

8.1 Extreme Values

We have learned earlier that when a function $f(x)$ can be written as an accumulation function, we can describe the behavior of the function in terms of its rate of accumulation $f'(x)$. We use sign analysis of $f'(x)$ to find the intervals of monotonicity of $f(x)$. And if $f'(x)$ can also be written as an accumulation function with rate $f''(x)$, the sign analysis of $f''(x)$ determines the intervals of concavity of $f(x)$.

In this section, we apply this information to describe the extreme values of a function. By identifying points where $f'(x)$ changes sign, we can find local maximum and minimum values. Global extreme values require comparing local extremes with end behaviors.

8.1.1 Local Extreme Values

When the derivative $f'(x)$ changes sign at a point where $f(x)$ is continuous, the function has a local or relative extreme value. We begin by focusing on what we mean by a local extreme value. A local extreme is a point where the function reaches either its highest or lowest point *on an interval around that point*. The function might exceed the value on some other interval, but the value needs to be the extreme in a neighborhood of the point.

Definition 8.1.1 Local (Relative) Extreme Values. A function $f(x)$ has a **local maximum** at a point $x = c$ in the domain if there is an interval (a, b) with $c \in (a, b)$ so that $f(x) \leq f(c)$ for all $x \in (a, b)$.

A function $f(x)$ has a **local minimum** at a point $x = c$ in the domain if there is an interval (a, b) with $c \in (a, b)$ so that $f(c) \leq f(x)$ for all $x \in (a, b)$. \diamond

The reason that the definition describes these extreme values as local extremes is that the function might go higher or lower at some point outside of the interval. A local or relative extreme value is only at the highest or lowest points relative to its immediate neighbors. The following graph illustrates a function with multiple local extreme values.

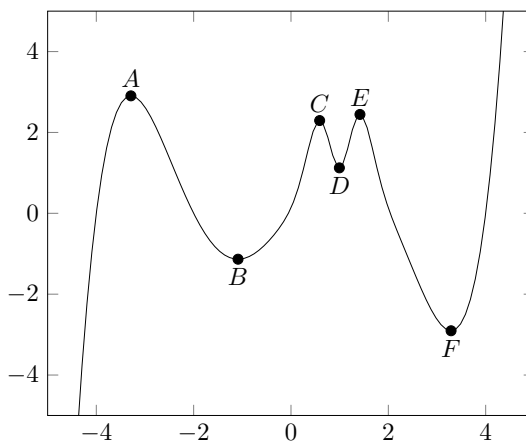


Figure 8.1.2 Illustration of a function with local extremes and no global extremes.

The function shows local maxima at points labeled A , C , and E and local minima at points labeled B , D , and F . Of the three maxima, the value at A is the greatest. Because the graph continues to increase after F , however, the function reaches values higher than all of the local maxima. Similarly, the

minimum at F is the lowest of the three local minima but the function reaches values even lower at points to the left of A .

One observation we should make is that the local extreme values occurred where the function transitioned between an interval of increasing to an interval of decreasing. Such points are called turning points. Sign analysis using the first derivative can often identify these turning points, so we use sign analysis to find local extreme values. The applicable theorem is called the first derivative test for extreme values.

Theorem 8.1.3 First Derivative Test. *Suppose that $f'(x)$ exists on an interval (a, b) , possibly except at $x = c$ with $a < c < b$ and that $f(x)$ is continuous at $x = c$.*

- If $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$, then $f(x)$ is decreasing on $(a, c]$ and increasing on $[c, b)$ so that f has a **local minimum** at $x = c$.
- If $f'(x) > 0$ for $x \in (a, c)$ and $f'(x) < 0$ for $x \in (c, b)$, then $f(x)$ is increasing on $(a, c]$ and decreasing on $[c, b)$ so that f has a **local maximum** at $x = c$.
- If $f'(x)$ does not change sign, then f does not have a local extreme value at $x = c$.

Because $f'(x)$ most frequently changes sign at points where $f'(x) = 0$, we call such points the **critical points** of $f(x)$. When we have a more precise definition of the derivative, we will learn that critical points also need to include points where $f'(x)$ is not defined.

