B.1 Right Triangles and Trigonometry

B.1.1 Right Triangles

We already reviewed the idea that similar triangles have equal ratios of corresponding sides. This is at the heart of trigonometry using right triangles. An important fact from geometry is that any two triangles that have equal angles are similar. A right triangle, by definition, has one angle that is perpendicular or 90 degrees. Because the sum of angles in a triangle always add to 180 degrees, it really only takes one of the other angles to establish similarity.

We consider an acute angle θ (less than 90 degrees). A right triangle with base angle θ in standard position (with the base horizontal and the angle on the left) is shown in the diagram below. The legs that join at right angles are identified as being adjacent (adj) to the angle or opposite (opp) to the angle, while the side opposite the right angle is the hypotenuse (hyp).

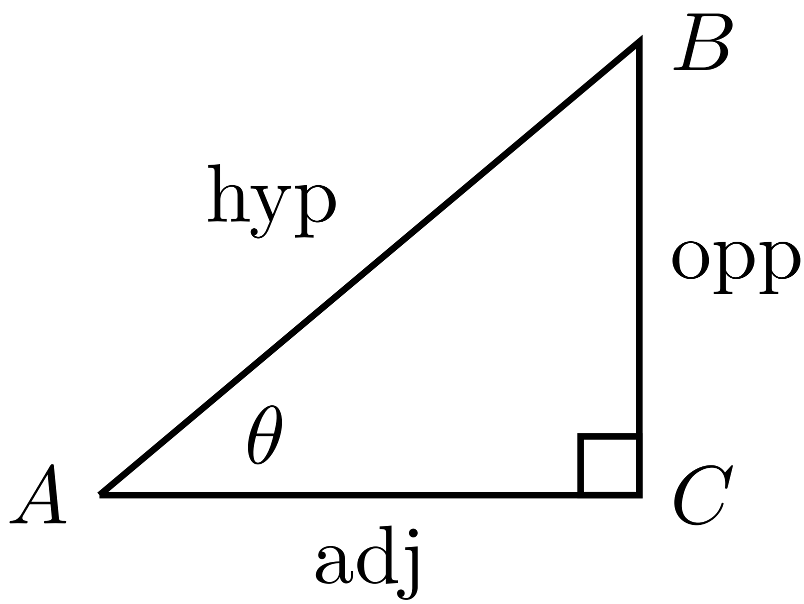


Figure B.1.1 An illustration of similar right triangles with base angle θ in standard position.

For any right triangle with the same base angle, the ratios of these three sides are always going to found in the exact proportions. These proportions define the six trigonometric values of the angle:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \text{(sine)}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \text{(cosine)}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \text{(tangent)}$$

$$\sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \text{(secant)}$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} \qquad \text{(cotangent)}$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \text{(cosecant)}.$$

The first three proportions are often introduced using a mnemonic ``SOH-CAH-TOA" where the three letters correspond to the first letter of the proportion name (*S*ine), the numerator side (*O*pposite) and denominator side (*H*ypotenuse).

It is sometimes useful to think of drawing special right triangles where one of the sides has unit length (length=1). When the hypotenuse has unit length, the triangle involves sine (opposite) and cosine (adjacent). When the adjacent leg has unit length, the triangle has sides with lengths given by tangent (opposite) and secant (hypotenuse). When the opposite leg has unit length, the triangle has sides with lengths given by cotangent (adjacent) and cosecant (hypotenuse). When I realized this simple fact, the naming of the proportions made much more sense.

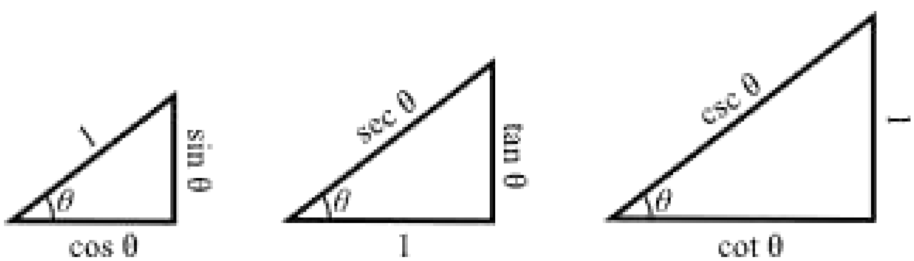


Figure B.1.2 Illustration of the three similar unit right triangles with angle θ.

