9.1 Derivative Rules

We have learned that the derivative is defined by the limit of an average rate of change as the gap between the two points goes to zero. For functions already defined as an accumulation function with a known, continuous rate of accumulation, the Fundamental Theorem of calculus guarantees that the derivative equals the rate of accumulation. Every time we need a derivative of any other function, we must use the definition and compute the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

and then go through the algebra and simplification to find the resulting formula.

It would be much nicer if we could look at the formula of $f(x)$ and know what the formula of the derivative $f'(x)$ should be. Computing the formula for $f'(x)$ based on the structure of the formula for $f(x)$ is a process called **differentiation**. Rules for derivatives will provide us with a methodical way to differentiate algebraic formulas.

Differentiation is a process of taking a function and using it to determine another function. That is, differentiation defines a map between functions, $f \mapsto f'$. The entire function $f(x)$, not just a value, is the input to the map and an entirely different function $f'(x)$ is the output. Maps that take numbers as inputs and give numbers as outputs are called **functions**; a map that takes an entire function as an input is called an **operator**. When the independent variable is x, the symbol for the differential operator is $\frac{d}{dx}$. The x in the symbol is replaced by the appropriate independent variable for the function of interest.

Definition 9.1.1 The differential operator $\frac{d}{dx}$ takes a function as its input and provides the derivative function as its output,

$$\frac{d}{dx}[f(x)] = \frac{df}{dx}(x) = f'(x).$$

If y is a dependent variable defined by a function $y = f(x)$, then we can also write

$$\frac{dy}{dx} = \frac{d}{dx}[y] = f'(x).$$

$\diamondsuit$

In this section, we establish some elementary rules of differentiation. The rules of differentiation begin with linearity, matching the corresponding properties of definite integrals. The similarity ends here. Differentiation also has rules for multiplication and division, where the definite integral has no such rules. Differentiation goes even further and has a rule for function composition, called the chain rule. Each rule is justified by returning to the definition of the derivative using a limit of the difference quotient that represents an average rate of change.

9.1.1 Derivative Building Blocks

In order to differentiate algebraic formulas, we need to know the derivatives of elementary functions that will be our building blocks. Because we have the Fundamental Theorem of Calculus, any rate of accumulation that we know for an accumulation function is automatically going to be a derivative. However, it is also useful to show derivatives of elementary functions directly.

Theorem 9.1.2 Derivative of a Constant. *For a constant k,*

$$\frac{d}{dx}[k] = 0.$$

Proof. The function of interest is $x \mapsto k$, or $f(x) = k$. Using the definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[k] &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{k-k}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0.\end{aligned}$$

The last step in this sequence of equations is a consequence of the Limit Rule for a Constant. ■

Theorem 9.1.3 Derivative of the Identity. *For the identity function $f(x) = x$,*

$$\frac{d}{dx}[x] = 1.$$

Proof. Using the definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[x] &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{x+h-x}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1.\end{aligned}$$

Again, the limit calculation at the last step uses the Limit Rule for a Constant. ■

The identity function and constant functions are special cases of linear functions.

Theorem 9.1.4 Derivative of Linear Functions. *For a linear function $f(x) = mx + b$,*

$$\frac{d}{dx}[mx+b] = m.$$

Proof. Using the definition of the derivative,

$$\begin{aligned}\frac{d}{dx}[mx+b] &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(m(x+h)+b) - (mx+b)}{h} \\ &= \lim_{h\to 0} \frac{mx+mh+b-mx-b}{h} \\ &= \lim_{h\to 0} \frac{mh}{h} \\ &= \lim_{h\to 0} m \\ &= m.\end{aligned}$$

■

To establish additional derivatives of elementary formulas, we need now to develop rules that are based on the algebra of combining other formulas.

9.1.2 Overview of the Derivative Rules

Derivative rules are theorems that take as a hypothesis that one or two functions have known derivatives and the conclusion tells how to find the derivative of some combination of those functions. We start by stating the basic rules together for convenience in finding them.

