5.4 Continuity of Functions

5.4.1 Overview

The elementary limit rules for functions tell us that the limit of an algebraic expression made from arithmetic operations will equal the value of the expression at the point in question, if that value exists. So why bother introducing limits at all if they are the same as function evaluation?

The fact of the matter is, they aren't the same thing at all. Recall that for piecewise functions, we can use limits to find the limiting value of a function to the left and to the right of a break point. Function evaluation would only allow us to look at the point itself. Having a function value agree with the limits is a characteristic of a function being continuous. A value for x where a function is not defined is an example of a discontinuity.

In this section, we consider the continuity of functions. We learn about removable and infinite discontinuities, which correspond to holes and vertical asymptotes in a graph. We learn to compute limits of functions at these discontinuities by looking at simplified, factored expressions. Sign analysis is used for infinite discontinuities to determine whether the discontinuity corresponds to unbounded positive or negative values.

5.4.2 Removable and Infinite Discontinuities

The intuitive idea of a continuous function is a function whose graph is connected. Sometimes, this is thought of as being able to draw the graph without lifting the pen. The technical definition of a continuity at a point, say at $x = c$, involves three parts. First, the limit on the left exists. This means that we can trace the graph on a branch with $x < c$. Second, the limit on the right exists. This means that we also can trace the graph on a branch with $x > c$. Third, both limits are equal to $f(c)$. This gives us the connection from the left branch to the right branch through the point.

Any time a function has a break, it has a discontinuity at that location. A break can be a simple hole, a jump between values, or an infinite discontinuity associated with a vertical asymptote. Discontinuities might also occur due to limits themselves not existing for any reason.

Consider two functions, $f(x) = \dfrac{1}{(x-3)(x+2)}$ and $g(x) = \dfrac{x^2 - 4x + 3}{x - 3}$. In both functions, the value of the function is not defined at $x = 3$; f and g are both **discontinuous** at $x = 3$. Consequently, the corresponding limits $\lim\limits_{x \to 3} f(x)$ and $\lim\limits_{x \to 3} g(x)$ can not be computed directly using the limit rules for functions.

If we look at the graphs of our functions, as shown below, we see that there is something fundamentally different about the behavior around $x = 3$. The function $f(x) = \dfrac{1}{(x-3)(x+2)}$ appears to have a vertical asymptote at $x = 3$. The function $g(x) = \dfrac{x^2 - 4x + 3}{x - 3}$ looks continuous, even though we know it has a break at $x = 3$.
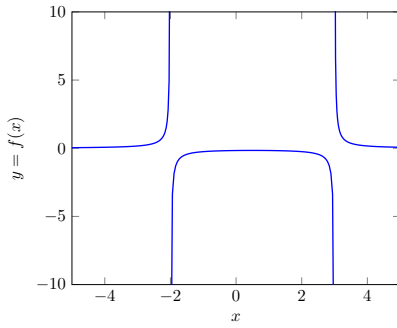
Figure 5.4.1 $y = \dfrac{1}{(x-3)(x+2)}$
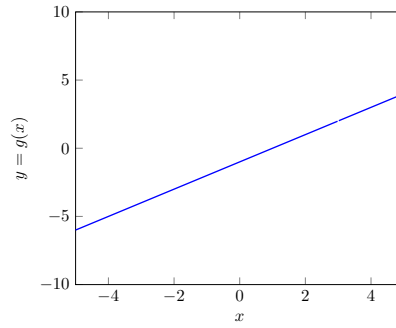
Figure 5.4.2 $y = \dfrac{x^2 - 4x + 3}{x - 3}$

If we factor the formula for $g(x)$, we discover that the formula simplifies.

$$\begin{aligned}
g(x) &= \frac{x^2 - 4x + 3}{x - 3} \\
&= \frac{(x-3)(x-1)}{x-3} \\
&= x - 1, \quad x \neq 3
\end{aligned}$$

Notice that we must include a domain restriction when we simplify. The original function is not defined for $x = 3$, but the simplified version is. To ensure the functions are the same, they must have the same domain. Because $x - 1$ is *continuous* at $x = 3$, $g(x)$ has a **hole** at $x = 3$ and we call this a **removable discontinuity**. A vertical asymptote at a point corresponds to a **infinite discontinuity**.

Example 5.4.3 The function $f(x) = \dfrac{3x^2 - x - 2}{x - 1}$ has a removable discontinuity at $x = 1$. What is the continuous function equivalent to $f(x)$?