Theorem B.1.3 Pythagorean Identities. *The Pythagorean theorem states that the sum of the squares of the legs of a right triangle must equal the square of the hypotenuse. Consequently, the trigonometric values of an angle must satisfy:*

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$
$$(\tan\theta)^2 + 1 = (\sec\theta)^2$$
$$1 + (\cot\theta)^2 = (\csc\theta)^2$$

Proof. Consider any triangle with base angle θ. The Pythagorean theorem guarantees

$$\text{opp}^2 + \text{adj}^2 = \text{hyp}^2.$$

The first identity is found by dividing both sides of this equation by hyp^2. The next two identities are found by dividing by adj^2 and opp^2, respectively. ■

B.1.2 Special Right Triangles

There are two right triangles that often appear in problems because they are geometrically simple. One of these is the isosceles right triangle where the base angle is exactly half of a right angle ($\theta = 45°$). This triangle is often called a 45–45–90 right triangle. The other simple triangle comes from dividing an equilateral triangle in half, so that the base angle is either 30° or 60°, and is called a 30–60–90 right triangle.

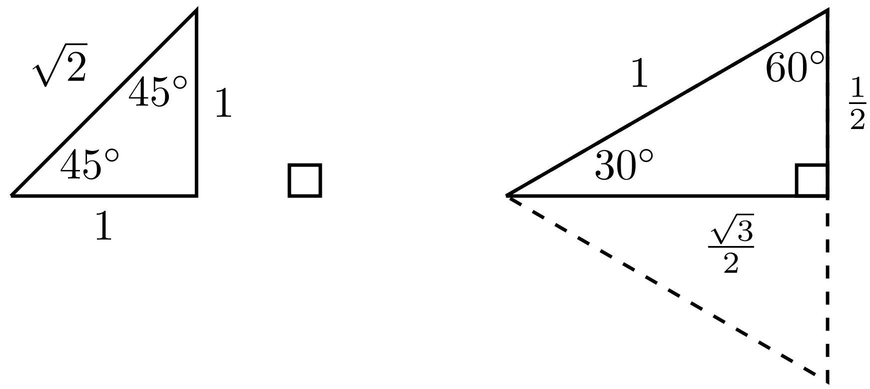


Figure B.1.4 Illustration of 45–45–90 and 30–60–90 right triangles.

It is handy to be able to reproduce these two triangles as quickly as possible. The key is to remember how the triangles were created. For the 45–45–90 triangle, draw an isosceles right triangle and label the lengths of both legs as 1. To find the length of the hypotenuse h, use the Pythagorean theorem:

$$\begin{aligned} 1^2 + 1^2 &= h^2 \\ h^2 &= 2 \\ h &= \sqrt{2} \end{aligned}$$

For the 30–60–90 triangle, recall that this is exactly half of an equilateral triangle. The hypotenuse will have length 1 while the leg opposite the 30 degrees is exactly half that length. Now use the Pythagorean theorem to find the length of the other leg b which bisected the triangle:

$$\begin{aligned} (\frac{1}{2})^2 + b^2 &= 1^2 \\ \frac{1}{4} + b^2 &= 1 \\ b^2 &= \frac{3}{4} \\ b &= \frac{\sqrt{3}}{2} \end{aligned}$$

Once you have the triangles drawn with lengths identified, you can use the triangles to find the proportions that define the trigonometric values. I strongly discourage trying to memorize this table. Learn how the table was created, and that will reinforce the more general principles and ultimately require less mental effort to recall.