Theorem 9.1.5 Differentiation Rules. *This theorem is a collection of multiple theorems. The hypothesis for any statement that involves $f(x)$ or $g(x)$ is that $\frac{d}{dx}[f(x)] = f'(x)$ or that $\frac{d}{dx}[g(x)] = g'(x)$. Further, k is assumed to be a constant.*

Table 9.1.6 Summary of the Differentiation Rules

$\frac{d}{dx}[k \cdot f(x)] = k \cdot f'(x)$	*Constant Multiple Rule*
$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$	*Sum Rule*
$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$	*Difference Rule*
$\frac{d}{dx}[\frac{1}{g(x)}] = \frac{-g'(x)}{(g(x))^2}$	*Reciprocal Rule*
$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$	*Product Rule*
$\frac{d}{dx}[\frac{f(x)}{g(x)}] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$	*Quotient Rule*
$\frac{d}{dx}[f\Big(g(x)\Big)] = f'\Big(g(x)\Big) \cdot g'(x)$	*Chain Rule*

One of these differentiation rules, the chain rule, will require its own section. That rule is focused on how to differentiate compositions of functions. The other rules focus on arithmetic combinations of functions and are the primary focus of this section. The chain rule was included for completeness in the listing of differentiation rules.

The proofs for these differentiation rules are based on applying the definition of a derivative to the formula in question while knowing that the limits that define the derivatives in the hypothesis are valid. To illustrate, we will look at four of the differentiation rules in detail.

9.1.3 Proofs of Algebraic Differentiation Rules

In each of our proofs for derivative rules, we are going to use the definition of the derivative for the function defined by the conclusion of the rule. We then use algebra to rewrite the difference quotient in a way that the divison by h (and recall $h \to 0$) only appears in expressions where a ratio converges to a known derivative.

In the proofs, in order to keep the algebra a little cleaner, we will use the notation Δf to represent

$$\Delta f = f(x+h) - f(x)$$

and similarly define

$$\Delta g = g(x+h) - g(x).$$

Then, because $f'(x)$ and $g'(x)$ are the derivatives of $f(x)$ and $g(x)$, respectively, we can substitute the following limits

$$\lim_{h \to 0} \frac{\Delta f}{h} = f'(x) \quad \text{and} \quad \lim_{h \to 0} \frac{\Delta g}{h} = g'(x).$$

The first rule we consider is the constant multiple rule. This rules states that if we know how to differentiation a function, then we can compute any constant multiple of that function by multiplying the derivative by the same constant.

Theorem 9.1.7 Constant Multiple Rule for Derivatives. *If* $\frac{d}{dx}[f(x)] = f'(x)$ *and* k *is a constant, then* $\frac{d}{dx}[kf(x)] = kf'(x)$.

Proof. The rule is interested in finding the rate of change of a new function $k \cdot f(x)$ knowing that $\frac{d}{dx}[f(x)] = f'(x)$. We begin by stating the definition of the derivative of the function $x \mapsto k \cdot f(x)$, and then we use algebra to factor k out as a common factor in the numerator:

$$\begin{aligned}\frac{d}{dx}[k \cdot f(x)] &= \lim_{h \to 0} \frac{k \cdot f(x+h) - k \cdot f(x)}{h} \\ &= \lim_{h \to 0} \frac{k(f(x+h) - f(x))}{h}. \\ &= \lim_{h \to 0} k \cdot \frac{\Delta f}{h}.\end{aligned}$$

Now notice that the formula is a product of the constant k and the average rate of change of f. The rules for limits act with the ordinary rules of arithmetic. In particular, the constant multiple rule for limits states that the limit of a constant times a function equals that constant times the limit of the function. Consequently, we have

$$\begin{aligned}\frac{d}{dx}[k \cdot f(x)] &= \lim_{h \to 0} k \frac{\Delta f}{h} \\ &= k \cdot \lim_{h \to 0} \frac{\Delta f}{h} \\ &= k \cdot f'(x).\end{aligned}$$

■

For our second example of proving a differentiation rule, we consider the reciprocal of a function. We know that $\frac{d}{dx}[x^2 + 3] = 2x$. This reciprocal rule will tell us how to compute the derivative $\frac{d}{dx}[\frac{1}{x^2+3}]$. We might be tempted to think the answer would be $\frac{1}{2x}$, but this is not correct. Derivatives do not follow simple rules for either division or multiplication.

