A.2 Algebra Review

A.2.1 Lines and Linear Functions

Lines are perhaps the most important elementary geometric object. A line captures the idea of following a given direction without turning. In ordinary language, we sometimes think of a line as a smooth curve that we could draw. Mathematically, a line would then be a *straight line* or a straight curve that does not bend. Algebraically, we can define a line using an equation involving two variables. This section reviews the basic principles of the algebraic properties of lines.

Definition A.2.1 General Equation of a Line. Every line in the (x, y) plane can be described as the set of points (x, y) that satisfy an equation

$$Ax + By = C \tag{A.2.1}$$

where A, B and C are constants. ♢

There are some special cases that describe horizontal and vertical lines. The (x, y) plane uses x as the horizontal axis (independent variable) and y as the vertical axis (dependent variable). So a horizontal line is a line where the dependent variable is constant while a vertical line is a line where the independent variable constant.

Definition A.2.2 Horizontal Line. A **horizontal line** in the (x, y) plane is the set of points that satisfy an equation

$$y = k \tag{A.2.2}$$

where k is a constant. ♢

Definition A.2.3 Vertical Line. A **vertical line** in the (x, y) plane is the set of points that satisfy an equation

$$x = h \tag{A.2.3}$$

where h is a constant. ♢

All other lines have an equation that involves both variables. We often wish to think of the line as describing the dependent variable as a function of the independent variable. These equations involve the calculation of the slope, which represents a rate (or ratio) of change.

Definition A.2.4 Slope as Rate of Change. Given any two points (x_1, y_1) and (x_2, y_2) on a non-vertical line, the change in the dependent variable $\Delta y = y_2 - y_1$ is proportional to the change in the independent variable $\Delta x = x_2 - x_1$, written $\Delta y = m \cdot \Delta x$. The proportionality constant m is called the **slope** of the line, calculated as the ratio of changes (rate of change)

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{A.2.4}$$

♢

Knowing the slope and one point is enough to quickly find an equation of a line.

Definition A.2.5 Point–Slope Equation of Line. Given that a line has slope m and passes through a point $(x, y) = (h, k)$, every point on the line satisfies the equation

$$y = m(x - h) + k. \tag{A.2.5}$$

We interpret k as the starting value for y and the expression $\Delta y = m(x - h)$ as the change in y given the change in x, $\Delta x = x - h$. ♢

A special case of the point–slope equation of the line occurs when the point is on the y-axis or, in other words, is a y-intercept.

Definition A.2.6 Slope–Intercept Equation of Line. Given that a line has slope m and passes through a y-intercept $(x, y) = (0, b)$, every point on the line satisfies the equation

$$y = mx + b. \tag{A.2.6}$$

♢

Remark A.2.7 Preparatory mathematics courses often emphasize the slope–intercept equation of a line as if it were the most important. However, the point–slope equation is the preferred equation to use in almost every circumstance.

Another special case of the point–slope equation of a line is when we know the slope and the x-intercept.

Definition A.2.8 Slope and X-Intercept Equation of Line. Given that a line has slope m and passes through an x-intercept $(x, y) = (a, 0)$, every point on the line satisfies the equation

$$y = m(x - a).. \tag{A.2.7}$$

♢

A.2.2 Quadratic Polynomials

Definition A.2.9 A **quadratic polynomial** in a variable x is an algebraic function that is equal to a formula of the form

$$f(x) = ax^2 + bx + c, \tag{A.2.8}$$

where a, b and c are constants called **coefficients**. ♢

The graph of a quadratic function, $y = ax^2 + bx + c$, is a **parabola**. Such a parabola has a mirror symmetry across a vertical line that passes through its **vertex** $x = -\frac{b}{2a}$. Depending on whether the vertex is above, on or below the x-axis and whether the parabola opens up or down, the graph can cross the x-axis twice, once or never. The location of these points are called x-intercepts, **roots** or **zeros** of the function. The values of the roots can always be found using the (((Unresolved xref, reference "thm-quadratic-formula"; check spelling or use "provisional" attribute)))quadratic formula.

Zeros are closely related to factoring. If we know the zeros, then we can immediately rewrite the polynomial in a factored form. On the other hand, if we know the factors, then we can quickly solve for the zeros without using the quadratic formula. This is a consequence of the fundamental properties of numbers in Theorem Theorem A.1.7.

