5.1 An Overview of Calculus

The previous chapter 13 studied sequences. There are three major concepts in calculus that we can use sequences to motivate. These are limits, derivatives, and integrals.

In this chapter, we will focus on the idea of the definite integral as a generalization of accumulating increments of change. Thinking of sequences in terms of their increments of change is simpler because the domain consists only of integers which are equally spaced. More general functions are defined with domains consisting of intervals of the real numbers. (Some functions can be defined on even more complex sets, which gives rise to even more advanced mathematics.) Consequently, we can not think only in terms of increments of change but in terms of a **rate of change**.

5.1.1 Derivatives and Integrals

For sequences, we learned to think of complementary ideas of accumulation sequences and increments. With a sequence x, we had a forward difference

$$\Delta x_n = x_{n+1} - x_n$$

and a backward difference

$$\nabla x_n = x_n - x_{n-1}.$$

These differences measure the change in the sequence x for consecutive values of the index, which plays the role of the independent variable.

For functions defined on intervals, there is no meaning to *consecutive* values of the independent variable. Near a point of interest $x = c$, there are infinitely many other values close to c. Consequently, when measuring the change of a function Δf, we must also specify the change in the independent variable Δx. Consider two values for the independent variable, say $x = a$ and $x = b$, and we define $\Delta x = b - a$ and $\Delta f = f(b) - f(a)$.

Different increments Δx will usually result in different function increments Δf. However, for many functions, the ratio $\Delta f/\Delta x$, called the **average rate of change**, has a limit as $\Delta x \to 0$. This limiting rate of change is called the **instantaneous rate of change** and in calculus is named the **derivative**.

Definition 5.1.1 Instantaneous Rate of Change. Given a function f that relates variables $x \overset{f}{\mapsto} y$, the **instantaneous rate of change** of y with respect to x is the **derivative** $\dfrac{dy}{dx}$ defined by

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

if the limit exists. Consequently, for sufficiently small increments Δx, we have

$$\Delta f \approx \frac{dy}{dx} \cdot \Delta x.$$

◊

The following example illustrates the role of the instantaneous rate of change to relate the increments of the independent variable with the increments of the dependent variable.

Example 5.1.2 A ball dropped from a tower has a height h (measured in feet) modeled as a function of time t (measured in seconds) given by

$$t \overset{f}{\mapsto} h = 40 - 16t^2.$$

At $t = 1$, the instantaneous rate of change is $\dfrac{dh}{dt} = -32$.

This rate of change is illustrated in the dynamic figure below. Thinking of $t_0 = 1$ as one value of the independent variable, you can adjust the second value t_1 to establish the increment $\Delta t = t_1 - t_0$. The function automatically computes $f(1)$ and $f(t_1)$ and shows $\Delta h = f(t_1) - f(1)$. The ratio $\Delta h/\Delta t$ will be close to $-32\,\frac{\text{ft}}{\text{s}}$ for small values of Δt.

A deprecated JSXGraph interactive demonstration goes here in interactive output.

Figure 5.1.3

We can recover this instantaneous rate of change using limits, as shown in the solution below.

Solution. We know that $\Delta t = t_1 - 1$ and $\Delta h = f(t_1) - f(1)$. Using the formula, this gives

$$\Delta h = (40 - 16t_1^2) - (40 - 16(1)^2) = -16(t_1^2 - 1).$$

The average rate of change is defined by the quotient

$$\frac{\Delta h}{\Delta t} = \frac{-16(t_1^2 - 1}{t_1 - 1},$$

which has a value for all $t_1 \neq 1$.

The instantaneous rate of change is the limit of the average rate of change as $\Delta t \to 0$, which in this case requires $t_1 \to 1$. Even though the quotient is not defined at $t_1 = 1$, we can simplify the formula used on the sides to a formula that is defined.

$$\begin{aligned}\frac{dh}{dt} = \lim_{\Delta t \to 0} \frac{\Delta h}{\Delta t} &= \lim_{t_1 \to 1} \frac{-16(t_1^2 - 1)}{t_1 - 1}\\ &= \lim_{t_1 \to 1} \frac{-16(t_1 + 1)(t_1 - 1)}{t_1 - 1}\\ &= \lim_{t_1 \to 1} -16(t_1 + 1)\\ &= -16(1 + 1) = -32\end{aligned}$$

The key step in this limit calculation was changing the limit expression from one in which the formula is not continuous to a new formula. When the formula is continuous, we can just evaluate it at the point of interest. The variable t_1 used in the limit could have been chosen to be any convenient name. □

At this point, our emphasis is understanding that the rate of change or derivative measures the limiting ratio for increments of change in the value of the function to corresponding increments of change in the independent variable. Not every function has a derivative. We will study the calculation of the derivative in more depth in later chapters.