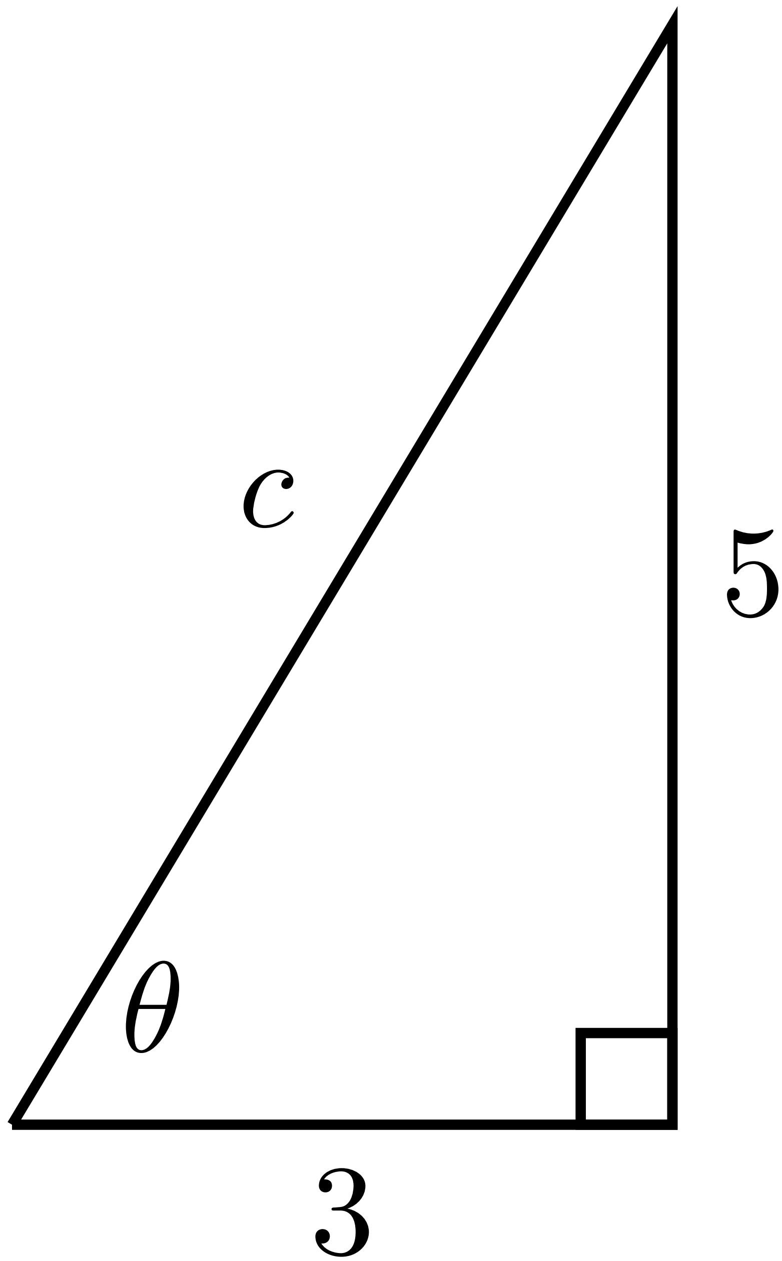
θ	$30°$	$45°$	$60°$
$\sin\theta = \frac{\text{opp}}{\text{hyp}}$	$\sin 30° = \frac{1}{2}$	$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\sin 60° = \frac{\sqrt{3}}{2}$
$\cos\theta = \frac{\text{adj}}{\text{hyp}}$	$\cos 30° = \frac{\sqrt{3}}{2}$	$\cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\cos 60° = \frac{1}{2}$
$\tan\theta = \frac{\text{opp}}{\text{adj}}$	$\tan 30° = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$	$\tan 45° = 1$	$\tan 60° = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

B.1.3 Examples

The first example illustates how you can use two known lengths for a triangle to find the trigonometric values for the angle.

Example B.1.5 Suppose a right triangle with a base angle θ has a base of length 3 and a height of length 5. What are the trigonometric values associated with θ?