Solution. A polynomial, like $3x^2 - x - 2$, will have a factor of $x - 1$ if and only if that polynomial has a value of 0 when $x = 1$. So we can see if it will cancel a factor by checking $3(1)^2 - (1) - 2 = 0$. Knowing this factor, we can soon find $3x^2 - x - 2 = (x - 1)(3x + 2)$. For all $x \neq 1$, we have

$$f(x) = \frac{3x^2 - x - 2}{x - 1} = \frac{(x-1)(3x+2)}{x-1} = 3x + 2.$$

We can only say this for $x \neq 1$ since the domain of f is $(-\infty, 1) \cup (1, \infty)$. That is,

$$f(x) = 3x + 2, \quad x \neq 1.$$

Our function $f(x)$ has the same graph as $y = 3x + 2$ except it has a hole at $x = 1$.

$\square$
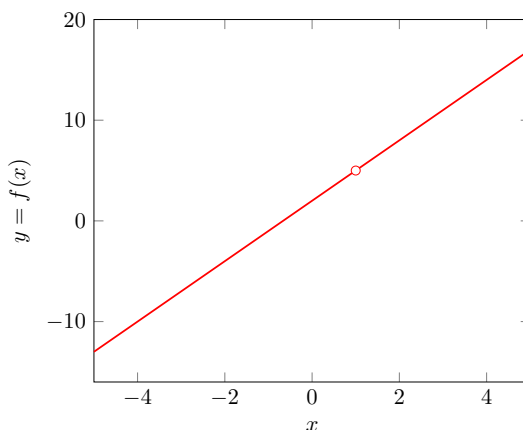
The previous example illustrates a basic feature of **rational functions** (i.e., a ratio or quotient of two polynomials). That is that there will be a canceling factor if the numerator and denominator have a common zero.

Theorem 5.4.4 *A rational function* $f(x) = \dfrac{p(x)}{q(x)}$ *where p and q are polynomial functions has a domain defined by*

$$D = \{x : q(x) \neq 0\}.$$

Further, p and q will have canceling common factors of the form $(x - a)$ *where a is a constant if and only if* $p(a) = 0$ *and* $q(a) = 0$.

For rational functions, the only possible discontinuities are holes and infinite discontinuities at vertical asymptotes. Holes correspond to points that are not in the domain but can be removed by canceling common factors. Any other points of discontinuity must be vertical asymptotes.

Example 5.4.5 Describe the discontinuities of the function

$$f(x) = \frac{x^3 - 5x^2 + 6x}{x^2 + x - 6}.$$

Solution.   The discontinuities are determined for a rational function by finding the zeros of the polynomial in the denominator, $q(x) = x^2 + x - 6$. We solve this by factoring:
$$q(x) = (x + 3)(x - 2).$$
There are discontinuities (breaks in the graph) at $x = -3$ and at $x = 2$.

We determine the type of discontinuity by seeing if common factors cancel. The numerator $p(x) = x^3 - 5x^2 + 6x$ can be tested even before factoring. At $x = -3$, we have $p(-3) = -27 - 5(9) + 6(-3) = -90$ so that $x + 3$ is not going to be a common factor. There must be a vertical asymptote at $x = -3$. At $x = 2$, we have $p(2) = 8 - 5(4) + 6(2) = 0$ so that there will be a common factor that cancels.

$$
\begin{aligned}
f(x) &= \frac{x^3 - 5x^2 + 6x}{x^2 + x - 6} \\
&= \frac{x(x^2 - 5x + 6)}{(x + 3)(x - 2)} \\
&= \frac{x(x - 2)(x - 3)}{(x + 3)(x - 2)}
\end{aligned}
$$

$$= \frac{x(x-3)}{x+3}, \quad x \neq 2.$$

Because the new formula has a natural domain $x \neq -3$, the discontinuity at $x = 2$ was removable. The graph has a hole at $x = 2$ and a vertical asymptote at $x = -3$. (Notice the addition of an explicit domain on the last step when we canceled, corresponding to the hole.)

