

5.3 Limits of Functions

We have previously studied limits of sequences. In the last section, we considered the continuity of piecewise functions as it depended on whether the function rules to the left and to the right of a point agreed with the value at the point. We used limit notation to describe the values coming from the left and from the right.

In this section, we seek to harmonize these two views of limits. We will introduce the idea that the limit of a function describes the limit of a sequence of output values for a converging sequence of input values. The behavior of limits of sequences justify our rules to calculate limits of functions. We will also discuss horizontal and vertical asymptotes of functions in the context of limits.

5.3.1 Limits

For sequences, we introduced the idea of limits as the value the sequence was approaching further and further in that sequence. We saw that the decimal approximations of the sequence values would eventually converge to the decimal approximation of the limiting value. The mathematical definition of the limit was stated in terms of the possibility of eventually waiting long enough in the sequence that the sequence values would approximate the limit value within *any* desired accuracy of approximation.

The only limits of interest in sequences were when the index went to infinity. For functions, in order to [understand continuity 5.2.19](#), we have found that we also need to think about limits as the independent variable approaches a value from either the left or the right. Sequences can give us a way to think about this possibility.

Definition 5.3.1 Limits of Function. For a function f defined on intervals to the left and right of c , we say

$$\lim_{x \rightarrow c} f(x) = L$$

to mean that for *every* independent sequence x such that $x_n \neq c$ and $x_n \rightarrow c$, the dependent sequence $y = (f(x_n))_{n=n_0}^{\infty}$ must have the limit L ,

$$\lim_{n \rightarrow \infty} f(x_n).$$

One-sided limits add constraints to the independent sequences, with $x \rightarrow c^+$ requiring $x_n > c$ and $x \rightarrow c^-$ requiring $x_n < c$. \diamond

Function limits are properties of the function itself and do not depend on the sequences chosen. If different independent sequences that converge to c result in different limits for the dependent sequence, then the function does not have a limit. The following example illustrates how this new definition relates our earlier concept of continuity of piecewise functions with the sequence definition of function limits. We create a table of sequence values, one column corresponding to the independent variable (input) and another column corresponding to the dependent variable (output). The input sequence is chosen to converge to the value c , and we examine what happens to the sequence of the dependent variable.

Example 5.3.2 A function is defined piecewise as

$$f(x) = \begin{cases} x + 2, & x < 3, \\ 4x - x^2, & x > 3. \end{cases}$$

Find $\lim_{x \rightarrow 3^-} f(x)$, $\lim_{x \rightarrow 3^+} f(x)$, and $\lim_{x \rightarrow 3} f(x)$ using sequences for approximation.

Solution. The left-sided limit needs to consider an independent sequence $x_n < 3$ with $x_n \rightarrow 3$. The following partial table illustrates an example with $x_n = 3 - 10^{-n}$.

n	x_n	$f(x_n)$
1	2.9	$f(2.9) = 2.9 + 2 = 4.9$
2	2.99	$f(2.99) = 2.99 + 2 = 4.99$
3	2.999	$f(2.999) = 2.999 + 2 = 4.999$

If we compare the values of the dependent sequence $f(x_n)$ with the value of the formula $x + 2$ evaluated at $x = 3$, which is $x + 2 = 5$, we can see that the dependent sequence is approaching that limit $f(x_n) \rightarrow 5$. We therefore write

$$\lim_{x \rightarrow 3^-} f(x) = 5.$$

In a similar way, a right-sided limit requires $x_n > 3$ with $x_n \rightarrow 3$, such as the sequence $x_n = 3 + 10^{-n}$.

n	x_n	$f(x_n)$
1	3.1	$f(3.1) = 4(3.1) - (3.1)^2 = 2.79$
2	3.01	$f(3.01) = 4(3.01) - (3.01)^2 = 2.9799$
3	3.001	$f(3.001) = 4(3.001) - (3.001)^2 = 2.997999$