Theorem 9.1.8 Reciprocal Rule for Derivatives. *If* $\frac{d}{dx}[g(x)] = g'(x)$, *then* $\dfrac{d}{dx}[\dfrac{1}{g(x)}] = \dfrac{-g'(x)}{(g(x))^2}$.

Proof. By hypothesis, $\dfrac{d}{dx}[g(x)] = g'(x)$. This means that $g'(x)$ is defined by its limit

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \to 0} \frac{\Delta g}{h} = g'(x).$$

The rule is interested in finding the rate of change of a new function $x \mapsto \dfrac{1}{g(x)}$. We will use that function to compute the derivative using the definition, which will require finding a common denominator.

$$\begin{aligned}\frac{d}{dx}[\frac{1}{g(x)}] &= \lim_{h \to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} \\ &= \lim_{h \to 0} \frac{\frac{g(x)}{g(x)g(x+h)} - \frac{g(x+h)}{g(x)g(x+h)}}{h}\end{aligned}$$

$$= \lim_{h\to 0} \frac{g(x) - g(x+h)}{g(x)g(x+h)} \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \frac{-\Delta g}{g(x)g(x+h)} \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \frac{-1}{g(x)g(x+h)} \cdot \frac{\Delta g}{h}$$

Since the limit involves $h \to 0$, $g(x)$ is a constant relative to the limit. In addition, because a differentiable function is continuous, we have $g(x+h) \to g(x)$ as $x \to h$. Consequently, the limit rules for reciprocals and constant multiples imply

$$\lim_{h\to 0} \frac{-1}{g(x)g(x+h)} = \frac{-1}{(g(x))^2}.$$

Using the limit rule for a product, we have

$$\frac{d}{dx}[\frac{1}{g(x)}] = \lim_{h\to 0} \frac{-1}{g(x)g(x+h)} \cdot \frac{g(x+h) - g(x)}{h}$$
$$= \frac{-1}{(g(x))^2} \cdot g'(x) = \frac{-g'(x)}{(g(x))^2}.$$

■

Example 9.1.9 Find $\dfrac{d}{dx}[\dfrac{1}{x^2+3}]$.

Solution. We start by recognizing the formula $\frac{1}{x^2+3}$ as the reciprocal of $g(x) = x^2 + 3$. We know $g'(x) = 2x$, so the Theorem 9.1.8 gives

$$\frac{d}{dx}[\frac{1}{x^2+3}] = \frac{-g'(x)}{(g(x))^2}$$
$$= \frac{-2x}{(x^2+3)^2}.$$

That is, if $f(x) = \frac{1}{x^2+3}$, the derivative is $f'(x) = \frac{-2x}{(x^2+3)^2}$. □

In the solution to the previous example, we introduced a name for a function for the sole reason of being able to refer to its derivative. This is one of the primary reasons for introducing the differentiation operator $\frac{d}{dx}$. It allows us to refer to derivatives using the operator with the original function as input. The work in the example could be rewritten

$$\frac{d}{dx}[\frac{1}{x^2+3}] = \frac{-\frac{d}{dx}[x^2+3]}{(x^2+3)^2}$$
$$= \frac{-2x}{(x^2+3)^2}.$$

The two rules of differentiation proved thus far involve operations on a single function. We now turn our attention to rules that combine multiple functions. The first rule we consider is the sum rule, which states that the derivative of a function formed by adding two functions will be the sum of those functions' derivatives.