$$y = \frac{x^3 - 5x^2 + 6x}{x^2 + x - 6}$$

□
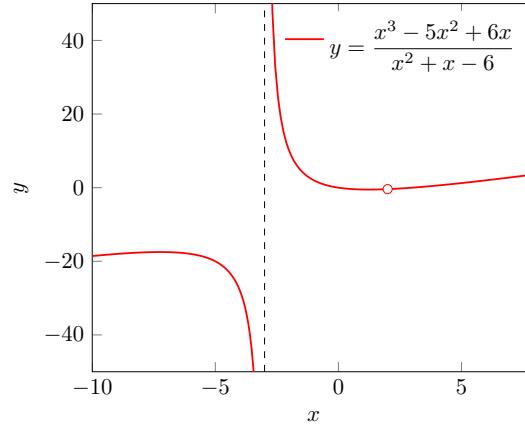
5.4.3 Limits at Discontinuities

The limit rules do not apply when substitution would result in division by zero. These precisely occur at points of discontinuity. Suppose a rational function $f(x) = \dfrac{p(x)}{q(x)}$ has $p(c) = 0$ and $q(c) = 0$. Immediate substitution of $x = c$ into $f(x)$ would result in $\frac{0}{0}$, which we have earlier identified as an **indeterminate limit form**. Because $p(c) = 0$ and $q(c) = 0$, $p(x)$ and $q(x)$ have a common factor $x - c$. Cancellation of that factor gives $f(x)$ a simplified form, and we can try again to evaluate the limit.

Example 5.4.6 Evaluate $\displaystyle\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4}$.
Solution. The formula is defined in terms of elementary arithmetic, so we try to evaluate the expression by substituting $x = 2$.

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4} \overset{?}{=} \frac{2^2 - 5(2) + 6}{2^2 - 4} = \frac{0}{0}$$

The limit has an indeterminate form. We can factor $x - 2$ from numerator and denominator and rewrite the expression.

$$\begin{aligned} f(x) &= \frac{x^2 - 5x + 6}{x^2 - 4} \\ &= \frac{(x-2)(x-3)}{(x-2)(x+2)} \\ &= \frac{x-3}{x+2}, \quad x \neq 2 \end{aligned}$$

Limits use the function to the side of the point in question. In this case, $f(x)$ uses the same formula on the left and the right of the discontinuity. Because the new formula is continuous, we can use substitution.

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4} = \lim_{x \to 2} \frac{x-3}{x+2}$$

$$= \frac{2-3}{2+2} = -\frac{1}{4}$$

$\square$

When a limit has a form $\frac{0}{0}$, we know to rewrite the formula in a simplified form. For sequences, we learned that if $a_n > 0$ and $a_n \to 0$, then $\frac{1}{a_n} \to +\infty$. That is, the reciprocal of a small positive number will be a large positive number. The smaller a_n becomes, the larger $\frac{1}{a_n}$ will be. Consequently, a rational function with a limit of the form $\frac{L}{0}$ has a vertical asymptote, and the limit will be unbounded. We use sign analysis to determine if the left- and right-limits are $+\infty$ or $-\infty$.

Example 5.4.7 Evaluate $\lim\limits_{x \to -2} \dfrac{x^2 - 5x + 6}{x^2 - 4}$.

Solution.   In the example above, we already found

$$f(x) = \frac{x^2 - 5x + 6}{x^2 - 4} = \frac{x-3}{x+2}, \quad x \neq 2.$$

Attempting substitution, we find

$$\lim_{x \to -2} f(x) \stackrel{?}{=} \frac{-2-3}{-2+2} = \frac{-5}{0}.$$

This is an undefined expression and indicates that $f(x)$ has an infinite discontinuity.

To find the limit as either $+\infty$ or $-\infty$, we do sign analysis on the simplified formula. The test intervals are separated by the *roots* and *discontinuities*. The roots are at solutions to $x - 3 = 0$; the discontinuities are at solutions to $x + 2 = 0$. We have a root at $x = 3$ and a discontinuity at $x = -2$, illustrated in the number line shown below.

For the limit, we need the signs of the function in each interval bordering the point $x = -2$. The intervals to test are $(-\infty, -2)$ and $(-2, 3)$.

$$f(-3) = \frac{-3-3}{-3+2} = 6$$
$$f(-1) = \frac{-1-3}{-1+2} = -4$$

We could update the number line with these signs.

