6.2 Properties of Definite Integrals

Overview. Motivated by the properties of total accumulated change and of area, the definite integral inherits several significant properties. These properties are stated as theorems. We will be interested in applying the results of the theorems. However, to prove these properties is outside the scope of this text. We essentially think of these properties as the axioms of definite integrals, basic properties which must always be true.

Because accumulation functions are defined in terms of definite integrals, we also develop properties of accumulation functions in terms of our knowledge of the rate of accumulation. We will learn how the sign of the rate of accumulation determines if the accumulation function is increasing or decreasing. We will also learn how the concavity of the accumulation function is related to the rate of accumulation.

6.2.1 Integrability

Before we talk about the properties of the definite integral, we need to establish some terminology about when the definite integral is even defined. From our introduction, we know that the definite integral will be defined if we can describe the geometric region as a finite number of rectangles and triangles. Such shapes will occur for functions that are defined piecewise constant or piecewise linear. However, more complicated functions might have potential issues.

One of the most significant developments of modern mathematics was developing an understanding of when functions can be integrated or not. Mathematicians would take an interpretation of the definite integral and then construct bizarre functions for which that interpretation would break. Then they would create new definitions for integrals that worked over progressively more complex circumstances. For our purposes, we will focus on the definition of the integral using limits of Riemann sums.

Definition 6.2.1 Integrability. A function $f : [a, b] \to \mathbb{R}$ is **integrable** (or, more simply, integrable) on $[a, b]$ if $\displaystyle\int_a^b f(x)\,dx$ is defined. ♢

The actual interpretation of when the definite integral is defined is described in (((Unresolved xref, reference "section-riemann-sums"; check spelling or use "provisional" attribute))) . The scope of mathematics for this text is not concerned with determining which functions are or are not integrable, with one exception. Continuous functions are integrable.

Theorem 6.2.2 Continuous Functions are Integrable. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable on $[a, b]$.*

In fact, we get a result a little better than this. The function can have a finite number of discontinuities on the interval as long as left- and right-limits exist at all of the discontinuities. We do not allow infinite discontinuities at this stage of learning, as that will require a concept called improper integrals.

6.2.2 Splitting Properties

Consider any region in the plane for which we can find its area. Suppose we could cut the region, like we might cut a shape in two parts with scissors. The area of the original region would be the sum of the areas of the subregions created by our cut. This fact of geometric area motivates the **splitting** property of the definite integral.

Splitting properties are motivated by considering adjacent intervals, say $[a, b]$ and $[b, c]$, and requiring that the definite integral on $[a, c]$ is the sum of the integrals over the two pieces,

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

As described, this would seem to require that $a < b < c$. However, the definite integral is defined in a way that the order does not matter, so long as we replace our idea about intervals into directed excursions through an interval.

Recalling that the definite integral is motivated as the mathematical tool to compute the total change in a quantity as the accumulated change resulting from its rate of change, this result could be interpreted as saying, "The total change in Q as x goes from a to c is equal to the change in Q as x goes first from a to b plus the change as x then goes from b to c."

Theorem 6.2.3 Splitting Property of Definite Integrals. *Suppose that f is integrable on an interval that contains a, b and c. Then*

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$$

Similarly, if x does not change, then the dependent quantity Q should also not change, regardless of the function defining the rate of change. This is the motivation for the next theorem.

Theorem 6.2.4 Integral on an Empty Interval. *For any function f,*

$$\int_a^a f(x)\,dx = 0.$$

Combining these theorems, we obtain a reversal property of definite integrals. If we switch the order of the limits of integration, then the value of the definite integral must change sign.

Theorem 6.2.5 Integral in Reverse. *For any integrable function f,*

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Proof. Because an integral starting and ending at a must equal zero, if we go from a to b and then back, there must be no overall change:

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0.$$

This means the two integrals are additive inverses to each other. ■

When we interpret a definite integral as signed area, we must take into account the direction of integration. The usual integration over an interval $[a, b]$ is interpreted as going from $x = a$ to $x = b$. Regions above the axis generate positive signed area; regions below the axis generate negative signed area. However, if we were to reverse direction, going from $x = b$ to $x = a$, then the signs are reversed. This potential stumbling block can be mitigated if we remember to think of dx as having a sign as well. Integrals going from left to right result in $dx > 0$; integrals going from right to left have $dx < 0$.

Example 6.2.6 Suppose that we know $\displaystyle\int_0^4 f(x)\,dx = 6$ and $\displaystyle\int_3^4 f(x)\,dx = 10$. Find $\displaystyle\int_0^3 f(x)\,dx$.