Theorem 9.1.10 Sum Rule for Derivatives. *If $\frac{d}{dx}[f(x)] = f'(x)$ and $\frac{d}{dx}[g(x)] = g'(x)$, then $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$.*

Proof. By hypothesis, $\dfrac{d}{dx}[f(x)] = f'(x)$ and $\dfrac{d}{dx}[g(x)] = g'(x)$. This means that

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\Delta f}{h} = f'(x),$$
$$\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} \frac{\Delta g}{h} = g'(x).$$

The sum rule is interested in finding the rate of change of a new function $x \mapsto f(x)+g(x)$. Because $\Delta f = f(x+h)-f(x)$, we can rewrite $f(x+h) = f(x)+\Delta f$. Similarly, we can rewrite $g(x+h) = g(x)+\Delta g$. When we use the definition of the derivative, we find

$$\begin{aligned} \frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\ &= \lim_{h\to 0} \frac{f(x)+\Delta f+g(x)+\Delta g-f(x)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{\Delta f+\Delta g}{h} \\ &= \lim_{h\to 0} \left[\frac{\Delta f}{h}+\frac{\Delta g}{h}\right]. \end{aligned}$$

Again, because limit rules satisfy the ordinary rules of arithmetic, the limit rule for sums implies

$$\begin{aligned} \frac{d}{dx}[f(x)+g(x)] &= \lim_{h\to 0} \left[\frac{\Delta f}{h}+\frac{\Delta g}{h}\right] \\ &= \lim_{h\to 0} \left[\frac{\Delta f}{h}\right] + \lim_{h\to 0} \left[\frac{\Delta g}{h}\right] \\ &= f'(x)+g'(x). \end{aligned}$$

■

For our final example of a proof of a differentiation rule, we consider the derivative of a product. Consider a function like $f(x) = (x^2-3)\cdot(2x+1)$. It is tempting to take derivatives of each formula in place and assume that $f'(x)$ would be $(2x)\cdot(2) = 4x$. We can see that this is incorrect if we rewrote $f(x)$ after expanding the product,

$$f(x) = 2x^3+x^2-6x-3.$$

Once written as a simple polynomial, our experience with accumulation functions and the Fundamental Theorem of Calculus allows us to recognize

$$f'(x) = 6x^2+2x-6.$$

Proper differentiation rules will be consistent regardless of how a function is represented. For a function that is represented as a product of two other functions, the product rule shows that the derivative is a sum of contributions resulting from the rate of change of each factor.

Theorem 9.1.11 Product Rule for Derivatives. *If* $\frac{d}{dx}[f(x)] = f'(x)$ *and* $\frac{d}{dx}[g(x)] = g'(x)$, *then* $\frac{d}{dx}[f(x)\cdot g(x)] = f'(x)\cdot g(x)+f(x)\cdot g'(x)$.

Proof. By hypothesis, $\dfrac{d}{dx}[f(x)] = f'(x)$ and $\dfrac{d}{dx}[g(x)] = g'(x)$. This means that

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\Delta f}{h} = f'(x),$$

$$\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{\Delta g}{h}=g'(x).$$

The product rule is interested in finding the rate of change of a new function $x\mapsto f(x)g(x)$. As we did in the sum rule, we will take advantage of rewriting $f(x+h)=f(x)+\Delta f$ and $g(x+h)=g(x)+\Delta g$. When $f'(x)$ and $g'(x)$ both exist, f and g are both continuous so that

$$\begin{aligned}\lim_{h\to 0}\Delta f&=\lim_{h\to 0}f(x+h)-f(x)=0,\\ \lim_{h\to 0}\Delta g&=\lim_{h\to 0}g(x+h)-g(x)=0.\end{aligned}$$