To the left of $x = -2$, we see that $f(x) > 0$ (positive), so a limit from the left at the vertical asymptote must be

$$\lim_{x \to -2^-} \frac{x-3}{x+2} = +\infty.$$

To the right of $x = -2$, we see that $f(x) < 0$ (negative), so a limit from the right at the vertical asymptote must be

$$\lim_{x \to -2+} \frac{x-3}{x+2} = -\infty.$$

On the graph of the function, shown below, we see that the graph is unbounded above $(+\infty)$ to the left of the vertical asymptote and unbounded below $(-\infty)$ to the right of the vertical asymptote.

$$y = \frac{x^2 - 5x + 6}{x^2 - 4}$$
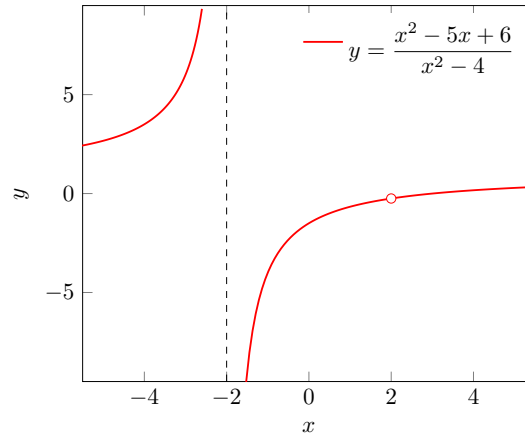
Motivated by our example, we are ready for a definition of a removable discontinuity. A removable discontinuity occurs when the graph to the left and to the right of a discontinuity approach the same point, but the function itself is not defined to match.

Definition 5.4.8 A function f has a **removable discontinuity** at $x = c$ if $\lim_{x \to c} f(x)$ exists (left- and right-limits have same value) *and* $\lim_{x \to c} f(x) \neq f(c)$, either because they are different values or $f(c)$ does not exist. ◇

An infinite discontinuity occurs at any point where the function has an infinite limit.

Definition 5.4.9 A function f has an **infinite discontinuity** at $x = c$ if one or both of $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ is infinite. The graph $y = f(x)$ has a vertical asymptote $x = c$. ◇

A jump discontinuity occurs when the limits on the left and right of a point both exist but have different values. We usually see these with piecewise functions.

Definition 5.4.10 A function f has an **jump discontinuity** at $x = c$ if $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ both exist but $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$. The graph $y = f(x)$ has a vertical gap between the branches to the left and to the right of $x = c$. ◇

5.4.4 Continuity on Intervals

Having discussed the continuity of functions at individual points, we introduce the idea of describing continuity on intervals. We want to be able to say that the graph of the function is connected over an entire interval.

Recall that a limit of a function $\lim_{x \to c} f(x)$ is defined in terms of sequences $x_n \to c$ with $x_n \neq c$. When thinking about continuity on an interval, we also require that the sequences stay in the interval.

We begin with open intervals. An open interval (a, b) is the set $\{x : a < x < b\}$. Open intervals have the feature that for every value in the set, say $c \in (a, b)$, there will be a sub-interval (a, c) to the left of the point and another sub-interval (c, b) to the right of the point. In relation to a sequence with $x_n \to c$, we can deal with left- and right-limits inside the interval.

Definition 5.4.11 A function f is **continuous on the open interval** (a, b) if for every $c \in (a, b)$, f is continuous at $x = c$. ◇

Closed intervals are a little trickier. A closed interval $[a, b] = \{x : a \le x \le b\}$ includes the end points. For values c strictly between a and b, we know that there are subintervals to the left and the right of c. However, at $x = a$, the interval only contains points to the right; and at $x = b$, the interval only contains points to the left. Continuity of a function on a closed interval must take this into account.

Definition 5.4.12 A function f is **continuous on the closed interval** $[a, b]$ if for every $c \in (a, b)$, f is continuous at $x = c$ and $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$. ◇

Continuity on an interval including only one end point requires one-sided continuity at that point using a limit that stays inside the interval. All of these definitions can be combined into a single definition.

Definition 5.4.13 A function f is **continuous on an interval** I if for every $c \in I$ and every sequence with values $x_n \in I$, $x_n \ne c$, and $x_n \to c$, we have $f(x_n) \to f(c)$. ◇

5.4.5 Extreme and Intermediate Value Theorems

There are two important theorems that describe what we know about functions that are continuous on closed intervals. The Extreme Value Theorem guarantees that any function that is continuous on a closed interval has a highest and lowest point within that interval. The Intermediate Value Theorem guarantees that a function that is continuous on a closed interval can not skip over any values between its values at the endpoints. The proofs for both of these theorems require advanced methods not taught at this level. We treat them essentially as axioms, statements that are true without proof.