Computing a derivative for a given function is analogous to computing the increments of a sequence. The complementary calculation for sequences is to compute the accumulation sequence for given increments. That is, if $x = (x_n)_{n=1}^{\infty}$ is a sequence of increments, then the accumulation sequence u

with increments $\nabla u_n = u_n - u_{n-1} = x_n$ and initial value u_0 was written

$$u_n = u_0 + \sum_{k=1}^{n} x_k.$$

The calculus analogue is to be given a function that represents a rate of change and use it to find a new function, the **accumulation function**, that has that rate of change as its derivative. Suppose $f(x)$ is the rate of change or derivative of a quantity Q with respect to x,

$$\frac{dQ}{dx} = f(x).$$

We are then interested in finding Q as a function of x if we know an initial value $Q(x_0) = Q_0$. The analogue of summation of increments is the **definite integral**, and we will write

$$Q = Q_0 + \int_{x_0}^{x} f(z)\,dz.$$

The rest of this chapter is focused on bringing meaning to the idea of the definite integral. We study definite integrals before derivatives because we have just studied summation and sequences. The calculations involved in developing the ideas of definite integrals apply these concepts. Ultimately, the Fundamental Theorem of Calculus will provide a connection between the definite integral and the derivative, showing that our two ideas of rate of change represent the same thing.

5.1.2 A Technological Aside

Computational tools play an important role in the real-world application of mathematics. It is increasingly common to have a tool perform actual computations with the user responsible to formulate the appropriate problem.

For example, you may have heard of the website WolframAlpha. This site acts like a search engine for mathematical content, and you can enter queries like “`factor x^2+3x`”. The ability extends to calculus tools as well. We might have asked for our earlier example “`derivative of 40-16t^2 at t=1`”.

Disadvantages of a site like WolframAlpha is that you are limited to a single query at a time and it can sometimes be hard to state precisely what you want. More powerful tools are available, including advanced programmable calculators and commercial software tools like Wolfram’s Mathematica and MapleSoft’s Maple programs.

A free, but similarly powerful tool is SageMath. A calculation in SageMath uses a **script** based on the Python programming language. Comments in the scripts follow the # symbol and are ignored by the computer but are useful to understand what is happening.

Example 5.1.4 To factor the formula $x^2 + 3x$, we would use the following script.

```
# Tell Sage that x is a variable
var("x")
# Ask Sage to factor. Include the multiplication *
factor(x^2+3*x)
```

```
(x+3)*x
```

□

Example 5.1.5 To find the derivative of $40 - 16t^2$ at $t = 1$, we would use the following script.

```
# Tell Sage that t is an independent variable
var("t")
# Define h as function of t
# -- Notice how every operation must be typed
h(t) = 40-16*t^2
show(h(t))
# The derivative is also a function
# but let Sage figure it out using the derivative operation.
Dh(t) = derivative(h(t), t)
show(Dh(t))
# Find the value of the derivative at t=1
Dh(1)
```

```
-16*t^2+40
-32*t
-32
```

□

Example 5.1.6 A container of water has a volume V. Suppose that the volume has an instantaneous rate of change with respect to time t given by

$$\frac{dV}{dt} = -40 + 3t.$$

When $\dfrac{dV}{dt}$ is negative, the volume is decreasing; when $\dfrac{dV}{dt}$ is positive, the volume is increasing. The expression defines exactly how fast the water is entering or leaving the container. Find the volume of water as a function of time if $V = 500$ when $t = 1$.