Example 8.1.4 Find the local extreme values of $f(x) = x^3 - 6x^2 - 36x + 5$.

Solution. The first step in a question about extreme values is to compute the rate of change $f'(x)$.

$$f'(x) = 3x^2 - 12x - 36.$$

To apply the [First Derivative Test](#), we need to complete sign analysis. Because $f'(x)$ is defined and continuous everywhere, our critical points of f are the roots of $f'(x)$. We solve the equation by factoring:

$$3x^2 - 12x - 36 = 0$$

$$3(x^2 - 4x - 12) = 0$$

$$3(x - 6)(x + 2) = 0.$$

There are two roots of f' , $x = 6$ and $x = -2$, which are the critical points of f .

Next, we perform sign analysis using the roots as the end points of the test intervals, which are $(-\infty, -2)$, $(-2, 6)$, and $(6, \infty)$. Using the factored function $f'(x) = 3(x + 2)(x - 6)$ makes it easier to find the signs without necessarily computing the final value. We can just look at the signs of each factor:

$$f'(-3) = 3(-3 + 2)(-3 - 6) = (+)(-)(-) = +,$$

$$f'(0) = 3(0 + 2)(0 - 6) = (+)(+)(-) = -,$$

$$f'(8) = 3(8 + 2)(8 - 6) = (+)(+)(+) = +.$$

We now know f is *increasing* on $(-\infty, -2]$, *decreasing* on $[-2, 6]$, and *increasing* on $[6, \infty)$.

By the [First Derivative Test](#), we now know that f has a local maximum at $x = -2$ and a local minimum at $x = 6$. The y -coordinate of the local maximum

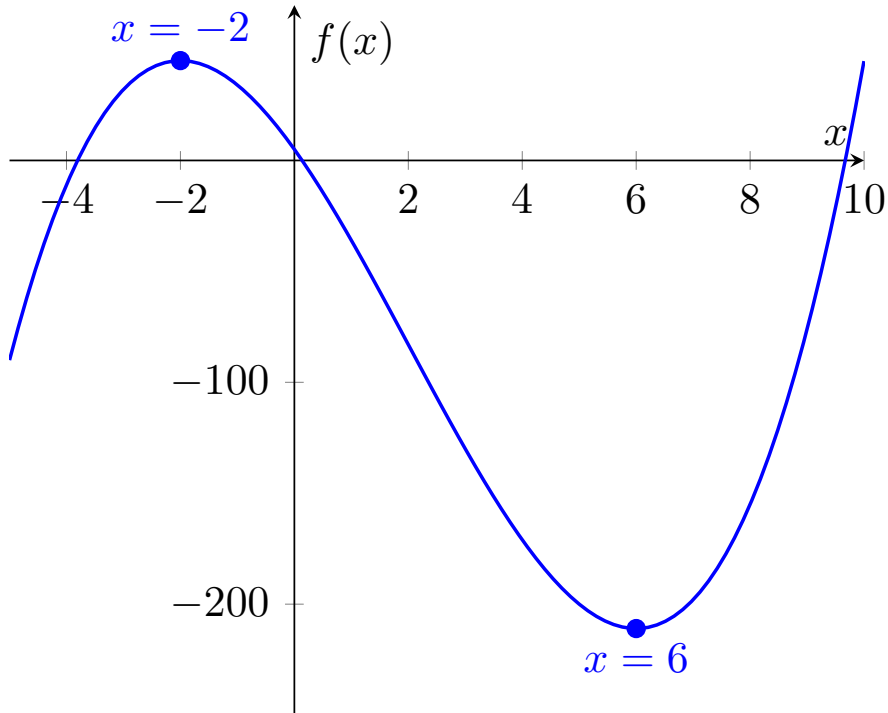
can be found using the formula for $f(x)$:

$$f(-2) = (-2)^3 - 6(-2)^2 - 36(-2) + 5 = 45.$$

This is guaranteed to be the maximum value over the interval $(-\infty, 6]$. The y -coordinate of the local minimum can also be found:

$$f(6) = (6)^3 - 6(6)^2 - 36(6) + 5 = -211,$$

which is guaranteed to be the minimum over the interval $[-2, \infty)$. If we wanted to graph this function and show the local extrema, we would know that our window would need to include the x -values of $x = -2$ and $x = 6$ as well these y -values.



