4.6 Calculating Sequence Limits

4.6.1 Overview

In the previous section, we learned about limits of sequences. Unfortunately, using a table of values to find a limit only allows us to estimate its value. A finite table alone can never clearly show whether a perceived pattern will continue or change after the values shown. It would be helpful to have some rules for finding limits based on the formula rather than numerical patterns.

This section establishes the rules of limits of sequences. Many limits can be calculated by identifying terms that are unbounded in the limit. We learn about how infinity behaves in the context of limit arithmetic. We also learn about indeterminate limit forms.

4.6.2 Infinite Limits

In order to compute limits of sequences, we begin with sequences that grow without bound, which is written $x_n \to \infty$ when the sequence grows in a positive direction or $x_n \to -\infty$ when the sequence grows in a negative direction. Arithmetic sequences with increments $\beta \neq 0$ (recall Theorem 13.2.8) must either steadily increase (positive increments $\beta > 0$) or steadily decrease (negative increments $\beta < 0$). The special case that $\beta = 0$ is somewhat boring, as this corresponds to a constant sequence so that the limit is just the constant value.

Theorem 4.6.1 Limit of an Arithmetic Sequence. *An arithmetic sequence with explicit formula $x_n = a + c \cdot n$ (for constants a and c) has unbounded growth when $c \neq 0$. The corresponding limit statements are*

$$\begin{aligned} \lim_{n\to\infty} (a + cn) &= +\infty \quad (c > 0) \\ \lim_{n\to\infty} (a + cn) &= -\infty \quad (c < 0) \\ \lim_{n\to\infty} (a) &= a \quad (c = 0) \end{aligned}$$

Geometric sequences are a little more complicated, depending on the ratio ρ (recall Theorem 13.2.10) and the initial value. Repeated multiplication by a number whose magnitude is larger than 1 makes the resulting magnitude increase without bound. Repeated multiplication by a number whose magnitude is smaller than 1 makes the resulting magnitude converge to 0. If the ratio ρ is negative, then the sign of the sequence values will alternate between positive and negative. This is summarized by another theorem.

Theorem 4.6.2 Limit of a Geometric Sequence. *A geometric sequence with explicit formula $x_n = a \cdot \rho^n$ and ratio ρ is unbounded when $|\rho| > 1$, meaning that $|x_n| \to \infty$.*

- *If $\rho \leq -1$, x_n alternates sign and the limit does not exist.*
- *If $\rho > 1$, then the limit depends on the sign of a:*

$$\begin{aligned} \lim_{n\to\infty} a \cdot \rho^n &= +\infty, \quad (a > 0), \\ \lim_{n\to\infty} a \cdot \rho^n &= -\infty, \quad (a < 0). \end{aligned}$$

- *If $|\rho| < 1$ (i.e., $-1 < \rho < 1$), then $\lim_{n\to\infty} a \cdot \rho^n = 0$.*

Example 4.6.3 Find the appropriate limits of the following sequences.

1. $\lim_{n\to\infty} 3 - 4n$
2. $\lim_{n\to\infty} 100 + 0.02n$
3. $\lim_{n\to\infty} -3 \cdot 1.05^n$
4. $\lim_{n\to\infty} 100 \cdot (-0.75)^n$
5. $\lim_{n\to\infty} 5 \cdot (-1.5)^n$

Solution.

