## 6.1 Accumulation Functions and the Definite Integral

**Overview.** The concept of the **definite integral** can be motivated by the notion of accumulated change. When we learn about linear functions, the idea of a constant slope or rate of change serves as the fundamental concept. For a definite integral, we generalize this notion to a changing rate.

In this section, we begin with an example of linear functions and piecewise linear functions as models of accumulation. Using these examples, we establish some basic principles that we want to hold in general. These principles become the fundamental properties of the definite integral.

### 6.1.1 Linear Functions as Accumulation

The word accumulation is defined as "the acquisition or gradual gathering of something" (Oxford Dictionary, , accessed August 27, 2019). Consider a tank of water that has water added at a constant rate of $5\,\frac{\mathrm{L}}{\mathrm{min}}$. At the start of our observation, the tank contains $200\,\mathrm{L}$ of water. We wish to think of the amount of water in the tank as an accumulation of the water that has flowed into the tank as a function of time.

Quantities that have a constant rate of change are modeled with linear functions, and the rate of change is used as the slope. If $V$ is the volume of water that the tank contains (in liters) and $t$ is the time of observation (in minutes), then the state of the tank is given by $(t, V)$. The equation that models the accumulation is then given by

$$V = 200 + 5t.$$

The $V$-intercept of 200 represents the starting value (when $t = 0$), and the product $5t$ represents the accumulation of additional water that is added during the interval of time $(0, t)$.

The point-slope equation of a line similarly captures the idea of accumulation. Suppose after 10 minutes, the water flowing into the tank stops and water begins to drain at a rate of $15\,\frac{\mathrm{L}}{\mathrm{min}}$. We can use our earlier model to find the volume of water after the 10 minutes of filling has completed,

$$V = 200 + 5(10) = 250.$$

This becomes a new initial value for the tank relative to the draining, which corresponds to a negative rate of change or slope. For $t > 10$, we have a new model,

$$V = 250 - 15(t - 10).$$

The expression $-15(t - 10)$ represents the accumulated loss of water. We multiply the rate $-15$ by the *increment* of time $t - 10$, since that is how long the tank was left to drain.

Putting our models together, we obtain a piecewise function that represents the accumulation of water in the tank.

$$V = \begin{cases} 200 + 5t, & 0 \le t \le 10, \\ 250 - 15(t - 10), & t > 10. \end{cases}$$

We can think of the rate of accumulation $R$ as another variable, which is also piecewise,

$$R = \begin{cases} 5, & 0 < t < 10, \\ -15, & t > 10. \end{cases}$$

We do not define $R$ when $t = 10$ because of the ambiguity in how the transition occurs. Because $R$ is the rate of accumulation corresponding to the accumulation $V$, we write $R = V'$ (read $V$-prime). We will later learn that $R$ is the derivative of $V$ at points where $R$ is continuous.

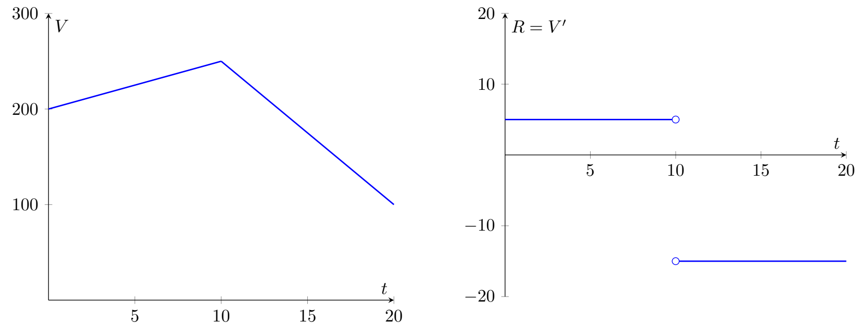



**Figure 6.1.1** The volume of the tank of water that fills for 10 minutes and then drains, and the corresponding rate of accumulation, as functions of time.

Given any piecewise constant rate of accumulation $f(x) = A'(x)$ for an accumulation $A(x)$, we can easily compute the formulas for $A(x)$ as a piecewise linear function. We repeatedly apply the point-slope equation of a line and require that $A(x)$ is continuous at each transition point. This will then help motivate some general properties that will relate to the definite integral.

**Example 6.1.2** Suppose $f(x) = A'(x)$ is defined as

$$f(x) = \begin{cases} 3, & x < 2, \\ -2, & 2 < x < 5, \\ 5, & x > 5. \end{cases}$$

If $A(0) = 2$, find $A(x)$ as a piecewise function.