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Although we do not yet know all of the rules that would allow us to compute derivatives, with the help of technology we can analyze many other functions.

Example 8.1.5 Use technology to find the derivative of the function

$$f(x) = \frac{x}{x^2 + 3}.$$

Then describe the local extreme values of $f(x)$.

Solution. In SageMath, we find the derivative formula using the `diff` command, which stands for the verb *differentiate*. The following script will define our function for SageMath and then ask it to show us the derivative where x is the independent variable.

```
# Define our function.
f(x)=x/(x^2+3)
# Show the derivative.
show( diff(f(x), x) )
```

$$-2x^2/(x^2+3)^2 + 1/(x^2+3)$$

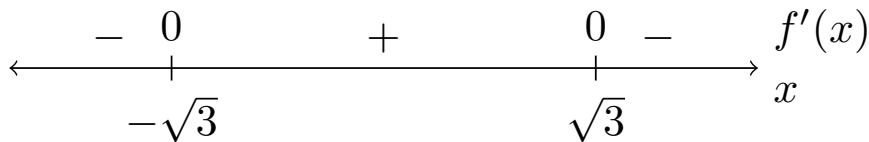
We see that $f(x)$ has a derivative

$$f'(x) = -\frac{2x^2}{(x^2+3)^2} + \frac{1}{x^2+3}.$$

We can simplify this if we get a common denominator.

$$\begin{aligned} f'(x) &= -\frac{2x^2}{(x^2+3)^2} + \frac{(x^2+3)}{(x^2+3)^2} \\ &= \frac{-2x^2 + x^2 + 3}{(x^2+3)^2} \\ &= \frac{3-x^2}{(x^2+3)^2} \end{aligned}$$

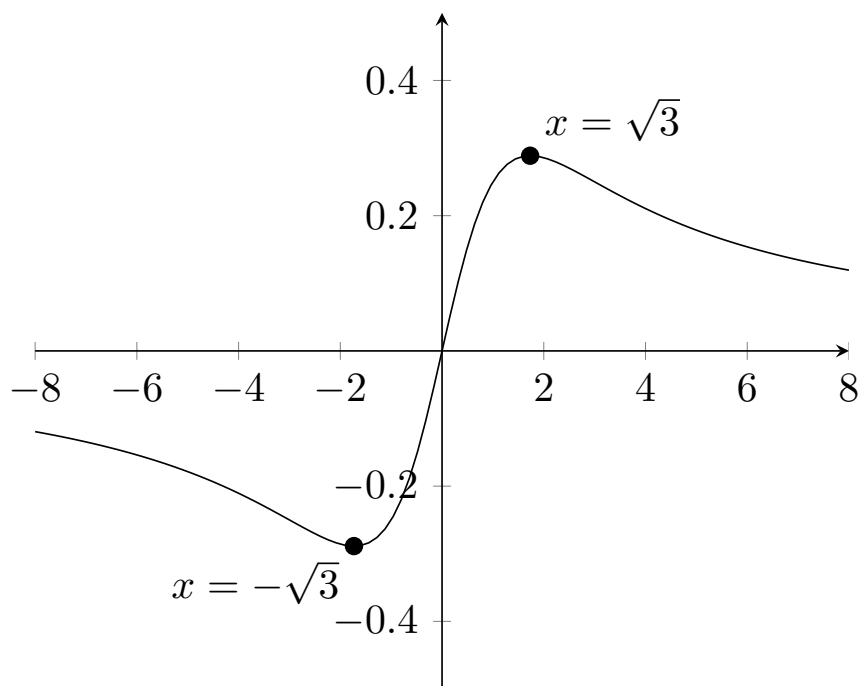
The denominator of $f'(x)$ is never zero because $x^2 + 3 \geq 3$ will never equal zero. So the sign can only change where $3 - x^2 = 0$ which occurs at two values, $x = \pm\sqrt{3}$. There are three intervals of interest to test, $(-\infty, -\sqrt{3})$, $(-\sqrt{3}, \sqrt{3})$, and $(\sqrt{3}, \infty)$. We can find the signs of $f'(x)$ using the values $x = -2$, $x = 0$ and $x = 2$. The signs are summarized in the number line summary below.