1. The sequence $x_n = 3 - 4n$ is recognized as the explicit formula of an arithmetic sequence with increment $c = -4$. Since this is a negative increment, the sequence decreases without bound. So we write
$$\lim_{n\to\infty} 3 - 4n = -\infty.$$
2. The sequence $x_n = 100 + 0.02n$ is arithmetic with increment $c = 0.02$. Since the increment is positive, the sequence increases without bound and we write
$$\lim_{n\to\infty} 100 + 0.02n = +\infty.$$
3. The sequence $x_n = -3 \cdot 1.05^n$ is a geometric sequence with ratio $\rho = 1.05$. Because $\rho > 1$, the sequence grows without bound. Furthermore, because the terms are all negative, we have a limit
$$\lim_{n\to\infty} -3 \cdot 1.05^n = -\infty.$$
4. The sequence $x_n = 100 \cdot (-0.75)^n$ is a geometric sequence with ratio $\rho = -0.75$. Because the ratio is negative, the signs of the terms alternate between positive and negative. However, since $|\rho| = 0.75 < 1$, the magnitude of the terms converges to zero so that
$$\lim_{n\to\infty} 100 \cdot (-0.75)^n = 0.$$
5. The sequence $x_n = 5 \cdot (-1.5)^n$ has a negative ratio $\rho = -1.5$. Since $|\rho| > 1$, the terms have alternating signs but grow in magnitude. Consequently, $\lim_{n\to\infty} 5 \cdot (-1.5)^n$ does not exist.

□

4.6.3 Arithmetic of Infinity

Once we know how to identify when sequences have unbounded terms, we can use that information to find limits of related sequences. We can think of this as the arithmetic of infinity. Infinities can add and multiply but should never be subtracted or divided from one another. The signs of arithmetic involving infinity behave like for numbers, such as having a negative times a positive be negative.

The most important principle to remember is that infinities should never cancel one another. Cases where the formula looks like infinities *might* cancel are called indeterminate. This includes trying to cancel infinity by multiplying by zero. An indeterminate limit form means that the value of the limit *can not be determined* without further analysis that resolves the apparent cancellation.

Theorem 4.6.4 Arithmetic Rules for Infinity. *Suppose unbounded sequences are combined using arithmetic operations. Then the following arithmetic relating to limits will be valid, where c will represent a positive number.*

- *Adding a number to infinity has no effect:*

$$+\infty \pm c = +\infty$$
$$-\infty \pm c = -\infty$$

- *Multiplying infinity by a non-zero number is still infinite, but changes sign if multiply by a negative number:*

$$c \cdot \pm\infty = \pm\infty$$
$$-c \cdot \pm\infty = \mp\infty$$

- *Adding infinities of the same sign are infinite. Don't cancel opposite infinities.*

$$+\infty + +\infty = +\infty$$
$$-\infty + -\infty = -\infty$$
$$+\infty + -\infty = \textit{indeterminate}$$

- *Multiplying infinities are infinite, and negative if opposite signs.*

$$+\infty \cdot +\infty = +\infty$$
$$-\infty \cdot -\infty = +\infty$$
$$+\infty \cdot -\infty = -\infty$$

- *The reciprocal of infinity is zero, but they can't cancel.*

$$\frac{1}{\pm\infty} = 0$$
$$0 \cdot \pm\infty = \textit{indeterminate}$$
$$\frac{\pm\infty}{\pm\infty} = \textit{indeterminate}$$
$$\frac{0}{0} = \textit{indeterminate}$$

The previous theorem was stated somewhat imprecisely in order to convey the idea of arithmetic of infinities without being bogged down by formal notation relating to limits. Each statement really is about a limit.

As an example, the arithmetic on infinity $+\infty + +\infty = +\infty$ would more carefully be stated as follows. Suppose that there are two sequences x_n and y_n such that $x_n \to +\infty$ and $y_n \to +\infty$. The sequence defined by $u_n = x_n + y_n$ has limit

$$\lim_{n\to\infty} x_n + y_n = +\infty.$$

The shorthand notation of performing arithmetic with infinity allows this to be simplified as writing

$$\lim_{n\to\infty} x_n + y_n = +\infty + +\infty = +\infty.$$

The intermediate step $+\infty + +\infty$ is not truly arithmetic, but points out that $x_n \to +\infty$ and $y_n \to +\infty$, and since those sequences were added, the final limit is also $+\infty$. We are substituting limits of individual terms into the formula defining the expression. As long as the arithmetic involves no cancellation of infinities, it will result in a correct statement.

To deal with indeterminate forms, we usually need to try to rewrite the formula in a new way so that the cancellation is avoided. The most common approach for rewriting is to factor out a **dominant term**. When there are infinities trying to cancel, we identify which of the terms should dominate and we factor that expression from both terms and simplify.