Theorem A.2.10 Factor–Root Theorem for Quadratics. *A quadratic function $f(x) = ax^2 + bx + c$ with real roots $x = r$ and $x = s$ ($r = s$ is possible) is equal to the factored equation*

$$f(x) = a(x - r)(x - s). \tag{A.2.9}$$

A quadratic polynomial that has complex roots is called **irreducible** because it can not be rewritten in a factored form involving only real roots.

There are some tricks to factoring that can be useful to know. Factoring is the reverse process of multiplying by distribution, so we start by noticing what happens when you multiply out two simple factors:

$$(x+a)(x+b) = x^2 + (a+b)x + ab.$$

Notice that the coefficient of x is the sum $a+b$ and the constant term is the product ab. When trying to factor a quadratic, look for numbers that multiply to give the product term and add to give the coefficient of x. This is often a matter of trial and error.

Knowing one root $x = r$ of a quadratic $f(x) = ax^2 + bx + c$ so that $f(r) = 0$, we know that $x - r$ is a factor. The other factor can be determined easily.

Theorem A.2.11 Using Roots to Factor Quadratics. *Suppose that $x = r$ is a root of $f(x) = ax^2 + bx + c$ so that $f(r) = 0$. Then*

$$f(x) = (x - r)(ax + d) \tag{A.2.10}$$

where $d = b + ar = -c/r$.

Synthetic division is a procedure that works for quadratics as well as higher order polynomials. This procedure uses a table that starts with the coefficients on the first row. For a more thorough discussion for higher-order polynomials, see Algorithm A.2.19.

Algorithm A.2.12 Synthetic Division (Quadratics). *To divide a quadratic polynomial $f(x) = ax^2 + bx + c$ by the factor $x - r$ (proposed root $x = r$), we will apply the following steps.*

1. *Create a table that will have three rows and three columns. The first row will have the coefficients a, b and c in the three columns. The first entry of the second row will always be 0. So the start of the table will look like the following.*

$$\begin{array}{ccc} a & b & c \\ 0 & \_\_\_\_ & \_\_\_\_ \\ \_\_\_\_ & \_\_\_\_ & \_\_\_\_ \end{array}$$

2. *We finish a column, starting with the first column, by adding the values from the first and second rows, which in this case gives a again.*

$$\begin{array}{ccc} a & b & c \\ 0 & \_\_\_\_ & \_\_\_\_ \\ a & \_\_\_\_ & \_\_\_\_ \end{array}$$

3. *We next update the second row of the next column by multiplying the most recent value in the third row by the proposed root value r, which in this case gives ar.*

$$\begin{array}{ccc} a & b & c \\ 0 & ar & \_\_\_\_ \\ a & \_\_\_\_ & \_\_\_\_ \end{array}$$

4. *Add the values in the second column to update the third row to define the new coefficient $d = ar + b$ (recall Theorem A.2.11) and multiply by r to update the second row.*

$$\begin{array}{ccc} a & b & c \\ 0 & ar & ar^2 + br \\ a & d = ar + b & \_\_\_\_ \end{array}$$

5. *When we finish updating the third row by adding the values in the third column, we discover that the last entry in the third row corresponds to* $f(r) = ar^2 + br + c$.

$$\begin{array}{ccc} a & b & c \\ 0 & ar & ar^2 + br \\ a & d = ar + b & f(r) = ar^2 + br + c \end{array}$$

Once the table is complete, we interpret the values in the third row as giving coefficients of the factored polynomial along with the value of the polynomial at $x = r$ *(called the remainder) and can write*

$$f(x) = (x - r)(ax + d) + f(r).$$

If $f(r) = 0$ *(remainder is 0), then this is an actual factorization*

$$f(x) = (x - r)(ax + d).$$

Because synthetic division is quick, this can be a simple way to test for roots and factor simultaneously.

Every quadratic can be rewritten in a form $y = a(x - h)^2 + k$ where (h, k) is the vertex of the parabola and a is the leading coefficient and scaling factor. The process of rewriting a quadratic $y = ax^2 + bx + c$ in this vertex form is called **completing the square**. It is based on noticing what happens with expanding the square of a binomial, $(x + a)^2 = x^2 + 2ax + a^2$. The strategy involves adding a term to form a perfect square and subtracting the same term to guarantee the expression does not change.