If we compare the values of the dependent sequence $f(x_n)$ with the value of the formula $4x - x^2$ evaluated at $x = 3$, which is $4x - x^2 = 3$, we can see that the dependent sequence is approaching that limit $f(x_n) \rightarrow 3$. We therefore write

$$\lim_{x \rightarrow 3^+} f(x) = 3.$$

The two-sided limit requires only $x_n \rightarrow 3$. The sequence values might be either above or below 3. Above, we found that when the independent variable values are on the left $x_n < 3$, we had $f(x_n) \rightarrow 5$. But when $x_n > 3$, we had $f(x_n) \rightarrow 3$. Because different sequences with $x_n \rightarrow 3$ result in different limits for $f(x_n)$,

$$\lim_{x \rightarrow 3} f(x) \text{ does not exist.}$$

□

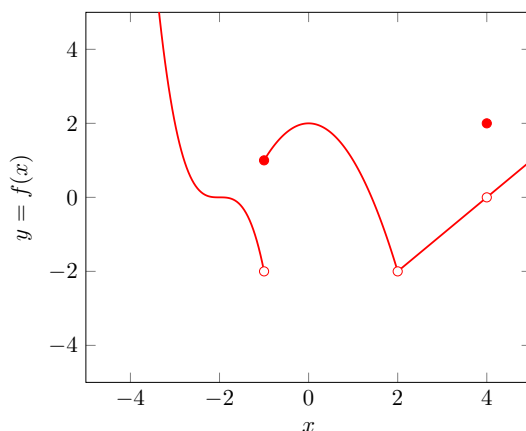
While the previous example attempted to connect our simpler understanding of limits with the limits of sequences, the given solution really only illustrated the first few terms from two out of infinitely many possible independent sequences. Using tables of sequence values might suggest possible values of the limits, but we need a more definitive reason that the limit agrees with simple evaluation of the formula.

When a function is visualized as a graph, a limit can be determined by looking at the branches of the graph immediately to the left or right of the point of interest. A limit of $f(x)$ with $x \rightarrow c^-$ means to look at the branch of the function with $x < c$ and identify what point that branch would lead to as $x \rightarrow c$. Similarly, a limit of $f(x)$ with $x \rightarrow c^+$ means to look at the branch of the function with $x > c$ and identify what point that branch would lead to as

$x \rightarrow c$.

Example 5.3.3 Consider the function f whose graph is shown below. Find the following limits based on the graph, assuming the coordinates of shown points are integers.

1. $\lim_{x \rightarrow -1} f(x)$
2. $\lim_{x \rightarrow 0} f(x)$
3. $\lim_{x \rightarrow 2} f(x)$
4. $\lim_{x \rightarrow 4} f(x)$



Solution.

1. $\lim_{x \rightarrow -1} f(x)$

We consider a sequence for x on the x -axis that converges to -1 . It is best to consider a sequence on the left and another on the right. For $x_n < -1$, our function will be using the cubic portion of the graph. As $x_n \rightarrow -1$ (from the left), the function will move closer and closer to the open point at $(-1, -2)$. The y -value of this point is the corresponding limit of the dependent sequence:

$$\lim_{x \rightarrow -1^-} f(x) = -2.$$

For $x_n > -1$, we will be somewhere to the right. As $x_n \rightarrow -1$ (from the right), we will eventually be on the portion of the function corresponding to the concave down parabola. The sequence will move us closer and closer to the filled-in point at $(-1, 1)$:

$$\lim_{x \rightarrow -1^+} f(x) = 1.$$

Because the left- and right-limits have different values, the two-sided limit *does not exist*.

2. $\lim_{x \rightarrow 0} f(x)$

We consider a sequence for x on the x -axis that converges to 0. Regardless of whether the sequence is to the left or the right of $x = 0$, the value x_n will eventually use the function defined by the concave down parabola.