Theorem 5.4.14 Extreme Value Theorem. *Suppose f is a function that is continuous on $[a, b]$. Then there must exist values $c_m, c_M \in [a, b]$ so that for any $x \in [a, b]$ we have*

$$f(c_m) \le f(x) \le f(c_M).$$

The values $f(c_m)$ and $f(c_M)$ are the minimum and maximum values, respectively, of the function f on $[a, b]$.

If a function is not continuous on $[a, b]$, then it does not necessarily have a maximum or minimum value. One way that this might happen is if f has a vertical asymptote within the interval. In that case, the values of f would be unbounded. Another way that this might happen is that f is bounded by what would be a maximum (or minimum) value but just doesn't reach it because of a sudden jump.
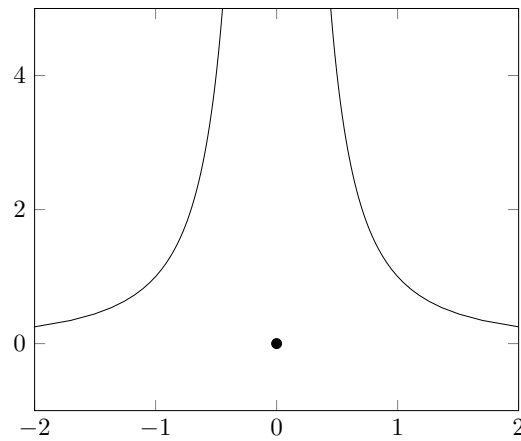
Example 5.4.15 Consider the function defined piecewise as

$$f(x) = \begin{cases} \frac{1}{x^2}, & x \ne 0, \\ 0, & x = 0. \end{cases}$$

This function has a non-removable discontinuity at $x = 0$, corresponding to a vertical asymptote. Because the formula has x^2 in the denominator (always positive), we have

$$\lim_{x \to 0} f(x) = +\infty.$$

This function is unbounded on the interval $[-1, 1]$ and has no maximum. It does have a minimum at $f(0) = 0$ since that is below the rest of the graph.
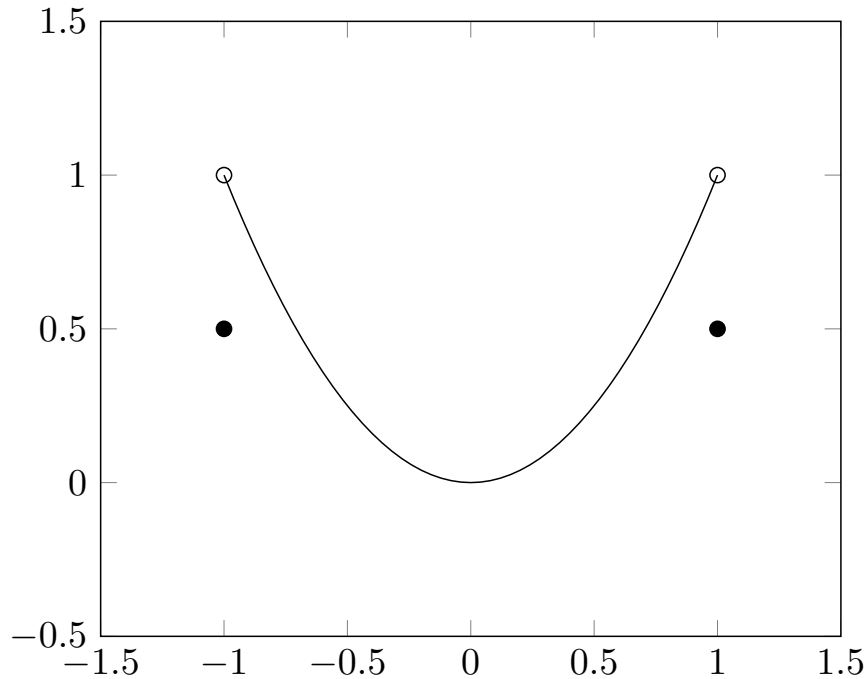
Example 5.4.16 Consider the function defined piecewise as

$$f(x) = \begin{cases} x^2, & -1 < x < 1, \\ \frac{1}{2}, & x = \pm 1. \end{cases}$$