The derivative in question is defined by

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)]&=\lim_{h\to 0}\frac{[f(x+h)g(x+h)]-[f(x)g(x)]}{h}\\ &=\lim_{h\to 0}\frac{(f(x)+\Delta f)(g(x)+\Delta g)-f(x)g(x)}{h}\\ &=\lim_{h\to 0}\frac{f(x)g(x)+\Delta f g(x)+f(x)\Delta g+\Delta f\Delta g-f(x)g(x)}{h}\\ &=\lim_{h\to 0}\frac{\Delta f g(x)+f(x)\Delta g+\Delta f\Delta g}{h}\\ &=\lim_{h\to 0}\left[\frac{\Delta f g(x)}{h}+\frac{f(x)\Delta g}{h}+\frac{\Delta f\Delta g}{h}\right]\\ &=\lim_{h\to 0}\left[\frac{\Delta f}{h}\cdot g(x)+f(x)\cdot\frac{\Delta g}{h}+\Delta f\cdot\frac{\Delta g}{h}\right].\\ &=f'(x)\cdot g(x)+f(x)\cdot g'(x)+0\cdot g'(x)\\ &=f'(x)\cdot g(x)+f(x)\cdot g'(x),\end{aligned}$$

using the Limit Rule of a Sum and the Limit Rule of a Product. ■

Example 9.1.12 Show that the derivative using the product rule for $\frac{d}{dx}[(x^2-3)\cdot(2x+1)]$ is consistent with first expanding and then differentiating the polynomial.

Solution. The function $f(x)=(x^2-3)\cdot(2x+1)$ is a product of $u=x^2-3$ and $v=2x+1$. The product rule informs us that $\frac{d}{dx}[uv]=\frac{du}{dx}\,v+u\,\frac{dv}{dx}$:

$$\begin{aligned}\frac{d}{dx}[(x^2-3)\cdot(2x+1)]&=\frac{d}{dx}[x^2-3]\cdot(2x+1)+(x^2-3)\cdot\frac{d}{dx}[2x+1]\\ &=(2x)\cdot(2x+1)+(x^2-3)\cdot(2)\\ &=4x^2+2x+2x^2-6\\ &=6x^2+2x-6\end{aligned}$$

We saw prior to the theorem that $f(x)=(x^2-3)(2x+1)=2x^3+x^2-6x-3$ has a derivative $f'(x)=6x^2+2x-6$, which is consistent with the result we obtained using the product rule. □

The quotient rule is a combination of the product rule and the reciprocal rule.

Theorem 9.1.13 Quotient Rule for Derivatives. *If* $\frac{d}{dx}[f(x)]=f'(x)$ *and* $\frac{d}{dx}[g(x)]=g'(x)$, *then* $\displaystyle\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$.

9.1.4 Applying the Rules to Formulas

In this section, we have established rules of differentiation for elementary formulas and for algebraic combinations of functions with known derivatives. We now consider how we can apply these rules together to compute derivatives from more complex formulas.

To illustrate how functions are formed as combinations of elementary functions, let us revisit the derivative of a linear function, $f(x) = mx + b$. This function is algebraically the *sum* of the expressions mx and b. Consequently, the derivative will be the sum of the derivatives of those expressions. The first, mx is a constant multiple of the identity function, so

$$\frac{d}{dx}[mx] = m\frac{d}{dx}[x] = m \cdot 1 = m.$$

The second, b is a constant function, so

$$\frac{d}{dx}[b] = 0.$$

Adding these together, we find

$$\frac{d}{dx}[mx + b] = \frac{d}{dx}[mx] + \frac{d}{dx}[b] = m + 0 = m,$$

exactly the same as we found applying the definition of the derivative in Theorem 9.1.4.

Example 9.1.14 Use the differentiation rules to show

$$\frac{d}{dx}[x^2] = 2x.$$

Solution. The function that is the input to the differentiation operator is x^2. The elementary building blocks so far only include constant functions, the identity function, and other linear functions. We need to see how x^2 is a combination of these elementary functions. To do this, we need to recognize that the square corresponds to multiplication,

$$x^2 = x \cdot x.$$

Once we recognize that our function is a product, we use the product rule with $f(x) = x$ and $g(x) = x$. The product rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x),$$

for which we use $f'(x) = 1$ and $g'(x) = 1$. Consequently,

$$\begin{aligned}\frac{d}{dx}[x^2] &= \frac{d}{dx}[x \cdot x] \\ &= 1 \cdot x + x \cdot 1 \\ &= x + x = 2x.\end{aligned}$$

□

We can repeat this process to find the derivative of x^3 and then x^4. The pattern generalizes to a rule that we call the power rule.