The point on the graph of the function will converge to the vertex of this parabola at $(0, 2)$. This is our limit.

$$\lim_{x \rightarrow 0} f(x) = 2.$$

3. $\lim_{x \rightarrow 2} f(x)$

For a sequence $x_n \rightarrow 2$ with $x_n < 2$, the point on the graph will eventually be on the parabola and approaching the open point at $(2, -2)$. For values with $x_n > 2$, the point on the graph will eventually be on the line and also approaching the point $(2, -2)$. Because the sequence always results in approaching the point $(2, -2)$, we have a limit

$$\lim_{x \rightarrow 2} f(x) = -2.$$

Notice that a limit does not depend on whether the point is included in the function or not. All that matters is whether the sequence of points converges to that point.

4. $\lim_{x \rightarrow 4} f(x)$

For a sequence $x_n \rightarrow 4$ and $x_n \neq 4$, eventually the function will be on the line to the left or right of $x = 4$. Either way, the corresponding point on the graph will be converging to $(4, 0)$:

$$\lim_{x \rightarrow 4} f(x) = 0.$$

The value of the function $f(4) = 2$ has no effect on the limit.

□

When we have a formula for a function, we already know that a table can be helpful but will not guarantee the value of the limit. We might think we can graph the function, but even our graphs of functions are ultimately based on a table of values. We need some methods to evaluate limits based on the formulas alone.

Our earlier motivation for limits involving piecewise functions suggest that we can find a limit by evaluating a function at the point of interest. In most cases, this is true. This is a consequence of the algebraic structure of expressions and the fact that limits behave very nicely with algebraic operations. The rest of this section explains why limits usually behave so well.

5.3.2 Limit Rules for Combining Sequences

We start by formalizing some rules about how sequence limits relate to the arithmetic of sequences. These rules are stated as theorems. We begin each theorem with one or more sequence that is given with a particular limit. We then define a new sequence using arithmetic involving those sequences. The conclusion of each theorem describes the limit of the new sequence. To apply a theorem, we must verify that the hypotheses are satisfied before we can use the conclusion.

For a converging sequence $x = (x_n)$ with $x_n \rightarrow L$, we can think of the sequence values as *approximating* L . The absolute error of approximation $|x_n - L|$ must vanish as $n \rightarrow \infty$. In other words, for any margin of error $\epsilon > 0$, there must be some index N so that $|x_n - L| < \epsilon$ once $n > N$. The proofs of these theorems rely on showing how the error of approximation for the arithmetic combination of the sequences can be related to the errors of

approximations of the given sequences to their limits in a way to show that the approximation error will eventually vanish.

The first three rules involve elementary operations involving constants on a single sequence. They correspond to the operations used to construct expressions involving another expression and a constant, as discussed in ((Unresolved xref, reference "subsubsection-elementary-arithmetic-operations"; check spelling or use "provisional" attribute)).

Theorem 5.3.4 Sequence Limit of a Constant Sum (SL:CS). *Given a sequence $u = (u_n)$ with $u_n \rightarrow L$ and any constant k , the transformed sequence $w_n = u_n + k$ has limit*

$$\lim_{n \rightarrow \infty} u_n + k = L + k.$$

Proof. The error of approximation for w_n from its proposed limit $L + k$ can be rewritten

$$|w_n - (L + k)| = |u_n + k - L - k| = |u_n - L|.$$

This is the same as the error of approximation for u_n from its limit L . As soon as $|u_n - L| < \epsilon$, we also have $|w_n - (L + k)| < \epsilon$. Because $|u_n - L| \rightarrow 0$, this proves

$$\lim_{n \rightarrow \infty} u_n + k = L + k. \quad \blacksquare$$

Theorem 5.3.5 Sequence Limit of a Constant Multiple (SL:CM). *Given a sequence $u = (u_n)$ with $u_n \rightarrow L$ and any constant k , the transformed sequence $w_n = k \cdot u_n$ has limit*

$$\lim_{n \rightarrow \infty} k u_n = k L.$$

Proof. The error of approximation for w_n from its proposed limit kL can be rewritten