We finish by interpreting our results.

- Because $f'(x) < 0$ on $(-\infty, -\sqrt{3})$ and $f'(x) > 0$ on $(-\sqrt{3}, \sqrt{3})$, we know $f(x)$ has a local minimum at $x = -\sqrt{3}$. (Minimum over the interval $(-\infty, +\sqrt{3})$)
- Because $f'(x) > 0$ on $(-\sqrt{3}, \sqrt{3})$ and $f'(x) < 0$ on $(\sqrt{3}, \infty)$, we know $f(x)$ has a local maximum at $x = \sqrt{3}$. (Maximum over the interval $(-\sqrt{3}, \infty)$)

A graph of the function is illustrated below.



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Having discussed how the first derivative $f'(x)$ allows us to identify local extreme values of $f(x)$, we should note that the second derivative $f''(x)$ will allow us to identify local extreme values of $f'(x)$. These points are the **inflection points** of the function f , where the concavity of f changes. Inflection points are significant as being extreme values in that they represent points where the rate of accumulation or rate of change reaches either a maximum or minimum rate.

8.1.2 Global Extreme Values

In Figure 8.1.2, we saw that when a function has local extreme values, there could still be other points that are not local extremes that exceed the extremes. This leads to the idea of **global extreme values**.

Definition 8.1.6 Suppose the function f has domain D .

- f has a **global maximum** at $c \in D$ if $f(c) \geq f(x)$ for all $x \in D$.
- f has a **global minimum** at $c \in D$ if $f(c) \leq f(x)$ for all $x \in D$.

◇

To identify global extremes of a function, we first need to find all of the local extreme values. Then we use additional information to test whether the function manages to reach beyond those values. The sign analysis used to analyze local extrema does give us some information about the intervals immediately to the left and right of an extremum. For example, we know that a local maximum will be greater than all points in the immediately adjacent intervals, but we may not know how far down the function goes.

Finishing the analysis for global extremes generally involves computing limits of the function on intervals not already accounted for by the local extremes. If a limit has a real (finite) value, the function values approach that limit but may not actually reach the limit as an actual function value. When a function approaches a value in a limit that *would be* an extreme value but never reaches it, we call that value a **bound** rather than an extreme.

Example 8.1.7 Find the global extreme values of $f(x) = 4x^2 - x^3$ restricted to each of the following domains.

1. $D = (-\infty, \infty)$
2. $D_1 = [-1, 3]$
3. $D_2 = (-2, 4]$
4. $D_3 = [-2, 2)$

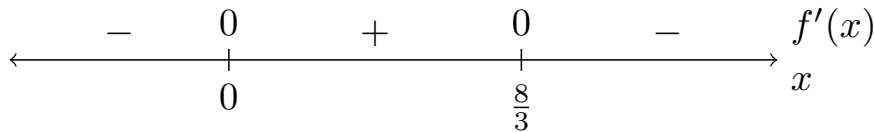
Solution. This question considers finding global extrema for a function when it is restricted to different domains. Regardless of which domain we use, we will need to find the local extreme values. Local extreme values are identified from sign analysis of $f'(x)$. We find

$$f'(x) = 4(2x) - (3x^2) = 8x - 3x^2.$$

Sign analysis begins by finding the roots, where $f'(x) = 0$.

$$\begin{aligned} 8x - 3x^2 &= 0 \\ x(8 - 3x) &= 0 \\ x = 0, \quad 8 - 3x &= 0 \\ x = 0, \quad x &= \frac{8}{3} \end{aligned}$$

Because $f'(x)$ is continuous, the roots determine the test intervals: $(-\infty, 0)$, $(0, \frac{8}{3})$, and $(\frac{8}{3}, \infty)$. Testing one value of x from each interval in $f'(x) = x(8 - 3x)$, we find the signs summarized on the following number line.