Algorithm A.2.13 Completing the Square. *A quadratic* $y = ax^2 + bx + c$ *can be rewritten in terms of its vertex by completing the following steps.*

1. *Factor the leading coefficient from the leading two terms*

$$y = a(x^2 + \frac{b}{a}x) + c.$$

2. *Think of the* x *term as being double half its value and add and subtract the square of the half-value:*

$$y = a\Big(x^2 + 2\frac{b}{2a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2\Big) + c.$$

3. *Recognize the square of a binomial, group those terms, and regroup the remaining terms:*

$$y = a\Big((x + \frac{b}{2a})^2 - (\frac{b}{2a})^2\Big) + c = a(x + \frac{b}{2a})^2 + \frac{-ab^2}{4a^2} + c.$$

4. *Interpret the results: The vertex is* (h, k) *where* $h = -\frac{b}{2a}$ *(because vertex form uses* $(x - h)^2$*) and* $k = -\frac{b^2}{4a} + c$. *The leading coefficient* a *is a scaling factor that determines the steepness of the parabola and whether the parabola opens up* $(a > 0)$ *or down* $(a < 0)$.

Example A.2.14 Complete the square for $3x^2 - 4x + 1$.

Solution.

1. Group the non-constant terms and factor out the leading coefficient.

$$\begin{aligned} 3x^2 - 4x + 1 &= (3x^2 - 4x) + 1 \\ &= 3(x^2 - \frac{4}{3}x) + 1 \end{aligned}$$

2. Recognize the coefficient $-\frac{4}{3}$ as double $-\frac{2}{3}$ and use this to complete the square.

$$\begin{aligned} 3x^2 - 4x + 1 &= 3\Big(x^2 + 2 \cdot \frac{-2}{3}x\Big) + 1 \\ &= 3\Big(x^2 + 2 \cdot \frac{-2}{3}x + (\frac{-2}{3})^2 - (\frac{-2}{3})^2\Big) + 1 \\ &= 3\Big(x^2 - \frac{4}{3}x + \frac{4}{9}\Big) - 3(\frac{4}{9}) + 1 \\ &= 3\Big(x - \frac{2}{3}\Big)^2 - \frac{4}{3} + 1 = 3\Big(x - \frac{2}{3}\Big)^2 - \frac{1}{3}. \end{aligned}$$

3. Interpret the results as saying $h = \frac{2}{3}$ (because the completed square is always of the form $(x-h)^2$) and $k = -\frac{1}{3}$. Thus the vertex of the parabola is at $(\frac{2}{3}, -\frac{1}{3})$. The leading coefficient $a = 3$ indicates that the parabola opens up and is three times steeper than the standard parabola $y = x^2$.

□

A.2.3 Polynomials

Linear and quadratic formulas are special cases of polynomials. This section gives an overview of principles about polynomials that are likely to appear in calculus. First, we introduce some basic definitions.

Definition A.2.15 Monomials. A **monomial** is an expression that is a constant multiple of a variable raised to a non-negative integer power, ax^k, where $k = 0, 1, 2, 3, \ldots$ and $a \in \mathbb{R}$.

Examples include $4x^2$ (with $a = 4$ and $k = 2$), $\frac{1}{3}x^7$ (with $a = \frac{1}{3}$ and $k = 7$) and 3 (where $a = 3$ and $k = 0$). The following are not monomials: $3\sqrt{x} = 3x^{1/2}$ (since not an integer power) and $\frac{3}{x^2} = 3x^{-2}$ since the power is a negative integer. ♢

Definition A.2.16 Polynomials. An algebraic expression that can be rewritten as a sum of monomials is called a **polynomial**. The monomials are called the **terms** of the polynomial. The monomial with the highest power is called the **leading term** and its power is called the **degree** of the polynomial. The constant multiples in the monomials are called **coefficients** and the coefficient in the leading term is called the **leading coefficient**.

We usually write a polynomial with terms ordered by decreasing powers, called **standard form**. An abstract representation of a polynomial with degree n is written

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where the symbols $a_n, a_{n-1}, \ldots, a_0$ represent the coefficients. A missing term is represented by a coefficient zero. ♢

Example A.2.17 $x^4 - 2x^2 + 3x + 1$ is a polynomial with degree $n = 4$. The coefficients are $a_4 = 1$, $a_3 = 0$ (since no x^3 term), $a_2 = -2$, $a_1 = 3$ and $a_0 = 1$.