$$|w_n - kL| = |k u_n - kL| = |k(u_n - L)| = |k| \cdot |u_n - L|.$$

If $k = 0$, then $w_n = 0$ for all n and $w_n \rightarrow 0$ must be true. If $k \neq 0$, then the error of approximation for w from its proposed limit is exactly $|k|$ times the error of approximation for u from its given limit. As soon as $|u_n - L| < \frac{1}{|k|}\epsilon$, we must have $|w_n - kL| < \epsilon$. Because $|u_n - L| \rightarrow 0$, this proves

$$\lim_{n \rightarrow \infty} k u_n = k L. \quad \blacksquare$$

Theorem 5.3.6 Sequence Limit of a Reciprocal (SL:MInv). *Given a sequence $u = (u_n)$ with $u_n \rightarrow L \neq 0$, the transformed sequence of multiplicative inverses $w_n = \frac{1}{u_n}$ has limit*

$$\lim_{n \rightarrow \infty} \frac{1}{u_n} = \frac{1}{L}.$$

Proof. Division is not defined when the denominator equals zero. Because $u_n \rightarrow L$ and $L \neq 0$, we know that $|u_n - L| < \frac{1}{2}|L|$ eventually. When $L > 0$, this means that $\frac{1}{2}L < u_n < \frac{3}{2}L$. If $L < 0$, then $\frac{3}{2}L < u_n < \frac{1}{2}L$. Either way, u_n is kept away from 0 and $w_n = \frac{1}{u_n}$ is guaranteed to be defined. (Before this point, we might have had $u_n = 0$ so that w_n is not defined.)

The error of approximation for w_n from its proposed limit $\frac{1}{L}$ can be rewritten using a common denominator

$$|w_n - \frac{1}{L}| = \left| \frac{1}{u_n} - \frac{1}{L} \right| = \left| \frac{L - u_n}{u_n L} \right| = |u_n - L| \cdot \frac{1}{|u_n| |L|}.$$

Because $|u_n| > \frac{1}{2}|L|$, we know that $\frac{1}{|u_n|} < \frac{2}{|L|}$. Thus,

$$\left|w_n - \frac{1}{L}\right| < |u_n - L| \cdot \frac{2}{|L|^2}.$$

The error of approximation for w_n is always smaller than $\frac{2}{|L|^2}$ times the error of approximation for u_n from its limit. Because $|u_n - L| \rightarrow 0$, this proves

$$\lim_{n \rightarrow \infty} \frac{1}{u_n} = \frac{1}{L}.$$

■

The second group of limit rules of combination allow us to take two limits that we know and combine them with arithmetic. Notice how the limit rules correspond exactly with the arithmetic operations used to construct expression, as discussed in [Subsection 2.2.2](#). The proofs of these theorems are more advanced and will not be given in this section.

Theorem 5.3.7 Sequence Limit of a Sum (SLC:Sum). *Given sequences $u = (u_n)$ with $u_n \rightarrow L$ and $v = (v_n)$ with $v_n \rightarrow M$, the sequence defined by the sum $w_n = u_n + v_n$ has limit*

$$\lim_{n \rightarrow \infty} [u_n + v_n] = L + M.$$

Theorem 5.3.8 Sequence Limit of a Difference (SLC:Diff). *Given sequences $u = (u_n)$ with $u_n \rightarrow L$ and $v = (v_n)$ with $v_n \rightarrow M$, the sequence defined by the difference $w_n = u_n - v_n$ has limit*

$$\lim_{n \rightarrow \infty} [u_n - v_n] = L - M.$$

Theorem 5.3.9 Sequence Limit of a Product (SLC:Prod). *Given sequences $u = (u_n)$ with $u_n \rightarrow L$ and $v = (v_n)$ with $v_n \rightarrow M$, the sequence defined by the product $w_n = u_n \cdot v_n$ has limit*

$$\lim_{n \rightarrow \infty} [u_n \cdot v_n] = L \cdot M.$$

Theorem 5.3.10 Sequence Limit of a Quotient (SLC:Quot). *Given sequences $u = (u_n)$ with $u_n \rightarrow L$ and $v = (v_n)$ with $v_n \rightarrow M$ and $M \neq 0$, the sequence defined by the sum $w_n = \frac{u_n}{v_n}$ has limit*

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{L}{M}.$$

In addition to algebraic operations combining sequences, we have operations associated with functions. This includes raising sequences to powers, applying exponential or logarithm functions, or using trigonometric functions. We will learn that each of these functions are continuous. Consequently, the following theorem will apply.