This function has a removable discontinuities at $x = \pm 1$, where the limits are 1 but the values are $\frac{1}{2}$. In this case, f is continuous on $(-1, 1)$ but not on $[-1, 1]$. The maximum value should have been $y = 1$, but the graph never reaches that value because of the discontinuity. The function does have a minimum value at $f(0) = 0$.
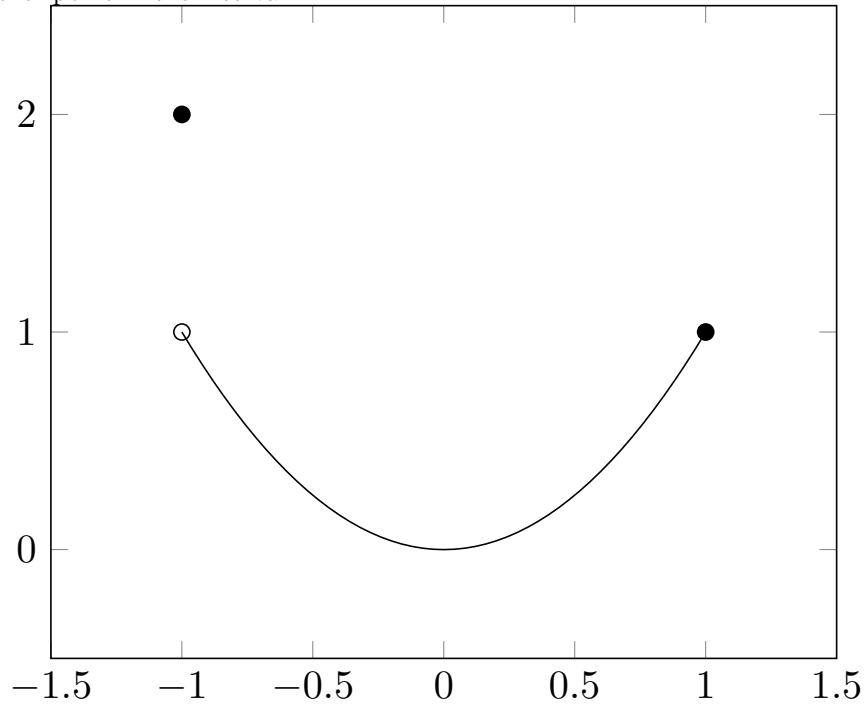

Example 5.4.17 Consider the function defined piecewise as

$$f(x) = \begin{cases} x^2, & -1 < x \leq 1, \\ 2, & x = -1. \end{cases}$$

This function has a removable discontinuity at $x = -1$. In this case, f is

continuous on $(-1, 1]$ but not on $[-1, 1]$. In spite of the discontinuity at $x = -1$, this function has a maximum value $f(-1) = 2$ because that value is above every other point in the interval.

$\square$

The previous example is included to emphasize that a theorem gives conditions that guarantee something is true. But those conditions are not always required. The extreme value theorem gives conditions that guarantee a function will have a maximum value. There are no exceptions for a continuous function on a closed interval to have both maximum and minimum values. But there are discontinuous functions that have them as well. It is just that there are also discontinuous functions that do not have extreme values.
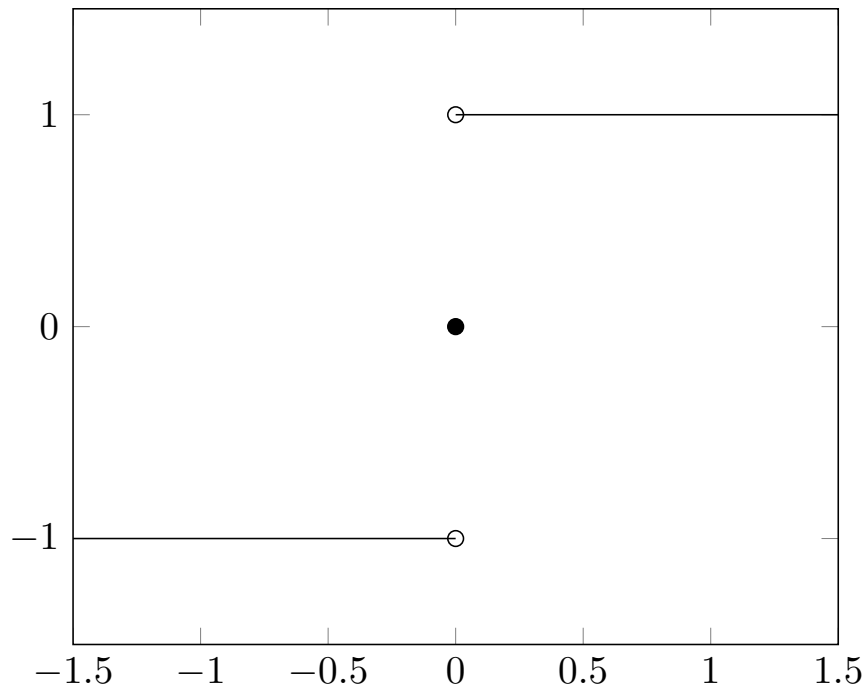
Theorem 5.4.18 Intermediate Value Theorem. *Suppose f is a function that is continuous on $[a, b]$. Then for every y between $f(a)$ and $f(b)$, there exists some $x \in (a, b)$ so that $f(x) = y$.*