Example 4.6.5 Determine the following limits, if possible.

1. $\displaystyle\lim_{n\to\infty} 3 + \frac{5}{2^n}$
2. $\displaystyle\lim_{n\to\infty} \frac{n^2 - 3n}{5n - 1}$
3. $\displaystyle\lim_{n\to\infty} \frac{1 + 2^n}{3 + 5^n}$

Solution.

1. The sequence $x_n = 3 + \frac{5}{2^n}$ is the sum of terms 3 and $\frac{5}{2^n}$. The constant sequence has a limit $3 \to 3$ (since it never changes). The geometric sequence $\frac{5}{2^n} = 5 \cdot (\frac{1}{2})^n$ has a ratio $|\rho| < 1$ so that $\frac{5}{2^n} \to 0$. The form of the limit (using the terms) is
$$\lim_{n\to\infty} 3 + \frac{5}{2^n} = 3 + 0,$$
and since this does not involve any cancelation of infinities, will give the correct limit,
$$\lim_{n\to\infty} 3 + \frac{5}{2^n} = 3 + 0 = 3.$$
2. The sequence $u_n = \dfrac{n^2 - 3n}{5n - 1}$ is a quotient of terms $n^2 - 3n$ and $5n - 1$. To find the limit, we explore the terms individually first.

   Because $n^2 = n \cdot n$, we know $n^2 \to +\infty \cdot +\infty = +\infty$. Similarly, the arithmetic sequence $3n \to +\infty$. However, the difference $n^2 - 3n$ would have a limit of the form $+\infty - \infty$, which is a cancellation of infinities. As written, $n^2 - 3n$ is an indeterminate form.

   Our strategy will be to rewrite this as a product, and the best practice is to factor out (divide out) the greatest power of n (dominant term),
$$n^2 - 3n = n^2(\frac{n^2}{n^2} - \frac{3n}{n^2}) = n^2(1 - \frac{3}{n}).$$
   From this, we find
$$\frac{3}{n} \to \frac{3}{+\infty} = 0 \quad \Rightarrow \quad 1 - \frac{3}{n} \to 1 - 0 = 1.$$
   Since we already know $n^2 \to +\infty$, we have the limit of the numerator
$$\lim_{n\to\infty} n^2 - 3n = \lim_{n\to\infty} n^2(1 - \frac{3}{n}) = +\infty \cdot 1 = +\infty.$$

The term in the denominator $5n - 1$ is an arithmetic sequence (linear function) with increment (slope) $c = 5$. We know

$$\lim_{n\to\infty} 5n - 1 = +\infty.$$

Unfortunately, that means our limit form as a quotient is itself an indeterminate form,

$$\lim_{n\to\infty} \frac{n^2 - 3n}{5n - 1} = \frac{+\infty}{+\infty}.$$

We can not cancel infinities, so we must rewrite our formula.

For this example, I worked out the numerator separately to make a point about that term itself being an indeterminate form. In practice, our strategy will be to apply that factoring principle to the entire formula at one step. This is illustrated below.

The problem can be solved up if we just factor out from the numerator and denominator the dominant term (greatest power) and simplify as needed.

$$\begin{aligned} \lim_{n\to\infty} \frac{n^2 - 3n}{5n - 1} &= \lim_{n\to\infty} \frac{n^2(1 - \frac{3}{n})}{n(5 - \frac{1}{n})} \\ &= \lim_{n\to\infty} \frac{n(1 - \frac{3}{n})}{5 - \frac{1}{n}} \\ &= \frac{+\infty \cdot (1 - \frac{3}{\infty})}{5 - \frac{1}{\infty}} \\ &= \frac{+\infty \cdot 1}{5} = +\infty. \end{aligned}$$