$2x^2(3x + 1)(x - 2)$ is a polynomial, but must be expanded (multiply out) to find the coefficients.

$$\begin{aligned} 2x^2(3x+1)(x-2) &= 2x^2(3x^2-6x+x-2) \\ &= 2x^2(3x^2-5x-2) \\ &= 6x^4-10x^3-4x^2 \end{aligned}$$

We see that the polynomial has degree $n=4$ and coefficients $a_4=6$, $a_3=-10$, $a_2=-4$ and $a_1=a_0=0$. □

Every polynomial $p(x)$ is a function whose domain is all real numbers $(-\infty,\infty)$. Values of x for which $p(x)=0$ are called zeros or roots of the polynomial. These roots are related to factors.

Theorem A.2.18 Root–Factor Theorem. *Suppose $p(x)$ is a polynomial of degree n for which $x=c$ is a root, $p(c)=0$. Then $p(x)$ can be written in a factored form*

$$p(x)=(x-c)\cdot q(x)$$

where $q(x)$ is a polynomial of degree $n-1$.

Synthetic division is an algorithm that can both test if a value $x=r$ is a root and determine the coefficients of the factored polynomial $q(x)$ at the same time. Synthetic division for quadratic polynomials (degree $n=2$) is a special case of this process, described in Algorithm A.2.12.

Algorithm A.2.19 Synthetic Division. *Given a polynomial $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+\cdots+a_1x+a_0$ and a test value $x=c$, synthetic division is an algorithm for finding coefficients $b_{n-1},\ldots,b_0$ of a polynomial $q(x)=b_{n-1}x^{n-1}+\cdots+b_1x+b_0$ and a remainder r such that*

$$p(x)=(x-c)q(x)+r.$$

The remainder is also the value of the original polynomial at $x=r$, $p(c)=r$, so that when $x=c$ is a root, $p(c)=0$ and $p(x)$ factors as

$$p(x)=(x-c)q(x).$$

The coefficients for $q(x)$ and the remainder are found using the steps below.

1. *We will create a table with three rows and $n+1$ columns. The first row consists of the coefficients of $p(x)$, using 0 for any skipped terms, ordered by decreasing power. The first value of the second row is always 0. The table will start as follows:*

$$\begin{array}{ccccc} a_n & a_{n-1} & \cdots & a_1 & a_0 \\ 0 & \_\_\_ & \cdots & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \cdots & \_\_\_ & \_\_\_ \end{array}$$

2. *A column will always be completed (finding the third row) by adding the values in the first and second rows.*

3. *Once a column is complete, the second row of the next column is found by multiplying the previous value in the third row by the test value c.*

4. *Repeat these two steps until the table is complete. The third row of the table gives the coefficients and remainder as follows:*

$$\begin{array}{ccccc} a_n & a_{n-1} & \cdots & a_1 & a_0 \\ 0 & c\cdot b_{n-1} & \cdots & c\cdot b_1 & c\cdot b_0 \\ b_{n-1} & b_{n-2} & \cdots & b_0 & r \end{array}$$

The interpretation is that

$$p(x) = (x - c)(b_{n-1}x^{n-1} + \cdots + b_1 x + b_0) + r.$$

Example A.2.20 Use synthetic division with the polynomial $p(x) = x^3 - 6x + 2$ with the test value $x = 2$ and interpret the result.

Solution. Start by identifying the coefficients. Any missed terms have a coefficient of zero,

$$p(x) = x^3 + 0x^2 + -6x + 2.$$

We start the synthetic division table using the coefficients in the first row.

1	0	−6	2
0	____	____	____
____	____	____	____

We then finish filling the table. To find values in the second row, we use the previous result in the third row and multiply by the test value 2. To find the values in the third row, we add the values in the column. The first value in the second row is always 0. The completed table is shown below.