Solution. Start by drawing a diagram, labeling the unknown side (the hypotenuse) with a variable, say c.



Using the Pythagorean theorem, we can find the length of the hypotenuse.

$$3^2 + 5^2 = c^2$$
$$9 + 25 = 34 = c^2$$
$$c = \sqrt{34}$$

Now that we know the lengths of all three sides, we can compute the trigonometric values for our angle.

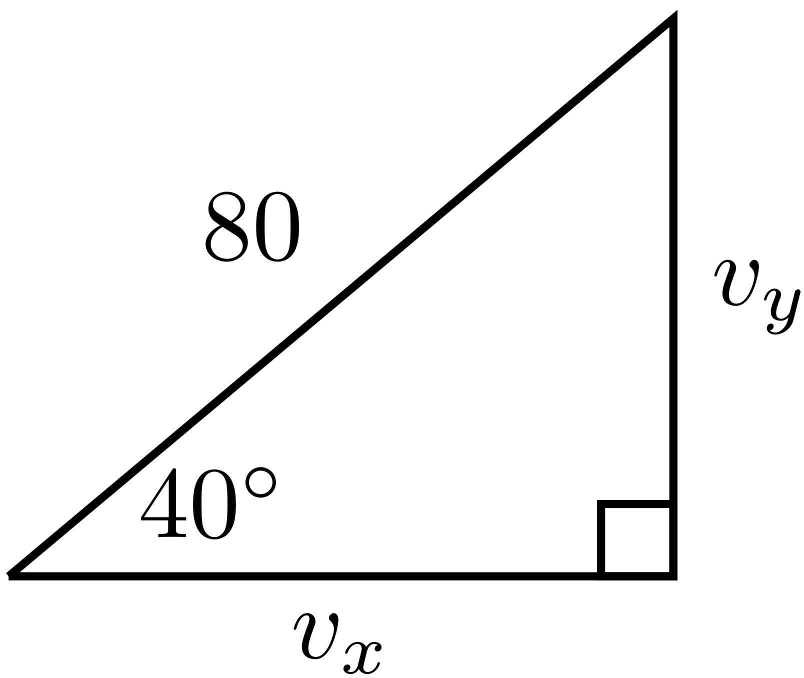
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{5}{\sqrt{34}}$$
$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3}{\sqrt{34}}$$
$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{5}{3}$$
$$\sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{34}}{3}$$
$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3}{5}$$
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{34}}{5}$$

□

Our second example illustrates how knowing the trigonometric values associated with an angle allow us to determine the lengths of sides. This occurs a lot in physics in the context of decomposing a force or velocity into two perpendicular components.

Example B.1.6 A golf ball is launched at a speed of 80 $\frac{\text{m}}{\text{s}}$ and at an angle of \(40^\circ\) from the ground. What are the horizontal and vertical components of the ball's velocity?

Solution. My preferred method of solution begins with a figure of a right triangle, where the angle $\theta = 40°$ is the base angle, the hypotenuse is labeled with length 80, and the legs are labeled with variables representing the horizontal velocity v_x and the vertical velocity v_y.



The trigonometric values for $\theta = 40°$ can be easily found using a calculator. (These once were found by looking up the angles in a table.) We can setup equations relating the ratios of lengths and the corresponding trigonometric values.

$$\sin 40^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{v_y}{80}$$
$$\cos 40^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{v_x}{80}$$

From the first equation, we can solve for v_y:

$$v_y = 80 \cdot \sin 40^\circ \approx 80 \cdot 0.6248 = 51.42.$$

We find the horizontal velocity v_x by solving the second equation

$$v_x = 80 \cdot \cos 40^\circ \approx 80 \cdot 0.7660 = 61.28.$$

So the ball is traveling with a vertical velocity approximately $51.42\,\frac{\text{m}}{\text{s}}$ and a horizontal velocity approximately $61.28\,\frac{\text{m}}{\text{s}}$. $\square$