**Solution**. Because the initial value is given as $A(0) = 2$, we begin our construction at $x = 0$. This point on the domain is inside the interval $x < 2$, so we start with a rate $A' = 3$. The formula for $A(x)$ with $x < 2$ is therefore

$$A(x) = 2 + 3x, \quad x < 2.$$

So that $A(x)$ is continuous, we must have $A(2) = 2 + 3(2) = 8$.

Having found the value of $A(x)$ on the interval $(-\infty, 2]$, we next consider the interval $(2, 5)$ where $A' = -2$. Using our value $A(2) = 8$ as an initial value, we can write

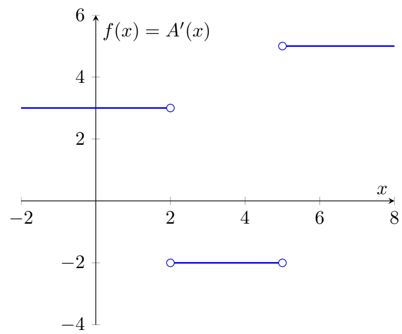
$$A(x) = 8 + -2(x - 2), \quad 2 < x < 5.$$

To have continuity at $x = 5$, we require $A(5) = 8 + -2(3) = 2$. Repeating the process on the last interval, $(5, \infty)$, where $A' = 5$, we obtain

$$A(x) = 2 + 5(x - 5), \quad x > 5.$$

Putting the pieces together, we obtain our final piecewise representation of $A(x)$:

$$A(x) = \begin{cases} 2 + 3x, & x \le 2, \\ 8 + -2(x - 2), & 2 < x \le 5, \\ 2 + 5(x - 5), & x > 5. \end{cases}$$



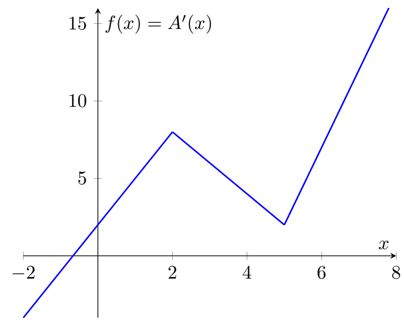




**Figure 6.1.3** Graphs of the piecewise constant rate of accumulation $f(x) = A'(x)$ and the piecewise linear accumulation $A(x)$.

□

### 6.1.2 A Geometric Interpretation of Accumulation

In our earlier example (Example 6.1.2), we had the point $A(0) = 2$ and a rate $f(x) = 3$ for $x < 2$. When we used the point-slope formula to find $A(2)$, we had

$$A(2) = 2 + 3(2) = 8.$$

Then, having found $A(2) = 8$ and knowing $f(x) = -2$ for $2 < x < 5$, we were able to compute $A(5)$ as

$$A(5) = 8 + -2(5 - 2) = 2.$$

In each case, we took a known starting value ($A(0) = 2$ or $A(5) = 8$) and then added an *increment of change*. With a constant rate of accumulation, these increments were calculated as the rate of change times the increment of change in the independent variable, $\Delta x$.

Expressing the increment of change as a product of two values has a useful geometric interpretation. The most basic geometric idea that is calculated as a product of two numbers is area. Can we interpret the increment of change as an area? Almost. An area is always a positive number, but our second increment of change $-2(3) = -6$ was a negative value. So we modify our idea to **signed area**.

How does the area geometrically appear? Consider the graph of the rate of accumulation, $y = f(x)$. The rate of change corresponds to the height of the graph from the axis. We should soon recognize that there are rectangles from which we can find the signed area. When the graph is below the axis, we have a signed height that is considered negative. When the graph is above the axis, the signed height is positive. The increment of $x$ depends on which direction we are going. We compute

$$\Delta x = x_{\text{end}} - x_{\text{start}}.$$

Consequently, if the increment moves to the right, we have $\Delta x > 0$; if the increment moves to the left, we have $\Delta x < 0$. The signed area is simply the product of the signed height times the signed increment of $x$.