1	0	−6	2
0	2	4	−4
1	2	−2	−2

Once the table is complete, we interpret the values in the third row as coefficients and a remainder. The last value is the remainder, $r = -2$, and the other values are the coefficients of a polynomial whose degree is one smaller than the original, in this case $n - 1 = 2$. That is, the quotient polynomial is $q(x) = x^2 + 2x - 2$. The original polynomial can be written

$$\begin{aligned} p(x) &= (x-2)q(x) + r \\ x^3 - 6x + 2 &= (x-2)(x^2 + 2x - 2) + -2 \end{aligned}$$

The non-zero remainder means that $x - 2$ is not a factor and also tells us that $p(2) = -2$. □

How do we know which numbers to try? If you have access to a graph of the polynomial, you should use the values for roots that you see. If you do not have access to a graph, then you might be able to use the results of the Integer Root Theorem or Rational Root Theorem so long as all of the coefficients of your polynomial are integers.

Theorem A.2.21 Integer Root Theorem. *If the coefficients of a polynomial*

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

has all integer coefficients, then the only possible integer roots are factors of the constant coefficient a_0.

Example A.2.22 The polynomial $p(x) = x^3 - 6x + 2$ has only integer coefficients and a constant coefficient $a_0 = 2$. The only factors of a_0 are ± 1 and ± 2. So the Integer Root Theorem guarantees that these four integers are the only four numbers that we need to check if they are roots. A quick test of each of those values (below) shows that $p(x)$ has no integer roots. (Without the theorem, we wouldn't know how many points we had to check.)

x	$p(x)$
1	-3
-1	-5
2	-2
-2	6

□

Theorem A.2.23 Rational Root Theorem. *If the coefficients of a polynomial*

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

has all integer coefficients, then a rational number $x = r/s$ (where r and s are integers) might be a root only if r is a factor of a_0 and s is a factor of a_n.

The Integer Root Theorem is a special case of the Rational Root Theorem where $s = 1$ (which is always a factor of a_n.

A.2.4 Absolute Value

The absolute value operation takes a number and finds its magnitude (or distance from zero). Because magnitude is a non-negative value and positive and negative pairs are the same distance from zero, we often imagine that the role of absolute value is to remove a negative sign, $|-3| = 3$. However, when a variable is involved, a negative sign means finding the inverse of a value for which we may not know if it is positive or negative. So it is incorrect to say that $|-x| = x$ (FALSE). The proper definition of absolute value uses a piecewise formula.

Definition A.2.24 Absolute Value.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

◊

As a function, the graph of absolute value $y = |x|$ gives two lines: $y = x$ when $x \geq 0$ and $y = -x$ when $x < 0$.

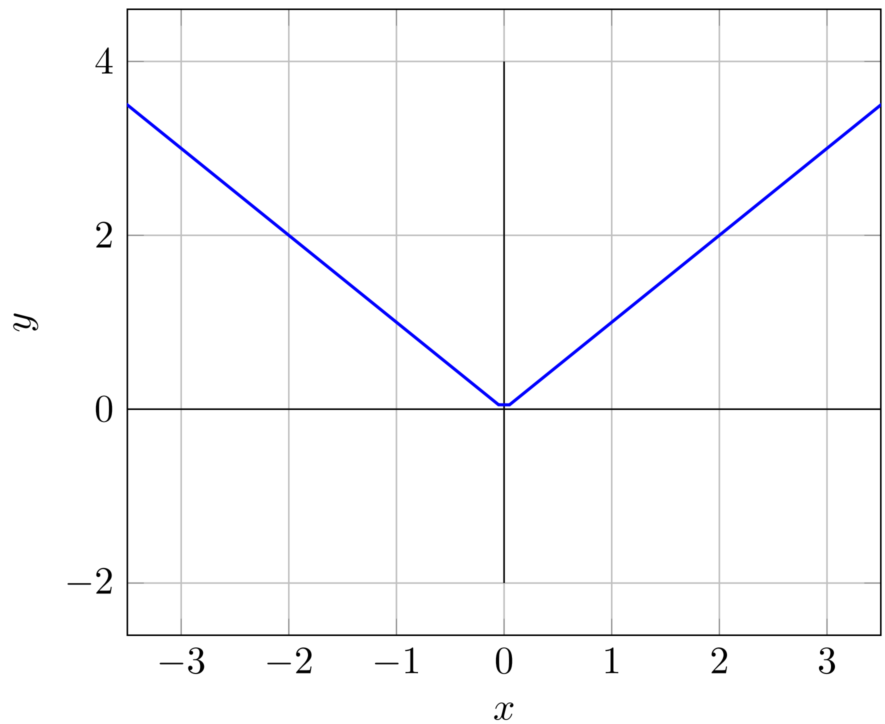


Figure A.2.25 Graph of the absolute value, $y = |x|$

It is sometimes useful to take advantage of an identity between the square root of a square and the absolute value. This is the source of the plus/minus when solving an equation with a square.