Theorem 5.3.11 Sequence Limit of a Continuous Function (SLC:CFxn). *Given a sequence $u = (u_n)$ with $u_n \rightarrow L$ and a function f that is continuous at L , the sequence defined by the sum $w_n = f(u_n)$ has limit*

$$\lim_{n \rightarrow \infty} f(u_n) = f(L).$$

5.3.3 Elementary Limit Rules for Functions

Having established the limit rules associated with sequences, we can apply those rules to create corresponding limit rules for functions.

The first collection of limit rules are some basic limits. We can think of them as our building blocks for more complicated limits. We begin by showing that constant functions and the identity functions are continuous.

Theorem 5.3.12 Limit of a Constant (LE:Const).

Hypothesis k is a real number.

Conclusion $\lim_{x \rightarrow a} k = k$.

Proof. For a constant function $f(x) = k$, the output sequence is a constant sequence regardless of the input sequence. ■

Theorem 5.3.13 Limit of the Identity (LE:Ident).

Hypothesis none

Conclusion $\lim_{x \rightarrow a} x = a$.

Proof. For identity function $f(x) = x$, the output sequence is will be the same as the input sequence. Since $x_k \rightarrow a$, the output sequence has limit a . ■

We include the limit of linear functions in our known limits of elementary functions.

Theorem 5.3.14 Limit of a Linear Function (LE:Line).

Hypothesis m and b are real numbers.

Conclusion $\lim_{x \rightarrow a} [mx + b] = ma + b$.

Proof. Given a sequence x_k with $x_k \neq a$ and $x_k \rightarrow a$, define the output sequence $y_k = mx_k + b$. This is a constant sum and constant multiple of x_k . By [SLC:CM](#), we know $mx_k \rightarrow ma$. By [SLC:CS](#), we then have $y_k = mx_k + b \rightarrow ma + b$. ■

5.3.4 Limit Rules of Combination

The second collection of limit rules tell us how we can take limits that we already know (starting with building blocks) and use them to compute more complicated limits. The first three rules take a single limit that is known to be valid and use arithmetic with a constant to find a new limit. Each of the theorem simply applies the corresponding limit rule for sequences on the sequence created by the function.

Theorem 5.3.15 Limit of a Constant Sum (LC:CS).

Hypothesis $\lim_{x \rightarrow a} f(x) = L$ and k is a real number.

Conclusion $\lim_{x \rightarrow a} [f(x) + k] = L + k$.

Theorem 5.3.16 Limit of a Constant Multiple (LC:CM).

Hypothesis $\lim_{x \rightarrow a} f(x) = L$ and k is a real number.

Conclusion $\lim_{x \rightarrow a} [k \cdot f(x)] = k \cdot L$.

Theorem 5.3.17 Limit of a Reciprocal or Multiplicative Inverse (LC:MInv).

Hypothesis: $\lim_{x \rightarrow a} f(x) = L$ and $L \neq 0$.

$$\text{Conclusion: } \lim_{x \rightarrow a} \frac{1}{f(x)} = \frac{1}{L}.$$

The next limit rules of combination allow us to take two limits that we know and combine them with arithmetic. In each of the cases, note that both limits in the hypothesis have $x \rightarrow a$ (i.e., x approaches the same value in both limits).

Theorem 5.3.18 Limit of a Sum (LC:Sum).

Hypothesis $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$.