Example 9.1.15 Continue to use the product rule of derivatives to show that

$$\frac{d}{dx}[x^3] = 3x^2,$$
$$\frac{d}{dx}[x^4] = 4x^3.$$

Solution. We rewrite the power as products:

$$x^3 = x \cdot x^2, \quad x^4 = x \cdot x^3.$$

We already know

$$\frac{d}{dx}[x] = 1,$$
$$\frac{d}{dx}[x^2] = 2x.$$

It is useful to remember the product rule using dependent variables instead of functions. That is, if $u = f(x)$ and $v = g(x)$, then the product rule becomes

$$\frac{d}{dx}[u \cdot v] = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}.$$

because this will guide our use of the differentiation operator.

The derivative of $x^3 = x \cdot x^2$ will use $u = x$ and $v = x^2$:

$$\begin{aligned}\frac{d}{dx}[x^3] &= \frac{d}{dx}[x \cdot x^2] \\ &= \frac{d}{dx}[x] \cdot x^2 + x \cdot \frac{d}{dx}[x^2] \\ &= 1 \cdot x^2 + x \cdot (2x) \\ &= 3x^2\end{aligned}$$

The derivative of $x^4 = x \cdot x^3$ uses $u = x$ and $v = x^3$, whose derivative we learned just above.

$$\begin{aligned}\frac{d}{dx}[x^4] &= \frac{d}{dx}[x \cdot x^3] \\ &= \frac{d}{dx}[x] \cdot x^3 + x \cdot \frac{d}{dx}[x^3] \\ &= 1 \cdot x^3 + x \cdot (3x^2) \\ &= 4x^3\end{aligned}$$

Notice that the derivative rules are self consistent. We could have written $x^4 = x^2 \cdot x^2$, and the product rule would still have given the same answer.

$$\begin{aligned}\frac{d}{dx}[x^4] &= \frac{d}{dx}[x^2 \cdot x^2] \\ &= \frac{d}{dx}[x^2] \cdot x^2 + x^2 \cdot \frac{d}{dx}[x^2] \\ &= 2x \cdot x^2 + x^2 \cdot 2x \\ &= 4x^3.\end{aligned}$$

□

We can continue to find more derivatives using these results. In particular, all of the polynomials that we learned earlier in terms of the rates of accumulation for accumulation functions, we now can justify as derivatives using the derivative rules.

Example 9.1.16 Find $\frac{d}{dx}[x^3+5x^2-8x+3]$.

Solution. It is helpful to give the original function a name, so we define $f(x)=x^3+5x^2-8x+3$. We start by using the sum rule of derivatives. That rule was formulated with adding two formulas together. Consequently, we need to repeat our use of the rule, breaking up the sum into two parts at a time.

$$\begin{aligned}\frac{df}{dx} &= \frac{d}{dx}[x^3+5x^2-8x+3] \\ &= \frac{d}{dx}[x^3] + \frac{d}{dx}[5x^2-8x+3] \\ &= \frac{d}{dx}[x^3] + \frac{d}{dx}[5x^2] + \frac{d}{dx}[-8x+3]\end{aligned}$$

We can stop at this point with the sum rule because $-8x+3$ is a linear function, and we have a derivative rule for any linear function. We can use the constant multiple rule to factor the 5 from the derivatives of x^2, and then use the derivatives that we know.

$$\begin{aligned}f'(x) &= \frac{d}{dx}[x^3] + 5\frac{d}{dx}[x^2] + \frac{d}{dx}[-8x+3] \\ &= 3x^2 + 5(2x) + -8 \\ &= 3x^2 + 10x - 8.\end{aligned}$$

We have shown $\frac{d}{dx}[x^3+5x^2-8x+3] = 3x^2+10x-8$. □

The previous example was one that we learned previously using accumulation functions. Because definite integrals also have constant multiple and sum rules, the process we used earlier is essentially the same. The new differentiation rules really show their value in finding derivatives of functions we that are not written as a sum.