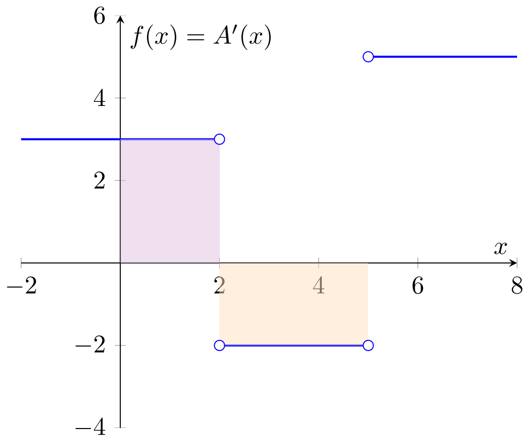



**Figure 6.1.4** A graph of the rate of accumulation function $y = f(x)$ showing the increments of change as signed areas, shaded in color. The increment as $x$ goes from $x = 0$ to $x = 2$ is shaded in purple. The increment as $x$ goes from $x = 2$ to $x = 5$ is shaded in orange.

We will generalize the idea of accumulation from constant rates of change to arbitrary rates of change using signed area. Suppose we are given the graph of a function $f(x)$ that is a rate of accumulation $f(x) = A'(x)$ for some quantity $A(x)$. We will require the function to be piecewise continuous and have no infinite limits. The increment of change for $A(x)$ as $x$ goes from $x = a$ to $x = b$ will be equal to the sum of the signed areas formed by the regions between $y = f(x)$ and the axis $y = 0$. This increment of change will be represented by a definite integral,

$$\Delta A = A(b) - A(a) = \int_a^b f(x)\,dx.$$

The notation for a definite integral is meant to be suggestive of this interpretation. The integral symbol $\displaystyle\int$ is drawn to look like the letter S to represent summation. The limits of integration $a$ and $b$ indicate the value for $x$ where we start on the bottom the the value for $x$ where we end on the top. What do we add? The increments $f(x)\,dx$ that are being accumulated. The expression $f(x)$ is the function giving the rate of accumulation and symbolically represents the signed height of incremental rectangles. The symbol $dx$ is called the **infinitesimal** and symbolically represents the signed increment of the independent variable or width of the rectangle. When $b > a$, we are integrating to the right and $dx > 0$; when $b < a$, we are integrating to the left (reverse) and $dx < 0$.

When the shape of the graph of $f(x)$ uses straight line segments or other simple geometric shapes, we can calculate the signed area using simple geometric formulas.

**Area Formulas for Common Geometric Regions.**

- Rectangle, length $\ell$ and width $w$.

$$A = \ell\, w$$

- Triangle, base $b$ and height $h$ (perpendicular to base).

$$A = \tfrac{1}{2} b\, h$$

- Parallelogram, base $b$ and height $h$ (perpendicular to base).
$$A = b\,h$$
- Trapezoid, parallel sides $b_1$ and $b_2$ and height $h$ (perpendicular to parallel sides).
$$A = \tfrac{1}{2}(b_1 + b_2)\,h$$
- Circle, radius $r$.
$$A = \pi r^2$$

We now have two different interpretations of the definite integral. First, the definite integral is the total accumulated change where the function in the integral is the rate of accumulation $f(x)$. Second, the definite integral is the sum of the signed areas between the graph of $f(x)$ and the axis. If we can calculate the definite integral, such as by geometric formulas for area, then we can interpret that value as the total accumulated change. This can allow us to compute additional values of the accumulation function:

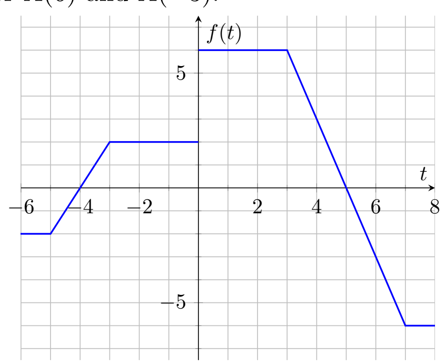
$$A(b) = A(a) + \int_a^b f(x)\,dx.$$

Notice how this equation for the accumulation function is similar to the point-slope equation of a linear function with slope $m$,

$$f(x) = f(a) + m(x-a).$$

There is an initial value. To that initial value, we add the increment of change. For a linear function, the rate of accumulation is constant and the increment is $m(x-a)$ For non-constant rates of accumulation, the increment of change is given by an integral.

**Example 6.1.5** Consider the graph of the function $f(t)$ shown below. Suppose that $f(t)$ is the rate of accumulation for $A(t)$, $f(t) = A'(t)$. If we know $A(2) = 5$, find the values for $A(6)$ and $A(-3)$.