3. The sequence $w_n = \dfrac{1 + 2^n}{3 + 5^n}$, by quick inspection, involves the geometric sequences 2^n and 5^n, both of which grow exponentially so that $w_n \to \frac{+\infty}{+\infty}$. This indeterminate form involves canceling infinities, so we must rewrite the formula. Following the method of the previous example, we factor out the dominant term, in this case the geometrically growing powers.

$$\begin{aligned} \lim_{n\to\infty} \frac{1 + 2^n}{3 + 5^n} &= \lim_{n\to\infty} \frac{2^n(\frac{1}{2^n} + \frac{2^n}{2^n})}{5^n(\frac{3}{5^n} + \frac{5^n}{5^n})} \\ &= \lim_{n\to\infty} \frac{2^n(\frac{1}{2^n} + 1)}{5^n(\frac{3}{5^n} + 1)} \end{aligned}$$

This is still indeterminate form $\frac{+\infty}{+\infty}$, so we rewrite $\dfrac{2^n}{5^n} = (\frac{2}{5})^n$, which is a geometric sequence with ratio $\rho = \frac{2}{5}$ satisfying $|\rho| < 1$.

$$\begin{aligned} \lim_{n\to\infty} \frac{1 + 2^n}{3 + 5^n} &= \lim_{n\to\infty} \frac{2^n(\frac{1}{2^n} + 1)}{5^n(\frac{3}{5^n} + 1)} \\ &= \lim_{n\to\infty} \frac{(\frac{2}{5})^n(\frac{1}{2^n} + 1)}{\frac{3}{5^n} + 1} \\ &= \frac{0 \cdot (0 + 1)}{0 + 1} = \frac{0}{1} = 0 \end{aligned}$$

□

Knowing how to find limits of sequences with explicit formulas, we can also find limits for recursive sequences whose explicit formulas are known.

Example 4.6.6 Find the limit of a recursive sequence u defined by

$$u_n = 0.8u_{n-1} + 20$$

and initial value $u_0 = 50$.

Solution. A sequence that has a linear projection function, involving both multiplication by a ratio and the addition of an increment, has a shifted geometric sequence as its explicit formula. The equilibrium value is found by solving the fixed point equation.

$$\begin{aligned} 0.8x + 20 &= x \\ 20 &= 0.2x \\ 100 &= x \end{aligned}$$

Thus, the equilibrium value is $u^* = 100$.

The explicit formula is a geometric sequence with ratio $\alpha = 0.8$ shifted by the equilibrium,

$$\begin{aligned} u_n &= u^* + (u_0 - u^*)(0.8)^n \\ &= 100 + (50 - 100)(0.8)^n \\ &= 100 + -50(0.8)^n \end{aligned}$$

Be careful not to violate the order of operations by adding the 100 and -50 or multiplying the -50 and 0.8,

Using this explicit formula, we can find the limit of the sequence. The geometric sequence has a limit 0 because the ratio has magnitude smaller than 1.

$$\lim_{n\to\infty} u_n = 100 + (-50)(0) = 100$$

That is, the sequence converges to the equilibrium value. □

Example 4.6.7 Find the limit of the sequence defined by the recursive equation

$$x_{n+1} = 1.05x_n - 20$$

and initial value $x_0 = 300$.

Solution. Find the fixed point by solving the equation $x = 1.05x - 20$.

$$\begin{aligned} x &= 1.05x - 20 \\ -0.05x &= -20 \\ x &= 400 \end{aligned}$$

Using the fixed point $x^* = 400$ and the growth factor $\alpha = 1.05$, we can write down the explicit formula,

$$\begin{aligned} x_n &= x^* + (x_0 - x^*)\alpha^n \\ &= 400 + (300 - 400)1.05^n \\ &= 400 - 100 \cdot 1.05^n \end{aligned}$$

The geometric term with ratio $\alpha = 1.05$ grows without bound. The limit of the sequence can be found:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 400 - 100 \cdot 1.05^n$$

$$= 400 - 100 \cdot +\infty$$
$$= 400 - \infty$$
$$= -\infty.$$

The sequence will decrease without bound. □

4.6.4 Limit Applications to Modeling

With these tools, we can analyze sequences associated with physically meaningful models. Limits tell us about the long-term behavior. As time progresses, the sequence will progressively get closer and closer to the limit.