Conclusion $\lim_{x \rightarrow a} [f(x) + g(x)] = L + M$.

Theorem 5.3.19 Limit of a Difference (LC:Diff).

Hypothesis $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$.

Conclusion $\lim_{x \rightarrow a} [f(x) - g(x)] = L - M$.

Theorem 5.3.20 Limit of a Product (LC:Prod).

Hypothesis $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$.

Conclusion $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = L \cdot M$.

Theorem 5.3.21 Limit of a Quotient (LC:Quot).

Hypothesis $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$ and $M \neq 0$.

Conclusion $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

In addition to the arithmetic of functions, composition of functions plays an important role in algebra and calculus. So we need a limit rule associated with composition. Recall that for composition, the output of one function becomes the input to another function.

Theorem 5.3.22 Limit of a Continuous Composition (LC:Comp).

Hypothesis $\lim_{x \rightarrow a} f(x) = L$ and g is continuous at L , or in other words,
 $\lim_{u \rightarrow L} g(u) = g(L)$.

Conclusion $\lim_{x \rightarrow a} g \circ f(x) = \lim_{x \rightarrow a} g(f(x)) = g(L)$.

If f is continuous at a so that $\lim_{x \rightarrow a} f(x) = f(a)$ then we have $\lim_{x \rightarrow a} g \circ f(x) = g \circ f(a)$. In other words, the composition of continuous functions is a continuous function.

5.3.5 Justifying Limit Calculations

There are two ways in which limit rules are applied. One way is to provide formal justification of limit calculations, or in other words, to write a proof of limit statements. The other way limit rules are used is to break a computation down into recognizable and manageable parts. This section focuses on the process of formal justification.

A mathematical proof is essentially a sequence of statements, each of which is demonstrably true based only on previously stated knowledge and logical arguments. This means that when writing a proof or any careful justification, we must be careful that when we write something down we have previously

established all of the necessary conditions at a previous step. In order to avoid circular reasoning, we should avoid referring to something as true before we actually show it is true.

For justification of limit statements, this means that we start from the small building blocks that create our formula and put them together one step at a time until we can justify the limit statement we are trying to prove.

Example 5.3.23 Compute and justify $\lim_{x \rightarrow 2} 3x^2(2x - 5)$.

Solution. Start by planning ahead. The formula $3x^2(2x - 5)$ is a product of $3x^2$ and $2x - 5$. This second factor $2x - 5$ is a linear function so there is a limit rule for that piece. But $3x^2$ is not linear, and we recognize it as a product of 3 (a constant) and x^2 . Finally, we see that $x^2 = x \cdot x$ is the product of the identity with itself. We will start with the elementary formulas and build them back up to the full function.

1. $\lim_{x \rightarrow 2} x = 2$ by LE:Ident.
2. $\lim_{x \rightarrow 2} x \cdot x = 2 \cdot 2 = 4$ by LC:Prod using limits of $f(x) = x$ (step 1) and $g(x) = x$ (step 1).
3. $\lim_{x \rightarrow 2} 3x^2 = 3 \cdot 4 = 12$ by LC:CM using constant $k = 3$ and limit of $f(x) = x^2$ (step 2).
4. $\lim_{x \rightarrow 2} 2x - 5 = 2(2) - 5 = -1$ by LE:Line ($m = 2$, $b = -5$).
5. $\lim_{x \rightarrow 2} 3x^2(2x - 5) = 12(-1) = -12$ by LC:Prod using the limits found in step 3 and step 4.

□

Example 5.3.24 Compute and justify $\lim_{x \rightarrow 3} x^3 + 4x^2 - 3x + 1$.