We can interpret the sign analysis of $f'(x)$ as characterizing the monotonicity of $f(x)$.

- $f(x)$ is decreasing on $(-\infty, 0)$.
- $f(x)$ is increasing on $(0, \frac{8}{3})$.
- $f(x)$ is decreasing on $(\frac{8}{3}, \infty)$.

Because $f(x)$ is decreasing on the left of $x = 0$ and increasing on the right, $f(x)$ has a *local minimum* at $x = 0$. The value of the function at this minimum is

$$f(0) = 4(0)^2 - (0)^3 = 0.$$

Similarly, because $f(x)$ is increasing on the left of $x = \frac{8}{3}$ and decreasing on the right, $f(x)$ has a *local maximum* at $x = \frac{8}{3}$. The value of the function at this maximum is

$$f\left(\frac{8}{3}\right) = 4\left(\frac{8}{3}\right)^2 - \left(\frac{8}{3}\right)^3 = \frac{256}{27} \approx 9.4815.$$

From our sign analysis of $f'(x)$, we know that $f(0)$ is the minimum value for the interval $(-\infty, \frac{8}{3}]$ and that $f(\frac{8}{3})$ is the maximum value for the interval $[\frac{8}{3}, \infty)$. To complete the analysis of global extreme values, we need use limits to compare $f(0)$ with points on the interval $(\frac{8}{3}, \infty)$ and $f(\frac{8}{3})$ with points on

the interval $(-\infty, 0)$. To evaluate limits involving $\pm\infty$, we need to factor out the dominant power,

$$f(x) = 4x^2 - x^3 = x^3\left(\frac{4}{x} - 1\right),$$

before using the [limit arithmetic of infinity](#).

$$\begin{aligned}\lim_{x \rightarrow -\infty} f(x) &= \lim_{x \rightarrow -\infty} x^3\left(\frac{4}{x} - 1\right) \\ &= (-\infty)^3 \cdot \left(\frac{4}{-\infty} - 1\right) \\ &= -\infty \cdot (0 - 1) = +\infty \\ \lim_{x \rightarrow +\infty} f(x) &= \lim_{x \rightarrow +\infty} x^3\left(\frac{4}{x} - 1\right) \\ &= (\infty)^3 \cdot \left(\frac{4}{\infty} - 1\right) \\ &= \infty \cdot (0 - 1) = -\infty\end{aligned}$$

Let us now address the question of the global extreme values on each of the requested restricted domains.

1. Find the global extremes on the interval $(-\infty, \infty)$.

We have learned that on the interval $(-\infty, 0)$, f reaches all values between $f(0) = 0$ and $\lim_{x \rightarrow -\infty} f(x) = \infty$. Clearly, $f(x)$ has *no maximum value* because it is unbounded above. Similarly, we learned that on the interval $(\frac{8}{3}, \infty)$, $f(x)$ reaches all values between $-\infty$ and $f(\frac{8}{3}) = \frac{256}{27}$ and is unbounded below. So $f(x)$ has *no minimum value*. The range of f has been shown to be $(-\infty, \infty)$ and f has no global extreme values on $(-\infty, \infty)$.

2. Find the global extremes on the interval $D_1 = [-1, 3]$.

We know f is decreasing on $[-1, 0)$, a subset of $(-\infty, 0)$, so the maximum value on that interval is

$$f(-1) = 4(-1)^2 - (-1)^3 = 5.$$

We also know f is decreasing on $(\frac{8}{3}, 3]$ so the minimum value on that interval is

$$f(3) = 4(3)^2 - (3)^3 = 9.$$

Comparing these to the local minimum $f(0) = 0$ and the local maximum $f(\frac{8}{3}) = \frac{256}{27}$, we see that $f(0) = 0$ is the *global minimum* and $f(\frac{8}{3}) = \frac{256}{27}$ is the *global maximum* for the interval $D_1 = [-1, 3]$. When restricted to this domain, the range of f becomes $[0, \frac{256}{27}]$.