For example, the concentration of a drug in a patient taking repeated doses can be modeled by a sequence. A limit of this sequence can tell us something about what will happen to that concentration if the dosing continues for an extended amount of time.

Example 4.6.8 Suppose a patient begins taking 500 mg of a drug every four hours. However, the body metabolizes 60% of the drug in the body every four hours. Find a formula for the amount of drug in the body immediately after each dose and then determine the limiting value.

Solution. The patient's body starts with no drug. Immediately after the first dose, there are 500 mg. Four hours later, 60% has been removed and then another dose is added in. If we let D_n be the sequence of drug mass in the body as a function of the number of doses n, then this is modeled recursively by the equation

$$D_{n+1} = D_n - 0.6D_n + 500,$$

with initial value $D_1 = 500$.

This model has a linear projection function $f(x) = 0.4x + 500$ and corresponding fixed point

$$0.4x + 500 = x \quad \Leftrightarrow \quad x = \frac{500}{0.6} = \frac{2500}{3}.$$

The explicit formula, using Theorem 4.3.7, is given by

$$D_n = \frac{2500}{3} + (500 - \frac{2500}{3}) \cdot 0.4^{n-1} = \frac{2500}{3} - \frac{1000}{3} \cdot 0.4^{n-1}.$$

Because the slope of the projection function $\alpha = 0.4$ has magnitude less than 1, the limiting value is the fixed point $x^* = \frac{2500}{3} \approx 833.33$. Thus, if the patient continues to take the drug, the amount in the body immediately after each dose will be approximately 833.33 mg. (Immediately before the dose, it must have been approximately 333.33 mg.) □

In the coming chapters, we will learn about the definite integral. The calculation of an integral is as a limit of a summation. The following example illustrates how such calculations are performed.

Example 4.6.9 Find $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \left(1 + \frac{3k}{n}\right) \cdot \frac{3}{n}$.

Solution. There are two major steps needed for this problem: (i) find an explicit formula for the summation that depends only on n and (ii) compute the limit of that explicit formula.

To compute the explicit formula for the summation, we need to remember that the variable n is a constant with respect to the summation index k. We will rewrite the formula of the sequence in summation to be a sum so that we can use the linearity property.

$$\begin{aligned}
\sum_{k=1}^{n}\left(1+\frac{3k}{n}\right)\cdot\frac{3}{n} &= \sum_{k=1}^{n}\frac{3}{n}+\frac{9k}{n^2} \\
&= \frac{3}{n}\cdot\sum_{k=1}^{n}1+\frac{9}{n^2}\cdot\sum_{k=1}^{n}k \\
&= \frac{3}{n}\cdot n+\frac{9}{n^2}\cdot\frac{n(n+1)}{2} \\
&= 3+\frac{9(n+1)}{2n} \\
&= 3+\frac{9n+9}{2n}
\end{aligned}$$

We see that the summation is itself a sequence involving an index n. We find the limit of that sequence. The fraction $\dfrac{9n+9}{2n}$ will be indeterminate form $\frac{\infty}{\infty}$, so we will factor out the dominant term of n from top and bottom.

$$\begin{aligned}
\lim_{n\to\infty}\sum_{k=1}^{n}\left(1+\frac{3k}{n}\right)\cdot\frac{3}{n} &= \lim_{n\to\infty} 3+\frac{9n+9}{2n} \\
&= \lim_{n\to\infty} 3+\frac{n(9+\frac{9}{n})}{2n} \\
&= \lim_{n\to\infty} 3+\frac{9}{2}+\frac{9}{2n} \\
&= 3+\frac{9}{2}+\frac{9}{\infty} \\
&= 3+\frac{9}{2}+0 \\
&= \frac{15}{2} = 7\frac{1}{2}
\end{aligned}$$