The Intermediate Value Theorem guarantees that the graph of $y = f(x)$ intersects every horizontal line between $y = f(a)$ and $y = f(b)$ at least once for values of x between a and b. Because continuity is essentially connectedness, the only way for the graph to go from $y = f(a)$ to $y = f(b)$ is to cross through all intermediate values. A discontinuous function has the ability to jump across values without touching them.

Example 5.4.19 Consider the function defined piecewise as

$$f(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

This function has a jump discontinuity at $x = 0$, and is otherwise constant. If we consider the interval $[-1, 1]$, the values at the endpoints are $f(-1) = -1$ and $f(1) = 1$. Except for $y = 0$, the function $y = f(x)$ has no solutions for $-1 < y < 1$ because of the jump.

The Intermediate Value Theorem allows us to know that a continuous function has a solution to an equation within a particular interval. If the interval is small, we have an approximation to the value of the solution. We say that the interval **brackets** the solution. Finding successively smaller bracketing intervals allows us to approximate the root to any needed precision. The Intermediate Value Theorem guarantees this works for continuous functions.

Example 5.4.20 The function $f(x) = x^3 - x - 3$ is continuous because it is a polynomial and defined everywhere. Because $f(1) = -3$ and $f(2) = 3$, we know that $f(x)$ must pass through every y-value between -3 and 3 for at least one value of x in the interval $(1, 2)$. In particular, if we are solving $f(x) = 0$, since $y = 0$ is between $f(1) = -3$ and $f(2) = 3$, we know that there is a solution x bracketed by the interval $[1, 2]$.
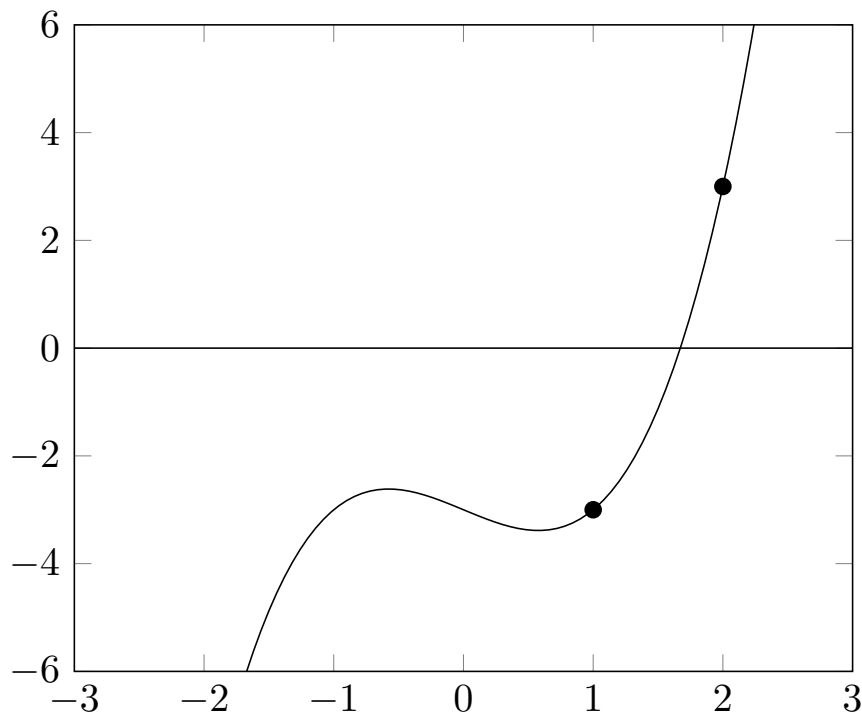
If we find a smaller interval, then we can know more precisely where the root occurs. In particular, since $f(1.6) = -0.504$ and $f(1.7) = 0.213$ and $y = 0$ is between those values, the Intermediate Value Theorem guarantees that our continuous function has a root bracketed by the interval $[1.6, 1.7]$. $\square$