3. Find the global extremes on the interval $D_2 = (-2, 4]$.

We know f is decreasing on $(-2, 0)$, so that the left end-point provides an upper bound

$$\lim_{x \rightarrow -2^+} f(x) = f(-2) = 4(-2)^2 - (-2)^3 = 24.$$

This is not a maximum value because $x = -2$ is not included in the domain. Because f is decreasing on $(\frac{8}{3}, 4]$ the minimum value on that interval is

$$f(4) = 4(4)^2 - (4)^3 = 0.$$

The global minimum occurs in two locations,

$$f(0) = f(4) = 0.$$

The value $f(-2) = 24$ is not a global maximum because $x = -2$ is not included in the domain. However, f does include all values up to that value through the limit so that f has an upper bound of 24. When f is restricted to $D_2 = (-2, 4]$, the range is $[0, 24)$.

4. Find the global extremes on the interval $D_3 = [-2, 2)$.

From the work above, we know $f(-2) = 24$ is the maximum value on $[-2, 0)$. Because the right end point $x = 2$ is to the left of the local maximum at $x = \frac{8}{3}$, we need to consider the interval of monotonicity $(0, 2)$. f is increasing on this interval and bounded above by the limit

$$\lim_{x \rightarrow 2^-} f(x) = f(2) = 4(2)^2 - 2^3 = 8.$$

Our work shows that f has a global minimum at $x = 0$ with $f(0) = 0$ and a global maximum at $x = -2$ with $f(-2) = 24$. The range of f restricted to $D_3 = [-2, 2)$ is $[0, 24)$.

The following figure illustrates the graph $y = f(x)$ restricted to each domain. Be sure to compare the analysis that identified the global extremes with the graphs.

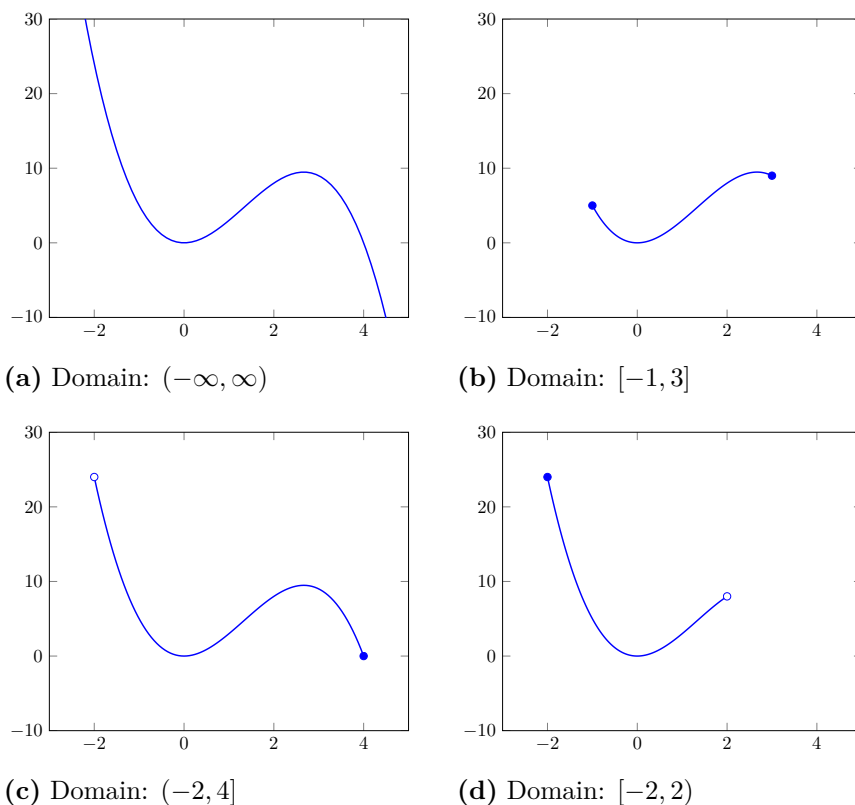


Figure 8.1.8

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The process to find global extrema is summarized as the following algorithm.