Theorem A.2.26

$$\sqrt{x^2} = |x|$$

Example A.2.27 Solve the equation $x^2 = 16$.

Solution. Applying a square root to both sides of the equation, we then get to use the absolute value identity.

$$\sqrt{x^2} = \sqrt{16} \quad \Leftrightarrow \quad |x| = \sqrt{16} = 4$$

The source of plus/minus is that there are two numbers with magnitude 4,

$$x = \pm 4.$$

□

The absolute value splits nicely with multiplication (and division). However, addition of two values with opposite signs shows that absolute values do not add: $|3 + -4| = |-1| \neq |3| + |-4| = 7$. Instead, we have an inequality called the triangle inequality.

Theorem A.2.28 Properties of Absolute Values.

$$|a \cdot b| = |a| \cdot |b|$$
$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$$

The triangle inequality is used to show that the absolute value of a sum (or difference) is bounded by the sum of the magnitudes of the individual terms.

Theorem A.2.29 Triangle Inequality.

$$|a+b| \leq |a| + |b| \tag{A.2.11}$$

$$|a-b| \leq |a| + |b| \tag{A.2.12}$$

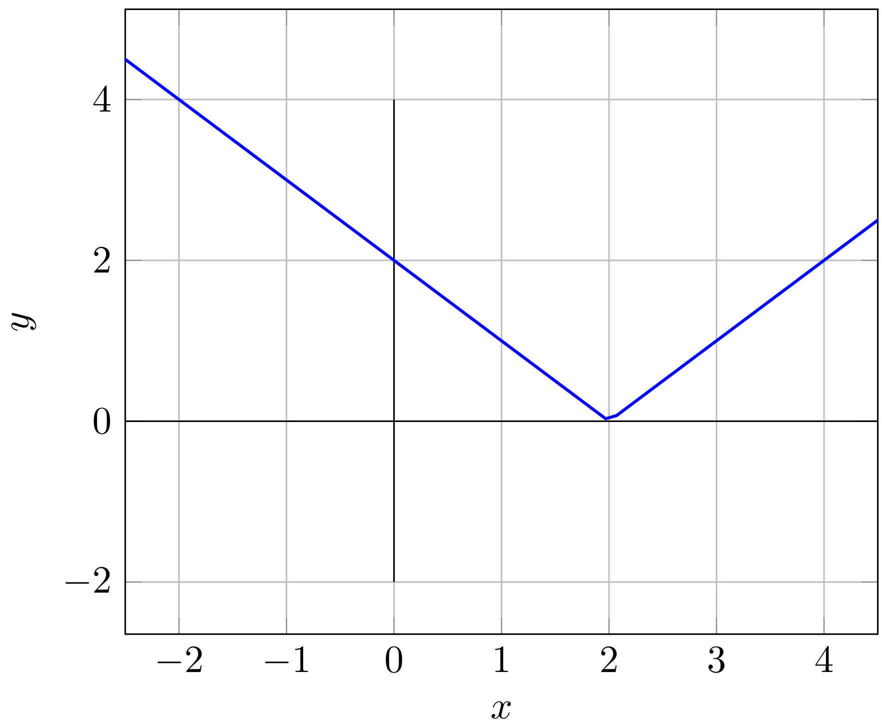
Occasionally we need to apply the triangle in reverse, showing that the absolute value of a sum (or difference) must be bigger than the difference in magnitudes of the parts.

Theorem A.2.30 Reverse Triangle Inequality.

$$|a+b| \geq \big||a| - |b|\big| \tag{A.2.13}$$

$$|a-b| \geq \big||a| - |b|\big| \tag{A.2.14}$$

Absolute value and subtraction is often used to describe the distance between two values. For example, the graph $y = |x-2|$ represents a shift of the graph $y = |x|$ two units to the right, so that instead of measuring the distance of x from 0 it measure the distance of x from 2.



Theorem A.2.31 *The expression $|x-a|$ measures the distance between x and the value a.*

Note that $|x+3| = |x - -3|$ so that it represents the distance between x and the value -3.