Solution. It is important to note that limit rules of combination only combine two formulas at a time. In this calculation, we will need the limit of x^3 . Writing this as $x^3 = x \cdot x \cdot x$ is not going to be as useful as writing $x^3 = x \cdot x^2$ because there are no rules to combine three limits at once. In addition, subtraction is always problematic, so it is best to rewrite subtraction as a sum,

$$x^3 + 4x^2 - 3x + 1 = x^3 + 4x^2 + -3x + 1.$$

1. $\lim_{x \rightarrow 3} x = 3$ by LE:Ident.
2. $\lim_{x \rightarrow 3} x \cdot x = 3 \cdot 3 = 9$ by LC:Prod using the limits in step 1 (twice).
3. $\lim_{x \rightarrow 3} x \cdot x^2 = 3(9) = 27$ by LC:Prod using the limits in step 1 and step 2.
4. $\lim_{x \rightarrow 3} 4x^2 = 4(9) = 36$ by LC:CM using $k = 4$ and the limit in step 2.
5. $\lim_{x \rightarrow 3} -3x + 1 = -3(3) + 1 = -8$ by LE:Line ($m = -3$, $b = 1$).
6. $\lim_{x \rightarrow 3} x^3 + 4x^2 = 27 + 36 = 63$ by LC:Sum using limits in step 3 and step 4.
7. $\lim_{x \rightarrow 3} x^3 + 4x^2 + -3x + 1 = 63 + -8$ by LC:Sum using limits in step 6 and step 5.

□

Because most expressions that we work with are defined strictly in terms of the basic arithmetic operations and elementary functions, the limit rules we have developed essentially allow us to replace the independent variable in the formula $f(x)$ with the limiting point $x \rightarrow c$. That is, whenever the expression involves basic arithmetic operations (addition, subtraction, multiplication, and division), we know that we *could* apply the limit rules step-by-step to justify

$$\lim_{x \rightarrow c} f(x) = f(c).$$

The exception is that our rule for quotients does not allow division by zero.

Theorem 5.3.25 *If $f(x)$ is an algebraic expression that involves only arithmetic operations, then $\lim_{x \rightarrow c} f(x) = f(c)$ so long as $f(c)$ is defined.*

Example 5.3.26 Determine $\lim_{x \rightarrow 2} \frac{2x + 3}{x^2 - 5}$.

Solution. At first glance, we might worry that the theorem does not apply because x^2 is a power and not an arithmetic operation. However, because $x^2 = x \cdot x$ is a product, we have a function $f(x) = \frac{2x + 3}{x \cdot x - 5}$ defined in terms of arithmetic operations. We evaluate $f(2)$:

$$f(2) = \frac{2(2) + 3}{2^2 - 5} = -7.$$

Consequently, by [Theorem 5.3.25](#), we have

$$\lim_{x \rightarrow 2} \frac{2x + 3}{x^2 - 5} = -7.$$

□

We will learn in the next section how to deal with expressions where the value is not defined.

5.3.6 Summary

- The limit of a function $\lim_{x \rightarrow c} f(x)$ represents the value L that is the limit of the dependent sequence $f(x_n)$ for *every* independent sequence (x_n) that satisfies $x_n \neq c$ and $x_n \rightarrow c$. One-sided limits add the constraint that the sequence must stay below c ($x \rightarrow c^-$) or above c ($x \rightarrow c^+$).
- Numerically, a function limit $\lim_{x \rightarrow c} f(x)$ can be approximated by testing the value of the function for values of the independent variable following a sequence $x \rightarrow c$.
- Graphically, a function limit $\lim_{x \rightarrow c} f(x)$ corresponds to the y -value of the point in the plane that the sequence of points $(x_n, f(x_n))$ approaches from the left and from the right as $x_n \rightarrow c$. If the two branches (left vs right) approach different points, the two-sided limit does not exist.
- Limit rules associated with all of the arithmetic operations justify applying the same operations with limits.
- An argument justifying limits using limit rules must demonstrate that the component limits are known prior to combining them with a limit rule.