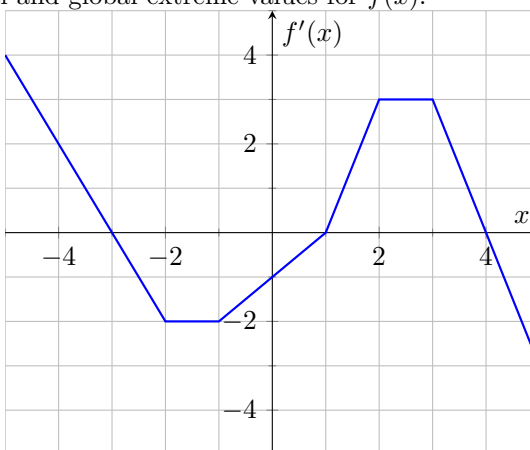
Algorithm 8.1.9 Finding Global Extreme Values. *To find the global extrema of a function $f(x)$ that is continuous on an interval:*

- Determine the derivative $f'(x)$ and perform sign analysis.
- Identify all of the local extreme values and compute the value of $f(x)$ at each extreme.
- Find extremes or bounds for $f(x)$ at the end points using values or limits, respectively.
- Identify the highest and lowest values out of the local extremes and the end points.
- If an extreme value occurs at a point in the interval, that value is a global maximum/minimum. If an extreme value occurs as a limit at an excluded end point, that value is a bound but not a global extremum.

8.1.3 Extreme Values involving Accumulation

In our examples above, we worked with functions with explicit formulas. However, most steps in the analysis involved only knowing information about the derivative. Here we consider examples where the derivative is given as the rate of accumulation and we do not know the explicit formula for the function in question.

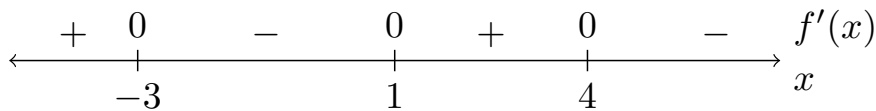
Example 8.1.10 Suppose f is defined as an accumulation function with an initial value $f(0) = 10$ and a rate of accumulation f' shown in the graph below. We assume the graph continues as shown outside of the viewing window. Determine local and global extreme values for $f(x)$.



Solution. Our function of interest is defined by an accumulation

$$f(x) = 10 + \int_0^x f'(z) dz.$$

We can find local extrema in the same way as before. However, instead of solving an equation $f'(x) = 0$, we can look at the graph to both find the roots and the signs of f' . The graph crosses the x -axis at $x = -3$, $x = 1$ and $x = 5$. The signs of $f'(x)$ are identified in the following sign analysis number line summary.



The [Theorem 8.1.3](#) allows us to conclude that $f(x)$ has a local maximum at $x = -3$, a local minimum at $x = 1$, and another local maximum at $x = 4$. It is possible to decide which maximum has a higher value by considering the signed area of the graph. In particular, because $f'(x)$ has linear segments, we can compute the areas in question using elementary geometry to find

$$\int_{-3}^1 f'(x) dx = -5,$$

$$\int_1^4 f'(x) dx = 6.$$

Using the splitting property of definite integrals, this implies

$$f(4) - f(-3) = \int_{-3}^4 f'(x) dx = -5 + 6 = 1.$$

Consequently, $f(4) = f(-3) + 1$ and f has a higher value at $x = 4$ than at $x = -3$.