Example 9.1.17 Find $\frac{d}{dx}[\frac{1}{x^3}]$.

Solution. We use the reciprocal rule of derivatives,

$$\frac{d}{dx}[\frac{1}{u}] = \frac{-\frac{du}{dx}}{u^2}.$$

So our derivative is

$$\begin{aligned}\frac{d}{dx}[\frac{1}{x^3}] &= \frac{-\frac{d}{dx}[x^3]}{(x^3)^2} \\ &= \frac{-(3x^2)}{x^3\cdot x^3} = \frac{-3x^2}{x^6} \\ &= \frac{-3}{x^4}.\end{aligned}$$

□

We have found derivative formulas for quite a few different elementary powers now. One of the themes in mathematics is to look for patterns and then determine whether that pattern would always hold. Consider the following sequence of statements that we have proved:

$$\frac{d}{dx}[x^2] = 2x,$$
$$\frac{d}{dx}[x^3] = 3x^2,$$
$$\frac{d}{dx}[x^4] = 4x^3.$$

There appears to be a pattern that the derivative of the power includes as a constant multiple the value of the power and another power of the variable that is one lower than the original function.

How does this relate to more other expressions? The identity function can also be thought of as a power, $x = x^1$, and we can rewrite the derivative rule for the identity as

$$\frac{d}{dx}[x^1] = 1x^0 = 1.$$

We can think of reciprocals of powers as equivalent negative powers. For example, the reciprocal rule guarantees

$$\frac{d}{dx}[\frac{1}{x}] = \frac{-1}{x^2}$$

and we just showed

$$\frac{d}{dx}[\frac{1}{x^3}] = \frac{-3}{x^4}.$$

If we rewrote these derivatives in the form of simple negative powers, we discover the pattern continues:

$$\frac{d}{dx}[x^{-1}] = -1x^{-2},$$
$$\frac{d}{dx}[x^{-3}] = -3x^{-4}.$$

When learning about the definition of the derivative, we found the derivative of the square root function,

$$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}.$$

By rewriting the square root as a fractional power, we discover that even this rule is following the same pattern.

$$\frac{d}{dx}[x^{1/2}] = \frac{1}{2}x^{-1/2}.$$

We have seen seven examples that appear to follow the same pattern. Inductive reasoning is the process of using examples to develop a generalization that we belief might be true. For this example, inductive reasoning would lead us to a **conjecture** that for *any* power,

$$\frac{d}{dx}[x^p] = px^{p-1}.$$

Deductive reasoning is the process of establishing the truth of such a statement built on a logical argument, or a proof, that applies the definitions and other proved conclusions to show whether or not that conjecture is true.

In this case, the claim can be proved true. We will prove the result in stages. First, we will generalize to all positive integer powers. Next, we will show that the result for positive integer powers implies a similar result for negative integer powers. As we continue to develop calculus, we will show our result is true for rational powers and ultimately for any real number.

Theorem 9.1.18 Power Rule for Derivatives. *For any real number p,*

$$\frac{d}{dx}[x^p] = px^{p-1}.$$

Proof. As indicated, we currently are only ready to prove this theorem for the integer powers. We start with positive integers. We have already proved the result for $p = 1, 2, 3, 4$ as part of our discovering the pattern. As we developed that pattern, we discovered that we were using a recursive argument each time. The powers $p = 3$ and $p = 4$ were based on knowing the results for $p = 2$ and $p = 3$, respectively. Our proof builds on this recursive argument to create a general statement.

