accumulation sequence is a sequence generated from an initial value and a given sequence of increments.

- If x is the sequence of increments and u is the accumulation sequence,

then u satisfies the recurrence relation

$$u_n = u_{n-1} + x_n.$$

- If two sequences share the same initial value and the same increments, then the sequences are identical (Theorem 4.3.3). More generally, if two sequences share the same initial value and sequence of recurrence relations involving the previous term, then the sequences are identical (Theorem 4.3.5).

- Summation notation (sigma notation) provides a method to communicate the sequence of increments as well as the range of index values. The index variable is sometimes called the dummy variable because any other variable could be used in its place.

- Every accumulation sequence u can be represented as the initial value added to the summation of its increments x_k with the index variable appearing as the upper limit,

$$u_n = u_m + \sum_{k=m+1}^{n} x_k.$$

4.3.5 Exercises

Find the first six terms of the indicated accumulation sequence for the given increment sequence. Clearly indicate the relevant index values.

1. Find u with increments defined by $x = (x_k)_{k=2}^{\infty} = (-4, -2, 0, 2, 4, 6, \ldots)$ and initial value 21.

2. Find w with increments defined by $y = (y_i)_{i=0}^{\infty} = (1, -1, 1, 1, -1, -1, 1, 1, 1, -1, \ldots)$ and initial value 0.

In the following problems, show that explicit definition and recursive definition define the same sequence. If not, explain why.

3. $x_n = 3n - 5$ for $n = -2, -1, 0, \ldots$ defines the same sequence as $x_n = x_{n-1} + 3$ with $x_{-2} = -11$.

4. $x_n = 4 + \frac{3^{(n+1)}}{4^n}$ for $n = 0, 1, 2, \ldots$ defines the same sequence as $x_n = \frac{3}{4}x_{n-1} + 1$ with $x_0 = 7$.

5. $x_n = 2^{n+1} - 1$ for $n = 0, 1, 2, \ldots$ defines the same sequence as $x_n = x_{n-1} + 2^n$ with $x_0 = 1$.

6. $x_n = n^2 - n$ for $n = 0, 1, 2, \ldots$ defines the same sequence as $x_n = x_{n-1} + n$ with $x_0 = 0$.

7. $x_n = \frac{1}{2}(3^n - 1)$ for $n = 0, 1, 2, \ldots$ defines the same sequence as $x_{n+1} = x_n + 3^n$ with $x_0 = 0$. Note: The recursive formula uses a forward recurrence, so either compare forward differences or rewrite the recursive equation as a backward recurrence.

Determine the intervals of monotonicity and concavity for each sequence defined by the given increments.

8. $\nabla x_n = 4n - 70$ for $n = 1, 2, \ldots$ with $x_0 = 20$.

9. $\nabla x_n = 50 - 3n$ for $n = 1, 2, \ldots$ with $x_0 = -10$.

10. $\nabla x_n = n^2 - 30n$ for $n = 1, 2, \ldots$ with $x_0 = 0$.

11. $\nabla x_n = -100 + 75n - n^2$ for $n = 1, 2, \ldots$ with $x_0 = 0$.

For each of the following summations, write down the sum of individual terms. Then compute the value of the sum. For example, $\sum_{k=2}^{5} 2k$ would be $2(2) + 2(3) + 2(4) + 2(5) = 4 + 6 + 8 + 10 = 28$.

12. $\displaystyle\sum_{k=12}^{15} 3k$

13. $\displaystyle\sum_{k=-2}^{2} 2^k$

14. $\displaystyle\sum_{k=2}^{5} \frac{2k+1}{5k}$

Rewrite the following sums in summation notation. Find an appropriate formula for the increment sequence and identify the correct lower and upper limits of the sum.

15. $15 + 20 + 25 + 30 + \cdots + 90$

16. $21 + 25 + 29 + 33 + \cdots + 61$

17. $\frac{1}{4} + \frac{2}{9} + \frac{3}{16} + \cdots + \frac{12}{169}$

18. The sum of all four digit odd numbers.

19. The sum of all two-digit squares, $16 + 25 + 36 + 49 + 64 + 81$.

20. The sum of all three-digit odd squares.

Show that the summation formulas below are valid for $n = 0, 1, 2, \ldots$ by showing that two sequences are equal to one another.

21. $\displaystyle\sum_{k=1}^{n} (2k) = n^2 + n$.

22. $\displaystyle\sum_{k=1}^{n} (4k - 3) = n(2n - 1)$.

23. $\displaystyle\sum_{k=1}^{n} (6k^2 - 2k) = 2n^2(n + 1)$.