□

4.6.5 Summary

- Arithmetic sequences with non-zero increments are unbounded. Geometric sequences are unbounded when the ratio has magnitude greater than 1 and converge to zero when the ratio has magnitude less than 1.
- Sequence limits obey the standard rules of arithmetic, including for infinite limits, with the exception that any formula that would cancel infinity are indeterminate. (See Theorem 4.6.4) Indeterminate means the limit can not be determined from simple arithmetic but requires rewriting in another form.
  - $\infty-\infty$: rewrite by factoring out dominant term
  - $\dfrac{\infty}{\infty}$: factor out dominant term in numerator and denominator, look to simplify
  - $\dfrac{0}{0}$: try to factor and simplify
  - $0\cdot\infty$: use negative powers to rewrite as fraction, then treat as $\frac{0}{0}$ or $\frac{\infty}{\infty}$

  Look for terms in the limit that vanish: $\dfrac{1}{\infty}=0$.

4.6.6 Exercises

Use limit arithmetic to find the exact value for each limit. If the limit does not exist, explain why.

1. $\lim_{n\to\infty} 3 - 4n$

2. $\lim_{n\to\infty} -5 + \dfrac{n}{200}$

3. $\lim_{n\to\infty} 3 \cdot (\frac{2}{5})^n$

4. $\lim_{n\to\infty} -3 \cdot (-0.8)^n$

5. $\lim_{n\to\infty} 4 - 5 \cdot (1.05)^n$

6. $\lim_{n\to\infty} 5 - (-1.5)^n$

7. $\lim_{n\to\infty} \dfrac{5 + 3n}{5 - n^2}$

8. $\lim_{n\to\infty} \dfrac{5 + 3n^2}{5 - n^2}$

9. $\lim_{n\to\infty} \dfrac{5 + 3n^3}{5 - n^2}$

10. $\lim_{n\to\infty} \dfrac{2^n - 3^n}{7 + 2 \cdot 3^n}$

11. $\lim_{n\to\infty} \dfrac{3 \cdot 2^n - 5 \cdot 3^n}{1 + 2 \cdot 5^n}$

Find an explicit formula for each sequence. Then determine the limit of the sequence.

12. $x_t = 1.05x_{t-1} - 10$ with $x_0 = 500$.

13. $x_{k+1} = 0.8x_k + 12$ with $x_1 = 5$.

14. $y_n = -1.5y_{n-1} + 5$ with $y_0 = 2$.

Use the properties and elementary formulas for summation to find a formula for the summation in order to compute the limit.

15. $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{5k}{n^2}$

16. $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{k^2}{n^3}$

17. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(2 + \dfrac{3k}{n}\right) \cdot \dfrac{3}{n}$

18. $\lim_{n\to\infty} \sum_{k=1}^{n} \left(1 + \dfrac{2k}{n}\right)^2 \cdot \dfrac{2}{n}$

Applications of sequences.

19. When a patient starts taking a drug, there is no drug in the blood. The prescription is to take 250 mg every six hours. Then, during the six hours between doses, the body metabolizes 20% of whatever drug is in the body.

Find a recursive formula using a linear projection function for the sequence of drug levels with an index counting the total number of doses. Clearly state the initial value. Find an explicit formula for the sequence and determine the limit. Is there a steady state amount of drug in the body?

20. A pond has been polluted by a stream. The pond holds 12,000 gallons of water and the stream replaces 900 gallons per day. The stream has been carrying the chemical pollutant at a concentration of $0.4\,\frac{\mathrm{g}}{\mathrm{gal}}$ and now the pond has that same pollution level. With stream cleanup, the chemical pollutant in the stream has just been reduced to a concentration of $0.1\,\frac{\mathrm{g}}{\mathrm{gal}}$.

Find a recursive formula using a linear projection function for the sequence of pollutant levels (total mass) in the pond with an index counting the number of days since the cleanup occurred. Clearly state the initial value. Find an explicit formula for the sequence and determine the limit. On what day will the total amount of pollutant in the pond be reduced to half of the pollution level prior to cleanup? Is there a steady state amount of pollutant in the pond?