Solution. We use the splitting property of definite integrals. The interval $[0, 4]$ can be split into $[0, 3]$ and $[3, 4]$ so that

$$\int_0^4 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^4 f(x)\,dx.$$

We know two of the integrals and can solve for the third:

$$6 = \int_0^3 f(x)\,dx + 10 \qquad \Leftrightarrow \qquad \int_0^3 f(x)\,dx = -4.$$

An alternate approach for finding the integral is to start with the integral that is wanted, using the interval $[0, 3]$, so that we start at 0 and end at 3. We will use the splitting property using out-of-order points and go from 0 to 4 and then from 4 to 3:

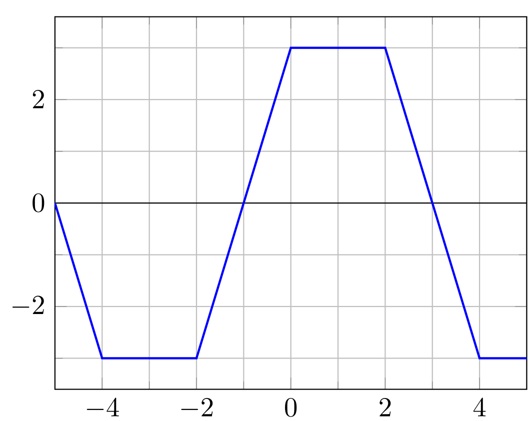
$$\int_0^3 f(x)\,dx = \int_0^4 f(x)\,dx + \int_4^3 f(x)\,dx.$$

The second integral is in a reversed order. If we switch the order to go left-to-right, then the integral is subtracted instead of added:

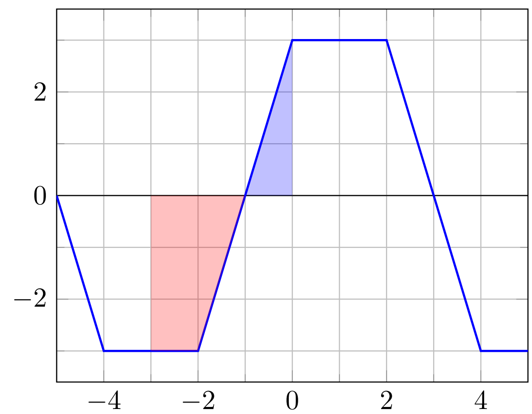
$$\int_0^3 f(x)\,dx = \int_0^4 f(x)\,dx - \int_3^4 f(x)\,dx = 6 - 10 = -4.$$

□

Example 6.2.7 Suppose that the graph below shows $y = f(x)$. Use the graph to find $\displaystyle\int_0^{-3} f(x)\,dx$.



Solution. Because the graph consists of straight lines, we can use geometry to calculate areas and use signed area to determine values of definite integrals. Shading the region between the graph $y = f(x)$ and the axis $y = 0$ and between $x = -3$ and $x = 0$, we get the figure shown below.



The region between $x = -3$ and $x = -1$ is a trapezoid that has area $\frac{1}{2}(1+2)(3) = \frac{9}{2}$. The region between $x = -1$ and $x = 0$ is a triangle with area $\frac{1}{2}(1)(3) = \frac{3}{2}$. Signed area corresponds to an integral from left-to-right so that

$$\int_{-3}^{0} f(x)\,dx = -\frac{9}{2} + \frac{3}{2} = -\frac{6}{2} = -3.$$

The integral of interest uses the opposite order, and so has the opposite sign:

$$\int_{0}^{-3} f(x)\,dx = -\int_{-3}^{0} f(x)\,dx = -(-3) = 3.$$

□

6.2.3 Summary

- **Integrability**: A function $f(x)$ is **integrable** on an interval if the definite integral $\displaystyle\int_a^b f(x)\,dx$ can be computed for any values a, b in the interval.

  We will need a clear definition for the definite integral to be more specific.

- Continuity implies integrability. If $f(x)$ is continuous on an interval, then it is guaranteed to also be integrable on that interval.

  In fact, so long as $f(x)$ is piecewise continuous with a finite number of discontinuities and $f(x)$ has one-sided limits at all of those discontinuities, then $f(x)$ will still be integrable. (The value at the end points don't matter to integrals.)

- **Splitting**: A definite integral $\displaystyle\int_a^b f(x)\,dx$ involves the independent variable x going through the values from $x = a$ (start) to $x = b$ (end). The splitting principle means that you can consider going from $x = a$ and take a diversion to any other point $x = c$, and then continue from $x = c$ to $x = b$ and get the same result,

  $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

  so long as $f(x)$ is integrable on an interval that includes all three points a, b, c.