We can find actual values if we use the initial value,

$$f(-3) = 10 + \int_0^{-3} f'(z) dz.$$

Because the integral goes right to left, we have $dz < 0$ and the signed area will be negated. Again using geometry, we find

$$f(-3) = 10 + -(-4.5) = 14.5.$$

Using this point and the integrals above, we can quickly find

$$f(1) = f(-3) + \int_{-3}^1 f'(z) dz = 14.5 - 5 = 9.5,$$

$$f(4) = f(1) + \int_1^4 f'(z) dz = 9.5 + 6 = 15.5.$$

To find global extrema, we need to think about what happens to the left and right of these local extrema. The sign analysis of $f'(x)$ shows that f is increasing on $(-\infty, -3)$. Thus, $f(-3)$ is the maximum value on $(-\infty, 3]$. The value of f is unbounded below on this interval,

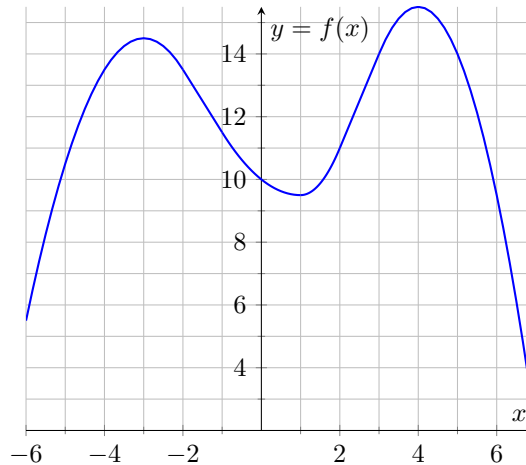
$$\lim_{x \rightarrow -\infty} f(x) = -\infty.$$

We can see this by considering the integral \int_{-3}^x for $x < -3$. Because the integral goes right to left, $dz < 0$, and $f'(z) > 0$ on this interval, the integral becomes more and more negative the further x goes to the left. With a similar argument using $\int_4^x f'(z) dz$, we find

$$\lim_{x \rightarrow \infty} f(x) = -\infty$$

and f is again unbounded below on the interval $(4, \infty)$.

We conclude that f has a global maximum $f(4) = 13.5$ and no global minimum. A graph of $y = f(x)$ is shown below.



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8.1.4 Summary

- A function f has a local maximum (plural: maxima) at $x = c$ if $f(c) \geq f(x)$ in a neighborhood of c . The function f has a local minimum (plural: minima) at $x = c$ if $f(c) \leq f(x)$ in a neighborhood of c .
- A function f has a global maximum at $x = c$ if $f(c) \geq f(x)$ for all x in the domain. The function f has a global minimum at $x = c$ if $f(c) \leq f(x)$ for all x in the domain.
- We can use sign analysis of the derivative f' to find local extreme values. Points where $f'(x)$ changes sign are local extrema. This is called the [First Derivative Test](#).
- Global extrema can occur at local extrema or at boundaries of intervals. We need to compare the value of the function at each of the local extrema with the end points. If an end point is not included, the value of the limit at that point can serve as a bound for the function but would not be an actual extreme value.

8.1.5 Exercises

For each function, identify all local extrema.

1. $f(x) = x^3 - 9x^2 - 48x + 60$
2. $f(x) = 120x + 3x^2 - 2x^3$
3. $f(x) = x^4 - 8x^2$
4. $f(x) = x^4 - 4x^3 + 3x^2 + 2$

For each function, find the global extrema on the given intervals.

5. $f(x) = 4x - x^2$ on (i) $D = [-1, 6]$, (ii) $D = (1, 4)$, and $(-\infty, \infty)$
6. $f(x) = x^2 + 3x$ on (i) $D = [-1, 2]$, (ii) $D = (-4, 1]$, and $(-\infty, \infty)$
7. $f(x) = x^3 - 9x^2 - 48x + 60$ on (i) $D = [-5, 5]$, (ii) $D = [-10, 10]$, and $(-\infty, \infty)$

8. $f(x) = x^4 - 12x^3 + 28x^2 - 17$ on (i) $D = [-1, 3]$, (ii) $D = (1, 8)$, and $(-\infty, \infty)$