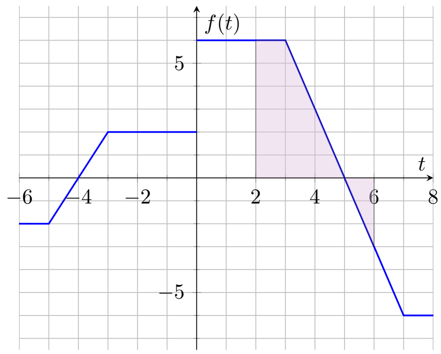



**Solution**. To find $A(6)$, we start at the known value $A(2) = 5$ and add the accumulated increment of change as $t$ goes from 2 to 6. That is, using the notation of a definite integral, we have

$$A(6) = A(2) + \int_2^6 f(t)\,dt.$$

We will calculate this definite integral using geometric shapes.

We identify our shapes by considering the regions vertically between the graph of $f(t)$ and the $t$-axis. At $t = 2$, the graph is above the axis and remains above the axis until $t = 5$. Our first region consists of a trapezoid bounded by $t = 2$ on the left, the $t$-axis below, and the graph of $f(t)$ above and to the right. From $t = 5$ to $t = 6$, the graph of $f(t)$ is below the axis. The second region consists of a triangle bounded by the $t$-axis on the top, the line $t = 6$ on the right, and the graph of $f(t)$ on the left. These shapes are illustrated in the graph below as shaded regions.




Having identified the relevant regions, we now calculate their signed area. We first recall that the direction of $t$ is left to right, so that horizontal signed lengths are positive. Vertical signed lengths depend on whether we are above (positive) or below (negative) the axis. The trapezoid between $t = 2$ and $t = 5$ has parallel bases of signed length 1 (top) and 3 (bottom) and a perpendicular height of 5 units. The resulting area for this trapezoid is

$$\text{area}_1 = \tfrac{1}{2}(1 + 3)(5) = 10.$$

The triangle has a base of signed length 1 and a height of signed length $-3$, with corresponding area

$$\text{area}_2 = \tfrac{1}{2}(1)(-3) = -\tfrac{3}{2}.$$

Note that we could instead have used a single rectangle for $t = 2$ to $t = 3$ and a triangle for $t = 3$ to $t = 5$ in place of the trapezoid.

The total accumulated increment of change is the sum of the signed areas,

$$\int_2^6 f(t)\,dt = 10 + -\tfrac{3}{2} = \tfrac{17}{2}.$$

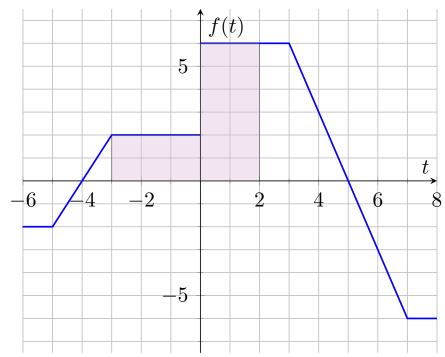
Consequently, we find

$$A(6) = A(2) + \int_2^6 f(t)\,dt = 5 + \tfrac{17}{2} = \tfrac{27}{2}.$$

To find $A(-3)$, we again start at the known value $A(2) = 5$ and add the accumulated increment of change as $t$ goes from 2 to -3. Using a definite integral, we have

$$A(-3) = A(2) + \int_2^{-3} f(t)\,dt.$$

Because $t$ is going backwards, our increments of $t$ will be negative. When we calculate geometric signed areas, illustrated in the graph below, our horizontal edges will have negative signed lengths.




This time, the regions consist of two rectangles. The signed area for the rectangle from $t = 2$ to $t = 0$, formed by a horizontal edge with signed length $-2$ and a vertical edge with signed length 6, is

$$\text{area}_1 = (-2)(6) = -12.$$

The signed area for the second rectangle from $t = 0$ to $t = -3$, formed by a horizontal edge with signed length $-3$ and a vertical edge with signed length 2, is