Suppose that we know for $p = n$,

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Next consider the power $p = n + 1$, and rewrite it $x^{n+1} = x \cdot x^n$. The product rule guarantees

$$\begin{aligned}
\frac{d}{dx}[x^{n+1}] &= \frac{d}{dx}[x \cdot x^n] \\
&= \frac{d}{dx}[x] \cdot x^n + x \cdot \frac{d}{dx}[x^n] \\
&= 1 \cdot x^n + x \cdot nx^{n-1} \\
&= x^n + nx^n \\
&= (n+1)x^n.
\end{aligned}$$

This general recursive argument shows that if the rule is satisfied for an initial value of $p = n$, the statement will be immediately known to be true for the sequence of values $p \in (n, n+1, n+2, \ldots)$. Having earlier shown the rule was true for $p = 1$, the recursive argument shows it will be true for all positive integers. In addition, because we know the rule is true for $p = \frac{1}{2}$, the same argument shows that the rule is true for all $p \in (\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots)$.

The reciprocal rule for derivatives allows us to use the rule for positive integers to show the pattern holds for negative integers. For a positive integer n, consider the corresponding negative power $p = -n$ as a reciprocal.

$$\begin{aligned}
\frac{d}{dx}[x^p] = \frac{d}{dx}[x^{-n}] &= \frac{d}{dx}[\frac{1}{x^n}] \\
&= \frac{-\frac{d}{dx}[x^n]}{(x^n)^2} \\
&= \frac{-nx^{n-1}}{x^{2n}} \\
&= -nx^{n-1} \cdot x^{-2n} = -nx^{n-1-2n} \\
&= -nx^{-n-1} = px^{p-1}.
\end{aligned}$$

The argument did not depend so much on n being an integer as it required that we knew the power rule was true for $p = n$. Consequently, the same argument proves that the power rule is also true for $p \in (-\frac{1}{2}, -\frac{3}{2}, \ldots)$. ■

9.1.5 Exercises

Each function is an algebraic combination of more elementary expressions. Identify the last operation in the expression and the component expressions that operation combines. Repeat the process for each of the component expressions.

For example, $3x^2 - 8x$ involves a final operation of subtraction involving the terms $3x^2$ and $8x$; $3x^2$ is a constant multiple of the expression x^2 with the constant 3; and $8x$ is a constant multiple of the identity x with constant 8.

1. $u(x) = 5x^4 + (2x+3)(3x-2)$

2. $f(x) = (x^2+5x)(x^3-7)$

3. $g(x) = \dfrac{2x^3}{4x+1}$

Show that the derivative of each product is the same whether the function is expanded into a sum before differentiation or the product rule is used on the original formula.

4. $P(x) = (2x+5)(3x-4)$

5. $Q(t) = (3t-1)(t^2+4t-5)$

6. $R(y) = (y^2-2)(y^2+2)$

Compute the derivatives.

7. $\dfrac{d}{dx}[4\sqrt{x}]$

8. $\dfrac{d}{dx}[2x^{5/2} - 5x^{3/2}]$

9. $\dfrac{d}{dt}[\dfrac{1}{t^2+4t}]$

10. $\dfrac{d}{dt}[\dfrac{5}{3t-1}]$

11. $\dfrac{d}{dr}[\dfrac{r}{r^2+4}]$

Applications

12. Find the tangent line for $y = \frac{x-1}{x+1}$ at $x = 2$.

13. Find the tangent line for $y = \sqrt{x}$ at $x = 100$.

14. Find all points on the graph $y = \dfrac{1}{x}$ such that the slope of the tangent line has slope -4.

15. The height y (feet) from the ground of an object tossed from a tower is a quadratic function of time t (seconds) given by

$$y = 50 + 40t - 16t^2.$$

(a) Determine the velocity at which the object is thrown. (Velocity is the instantaneous rate of change of height.)

(b) Find the time when the object is traveling at the same speed but opposite direction as when it was thrown.

(c) Find the time such that the velocity is equal to the average velocity over the first two seconds of flight.