5.3.7 Exercises

Use appropriate tables to approximate the following function limits. For a two-sided limit, be sure that your work verifies that both sides approximate the same value

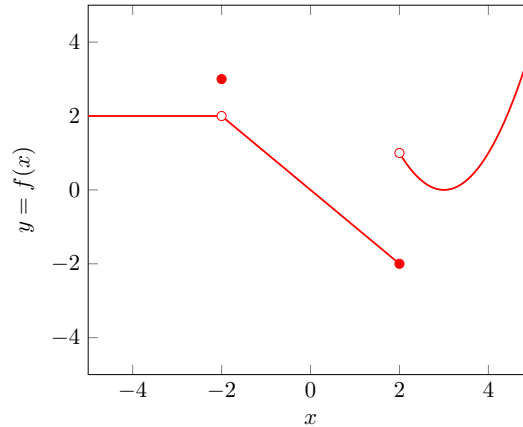
1. $\lim_{x \rightarrow 3^-} \frac{2^x - 8}{x - 3}$

2. $\lim_{x \rightarrow 3^+} \frac{2^x - 8}{x - 3}$

3. $\lim_{x \rightarrow 2} \frac{x^2 - 4}{2^x - 4}$

4. $\lim_{x \rightarrow 1} \frac{x^2 - 1}{|x - 1|}$

Consider the function f whose graph is shown below. Find the following values, if they exist, based on the graph and assuming the coordinates of shown points are integers.



5.

(a) $f(-2)$

(b) $\lim_{x \rightarrow -2^-} f(x)$

(c) $\lim_{x \rightarrow -2^+} f(x)$

(d) $\lim_{x \rightarrow -2} f(x)$

6.

(a) $f(0)$

(b) $\lim_{x \rightarrow 0^-} f(x)$

(c) $\lim_{x \rightarrow 0^+} f(x)$

(d) $\lim_{x \rightarrow 0} f(x)$

7.

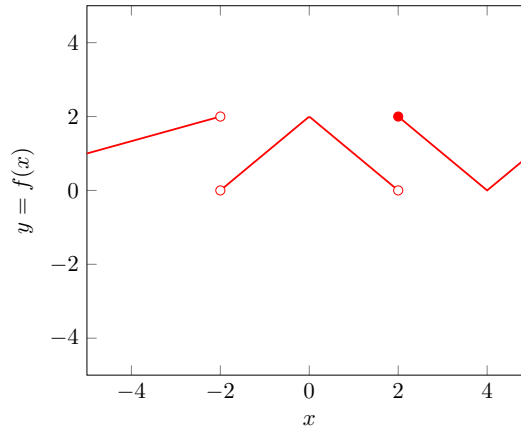
(a) $f(2)$

(b) $\lim_{x \rightarrow 2^-} f(x)$

(c) $\lim_{x \rightarrow 2^+} f(x)$

(d) $\lim_{x \rightarrow 2} f(x)$

Consider the function f whose graph is shown below. Find the following values, if they exist, based on the graph and assuming the coordinates of shown points are integers.



8.

(a) $f(-2)$

(b) $\lim_{x \rightarrow -2^-} f(x)$

(c) $\lim_{x \rightarrow -2^+} f(x)$

(d) $\lim_{x \rightarrow -2} f(x)$

9.

(a) $f(1)$

(b) $\lim_{x \rightarrow 1^-} f(x)$

(c) $\lim_{x \rightarrow 1^+} f(x)$

(d) $\lim_{x \rightarrow 1} f(x)$

10.

(a) $f(2)$

(b) $\lim_{x \rightarrow 2^-} f(x)$

(c) $\lim_{x \rightarrow 2^+} f(x)$

(d) $\lim_{x \rightarrow 2} f(x)$

Compute and justify the value of each limit applying the limit rules for functions step-by-step.

11. $\lim_{x \rightarrow -3} \frac{4x + 1}{2x + 3}$

12. $\lim_{x \rightarrow 2} 3x^2 - 4x + 5$

13. $\lim_{x \rightarrow 4} \frac{5x^2}{2x^2(3x - 1)}$

14. $\lim_{x \rightarrow -2} x^3 - 4x^2 + 5x - 7$