The Intermediate Value Theorem is our justification for performing sign analysis by testing intervals at single points. If we have solved for all of the roots (zeros) and all of the discontinuities of a function f, then f can not change sign on any interval containing none of the roots or discontinuities. Suppose that $f(a)$ and $f(b)$ have opposite sign with $a < b$. Then $y = 0$ is between $f(a)$ and $f(b)$. If a and b were chosen from an interval with no discontinuities, f must be continuous on $[a, b]$. The Intermediate Value Theorem would then guarantee that $f(x) = 0$ has a solution with $a < x < b$. Because the interval contained no roots, $f(a)$ and $f(b)$ must not have had opposite signs. Thus, f can never change sign on an interval containing no roots or discontinuities.

5.4.6 Summary

- A function defined by an algebraic formula has discontinuities at every point for which the formula is undefined.

- A rational function is defined as the quotient of two polynomials. Discontinuities of rational functions only occur at the zeros of the denominator. If the numerator and denominator have a zero at the same location $x = c$, then $x - c$ is a common factor that can be cancelled.

- A limit of the form $\frac{0}{0}$ is indeterminate. For rational functions with a limit of this form, we must factor and simplify to continue. If the limit ultimately exists (as a number), the discontinuity is removable and the limit corresponds to having a hole in the graph.

- A rational function with a limit of the form $\frac{L}{0}$ where $L \neq 0$ has an infinite discontinuity. The graph of such a function has a vertical asymptote. The

left- and right-side limits have signs that are based on sign analysis of the function in the intervals to the left and right of the point of interest.

- A function is continuous on an interval if it continuous at every point in the interval. If an end point is included in the interval, the function must be one-sided continuous from the side contained in the interval.

- The Extreme Value Theorem guarantees that whenever a function f is continuous on a closed interval $[a, b]$, there are points in the interval where f reaches its maximum and minimum (extreme) values restricted to that interval.

- The Intermediate Value Theorem guarantees that whenever a function f is continuous on a closed interval $[a, b]$, the equation $f(x) = y$ has a solution with $a < x < b$ for any y between $f(a)$ and $f(b)$.

- The Intermediate Value Theorem guarantees that a function can only change sign at its roots or discontinuities.

5.4.7 Exercises

Compute each of the following limits. If the limit is infinite, state both left- and right-side limits.

1. $\displaystyle\lim_{x \to 3} \frac{2x - 6}{x - 3}$

2. $\displaystyle\lim_{x \to 3} \frac{2x}{x - 3}$

3. $\displaystyle\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$

4. $\displaystyle\lim_{x \to -1} \frac{x - 2}{x^2 - x - 2}$

5. $\displaystyle\lim_{x \to 2} \frac{x^2 - 2x - 8}{x^2 - 4}$

6. $\displaystyle\lim_{x \to -2} \frac{x^2 - 2x - 8}{x^2 - 4}$

7. $\displaystyle\lim_{x \to 4} \frac{x^2 - 2x - 8}{x^2 - 4}$

8. $\displaystyle\lim_{x \to 0} \frac{3x^2 - 5x + 2}{2x^2 - x - 1}$

9. $\displaystyle\lim_{x \to -\frac{1}{2}} \frac{3x^2 - 5x + 2}{2x^2 - x - 1}$

10. $\displaystyle\lim_{x \to 1} \frac{3x^2 - 5x + 2}{2x^2 - x - 1}$

Classify the discontinuities for each function, if any. State the limits at each discontinuity.

11. $f(x) = \dfrac{3}{x^2 - 5}$

12. $f(x) = \dfrac{2x}{x^2 + 3x}$

13. $f(x) = \dfrac{x^3 - x}{2x - 2}$

14. $f(x) = \dfrac{x^3 + 7x^2 + 12x}{x^2 + 3x}$

15. $f(x) = \begin{cases} 3x, & x < 1, \\ 2, & x = 1, \\ 4 - x^2, & x > 1. \end{cases}$

16. $f(x) = \begin{cases} \frac{3}{x-3}, & x < 2, \\ 2x - 7, & x > 2. \end{cases}$