$$\text{area}_2 = (-3)(2) = -6.$$

The total accumulated increment is the sum,

$$\int_2^{-3} f(t)\,dt = -12 - 6 = -18.$$

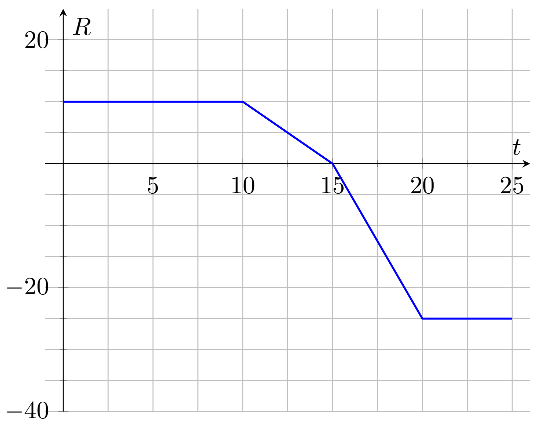
Consequently, we find

$$A(-3) = A(2) + \int_2^{-3} f(t)\,dt = 5 - 18 = -13.$$

□

**Example 6.1.6** A large tank of water initially contains 400 liters of water. For ten minutes, water is added at a constant rate of $10\,\frac{\text{L}}{\text{min}}$. The rate of water flow then steadily declines for the next five minutes from $10\,\frac{\text{L}}{\text{min}}$ to $0\,\frac{\text{L}}{\text{min}}$. At this point, a pump starts draining the tank, ramping its progress over five minutes so that the rate of draining goes from $0\,\frac{\text{L}}{\text{min}}$ to $25\,\frac{\text{L}}{\text{min}}$. The pump then drains water at this steady rate of $25\,\frac{\text{L}}{\text{min}}$ for another five minutes. How much water is in the tank at the end of this procedure?

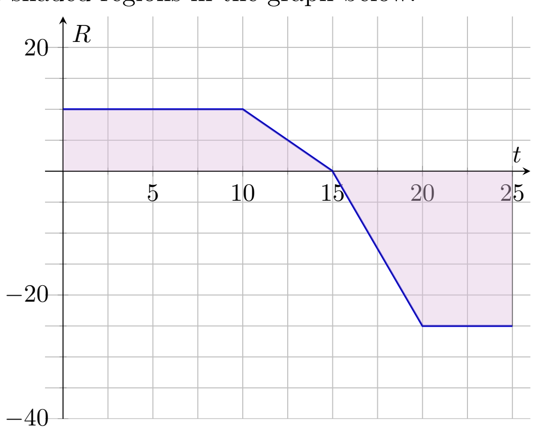
**Solution**. We start by describing the state variables. Let $V$ represent the volume of water in the tank, measured in liters. Let $R$ represent the rate of accumulation of water, which will be positive when water is flowing into the tank and negative when it is pumped out, measured in liters per minute. Let $t$ represent the time since the situation begins, measured in minutes. As we read the description of the problem, we should note that most of the information is describing the rate of accumulation $R = V'$ for the volume of water in the tank. We can sketch a graph of $R$ from the description, and we make the assumption that the description implies that this graph should be formed with line segments.




Now that we have a graph of the rate of accumulation $R(t) = V'(t)$, we can use geometric methods to calculate the area of regions to compute the total increment of change in the volume.

$$V(25) = V(0) + \int_0^{25} R(t)\,dt.$$

We consider the shaded regions in the graph below.




We could interpret the regions as either two trapezoids or as rectangles and triangles. The signed area of the trapezoid above the axis corresponding to times $t = 0$ to $t = 15$ is

$$\text{area}_1 = \tfrac{1}{2}(10 + 15)(10) = 125,$$

meaning that there were 125 liters added to the tank during the first 15 minutes. The signed area of the trapezoid below the axis corresponding to times $t = 15$ to $t = 25$ is

$$\text{area}_2 = \tfrac{1}{2}(10 + 5)(-25) = -\tfrac{375}{2} = -187.5.$$

This means that there were 187.5 liters drained from the tank during the last 10 minutes. Combining the results gives us the definite integral and overall increment of change in the volume of the tank. Starting with our initial tank level, we have

$$V(25) = V(0) + \int_0^{25} R(t)\,dt = 400 + (125 - 187.5) = 337.5.$$

The tank ends with 337.5 liters. □

### 6.1.3 Summary

- Piecewise constant rates of change correspond to continuous, piecewise linear functions.
- Given a function $f(x)$ that provides the rate of change for another function $A(x)$, the function $A$ is called the accumulation function with rate $f$ and the function $f$ is called the rate of accumulation for $A$. We write $f(x) = A'(x)$.
- The change in an accumation function $A(x)$ as $x$ goes from $x = a$ to $x = b$, calculated from $A$ by $A(b) - A(a)$, is represented by a **definite integral** of its rate of accumulation,

$$A(b) - A(a) = \int_a^b f(x)\,dx.$$

- The definite integral calculates a sum of increments, each represented by the product $f(x)\,dx$.
- The geometric interpretation of the definite integral is the sum of the increments of signed area of regions bounded by the graph of the rate of accumlation $f(x)$ and the $x$-axis.

### 6.1.4 Exercises

**1.**