The following SageMath script will start by defining the formula for the rate of change. It then uses a definite integral to create the variable for the volume,

$$V(t) = 500 + \int_1^t -40 + 3z\,dz.$$

```
# Define the independent variable.
var("t")
# Define dV as a function for rate
DV(t) = -40+3*t
show(DV(t))
# Define the V using integral, but need dummy variable
var("z")
V(t) = 500 + integrate(DV(z), [z, 1, t])
show(V(t))
```

```
3*t-40
3/2*t^2-40*t+1077/2
```

The integration variable z was needed in the integral for the same reason that a summation in sequence accumulations requires a dummy index variable. The formula `DV(z)` represents the formula for the rate of change evaluated at this integration variable instead of t, $-40+3z$. This could have been computed in WolframAlpha with the query `integrate -40+3z with respect to z from 1 to t`. □

5.1.3 Summary

- Calculus is developed using ideas similar to those for sequences—limits, increments, and accumulation— to limits of functions, derivatives, and integrals.

- The **derivative** $\dfrac{dQ}{dx}$ measures the **instantaneous rate of change** of a quantity Q with respect to the independent variable x, represented by a limit,
$$\frac{dQ}{dx} = \lim_{\Delta x \to 0} \frac{\Delta Q}{\Delta x}.$$
Consequently, increments of change in Q, ΔQ, will be approximately proportional to the increment in x,
$$\Delta Q \approx \frac{dQ}{dx} \cdot \Delta x,$$
for sufficiently small Δx.

- Given a function f' for the rate of change of a quantity Q, $x \overset{f'}{\mapsto} \frac{dQ}{dx}$, and an initial value Q_0 when $x = x_0$, the **accumulation function** will be that function with derivative $\frac{dQ}{dx} = f'(x)$, represented by the integral
$$Q = Q_0 + \int_{x_0}^{x} f'(z)\,dz.$$

- Computational tools, such as WolframAlpha and SageMath, are available to perform these calculations, leaving us the responsibility of formulating problems and interpreting the results.

5.1.4 Exercises

Use appropriate tables to approximate the following function limits. For a two-sided limit, be sure that your work verifies that both sides approximate the same value

1. $\displaystyle\lim_{x \to 3^-} \frac{2^x - 8}{x - 3}$

2. $\displaystyle\lim_{x \to 3^+} \frac{2^x - 8}{x - 3}$

3. $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{2^x - 4}$

4. $\displaystyle\lim_{x \to 1} \frac{x^2 - 1}{|x - 1|}$

Find the instantaneous rate of change for the relationship described in each problem using the limit of the average rate of change between the given point and a second variable point. Compare the instantaneous rate to the average rate for the specified increments.

5. An object tossed into the air has a height that changes in time. Let h measure the height from the ground in feet and let t measure the time since the object was tossed in seconds. Then h has a model
$$t \mapsto h = 4 + 30t - 16t^2.$$

Find $\frac{dh}{dt}$ at $t = 1$ and compare this to the average rate $\frac{\Delta h}{\Delta t}$ with $\Delta t = 0.1$.

6. The material cost for producing an aluminum box the shape of a cube is a function of the size of the cube. Let C be the cost in dollars and let s measure the length of each side of the box in centimeters. Then C has a model

$$s \mapsto C = 0.03s^2.$$

Find $\frac{dC}{ds}$ at $s = 10$ and compare this to the average rate $\frac{\Delta C}{\Delta s}$ with $\Delta s = -0.2$.

7. For a circle of radius r, the area A satisfies a relation

$$r \mapsto A = \pi r^2.$$

Find $\frac{dA}{dr}$ at $r = 2$ and compare this to the average rate $\frac{\Delta A}{\Delta r}$ with $\Delta r = 0.05$.

For each problem, write down the formula involving an integral for the quantity whose derivative and initial value are given. Use technology to find the algebraic formula of the quantity.

8. Given $\frac{dy}{dx} = 4$ and $y = 5$ when $x = 2$. Find y as a function of x.

9. Given $\frac{dy}{dx} = 2 + 3x$ and $y = 4$ when $x = 1$. Find y as a function of x.

10. Given $\frac{dQ}{dt} = t^3$ and $Q = 2$ when $t = 1$. Find Q as a function of t.

11. Given $\frac{dP}{dt} = 500e^{0.2t}$ and $P = 4000$ when $t = 0$. Find P as a function of t.