B.2 Measuring Arbitrary Angles

B.2.1 Angles and Rotation

In the previous section, we focused on the trigonometry of right triangles, which involve angles smaller than a right angle. The more significant use of measuring angles involve angles of rotation which can be significantly greater than 90 degrees. Before we proceed, we need to discuss how angles are measured.

Have you ever considered why we measure angles in degrees? What is so special about a degree? One thing nice about the number 360 is that it has so many factors, including 2, 3, 4, 5, 6, 8, 9, 10 and 12. (That only misses 7 and 11!) This makes it easy to divide into fractions that simplify cleanly with integers. Having an easily divisible number that is so close to the number of days in the year, so that 1 degree is close to the change in the daily position of stars in the sky, would have been convenient to ancient astronomers.

What other ways are there to measure rotations?

If a right angle is considered the fundamental unit, then we might consider measuring an angle as a percentage of a right angle, so that 90 degrees counts as 100 percent. This idea led to the development of what is called a gradian. So an angle of 1 gradian is exactly 1 percent of a right angle, and there are 400 gradians in a circle.

Alternatively, if we think of a full rotation as the fundamental unit, then we might measure an angle as a fraction of a rotation. I especially like this perspective when dealing with trigonometry on the unit circle. We will let τ (the Greek letter *tau*) be the unit of a complete rotation ($1\tau = 360°$). Since 90 degrees is one quarter rotation and 60 degrees is one sixth rotation, we have $90° = \frac{1}{4}\tau$ and $60° = \frac{1}{6}\tau$. Other conversions can be determined using standard techniques.

All of the preceding methods for measuring angles are based on natural activity of counting. And none of these methods are mathematically superior to one another. Curiously, the mathematically best way to measure an angle is not based on counting integer divisions of a rotation but is based on a unit called the radian which involves measuring arc length in terms of the radius.

B.2.2 Arc Length and Radian Measure of Angles

An arc is a path traced along the circumference of a circle. We can describe an arc by either measuring the length of the path (arc length) or by measuring the angle subtended by the arc. It should be obvious that the arc length is proportional to the angle.

Let s represent the arc length and let θ represent the angle of the arc. To say that $s \propto \theta$ is to say that there is a proportionality constant so that $s = k\theta$. Because s is a measure of length, by geometric similarity, the arc length must also be proportional to the radius of the circle r (or $k \propto r$). This means that there is a constant α so that

$$s = \alpha\theta r.$$

The value of the constant α depends on how we measure angles. To see this, we will use an arc of a complete rotation which has an arc length equal to the circumference of the circle $s = 2\pi r$. If the angle is measured in degrees, $\theta = 360$, then the proportionality constant will be α_{deg} defined by

$$2\pi r = \alpha_{\text{deg}}(360)r \quad \Leftrightarrow \quad \alpha_{\text{deg}} = \frac{2\pi}{360}.$$

If the angle is measured in gradians, $\theta = 400$, then the proportionality constant α_{grad} satisfies

$$2\pi r = \alpha_{\text{grad}}(400)r \quad \Leftrightarrow \quad \alpha_{\text{grad}} = \frac{2\pi}{400}.$$

In a similar way, if the angle is measured in rotations τ, then

$$\alpha_\tau = \frac{2\pi}{1} = 2\pi.$$

The mathematically defined measure of angle called the radian is determined by choosing the proportionality constant α as being convenient instead of choosing the measurement of the angle as being convenient. If we choose $\alpha = 1$, then this requires that we measure a full rotation θ to satisfy

$$2\pi r = 1\theta r \quad \Leftrightarrow \quad \theta = 2\pi.$$

That is, a full rotation is defined as 2π radians.

With radians as the measure of angle, the arc length formula simplifies to

$$s = \theta r.$$

In other words, the angle θ is determined by measuring the length of a subtended arc using the radius as the unit of length. An angle $\theta = 1$ radian has an arc length equal to the radius of the arc. When angles are measured in radians, no units are used; so we would just say $\theta = 1$. We also have, for example,

$$\frac{1}{4}\tau = \frac{\pi}{2}, \tag{B.2.1}$$

$$\frac{1}{2}\tau = \pi, \tag{B.2.2}$$

$$1\tau = 2\pi. \tag{B.2.3}$$

How did you read that last sentence? Did you visualize and interpret what it says, and not just read the symbols? What does each equation mean? Below is a figure illustrating the examples $\theta = 1$, $\theta = \frac{1}{4}\tau$ and $\theta = \frac{1}{2}\tau$. Could you draw a similar figure showing angles corresponding to 30, 45 and 60 degrees? What is the radian measure of those angles?

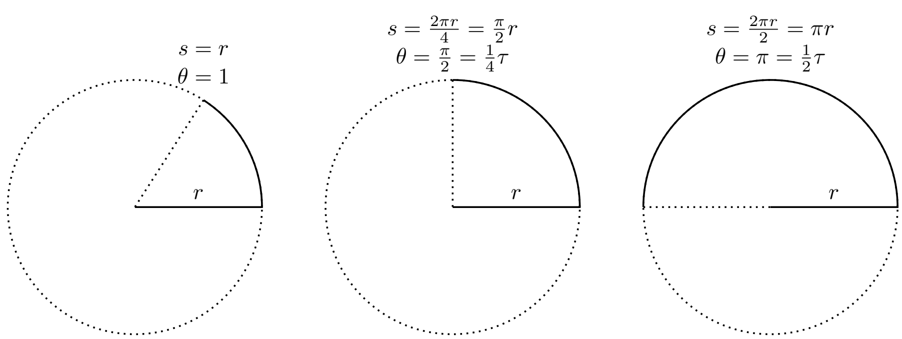

Figure B.2.1 Important arcs measured in radians.

B.2.3 Unit Circle and Standard Position

In order to have a universal reference for an angle, we introduce the standard position of an angle in terms of a unit circle. The equation for a circle with radius $r = 1$ and a center at the origin $(0,0)$ is $x^2 + y^2 = 1$. Standard position for an angle always forms an arc on the unit circle that starts on the x-axis at $(1,0)$ and moves along the circle in the counter clockwise direction for positive angles (clockwise for negative angles).

An angle greater than 2π will wrap completely around the circle and continue. Multiples of 2π $(0, \pm 2\pi, \pm 4\pi, \ldots)$ all end at the same point $(1,0)$ on the unit circle but correspond to a different number of rotations. In fact, every point on the unit circle has infinitely many different angles (that differ by multiples of 2π) that end at that point when in standard position.

Example B.2.2 Find all angles that terminate at the point $(0,1)$.

Solution. The point $(0,1)$ is a quarter turn in the positive direction or three quarters turn in the negative direction. So the angles $\theta = \frac{1}{4}\tau = \frac{1}{4}(2\pi) = \frac{\pi}{2}$ and $\theta = -\frac{3}{4}\tau = -\frac{3}{4}(2\pi) = -\frac{3\pi}{2}$ are two of the angles desired.

Notice that these angles are 2π apart. In fact, we can add any integer multiple of 2π and end at the same spot. One of writing this is to say

$$\theta = \frac{\pi}{2} + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots.$$

□

Example B.2.3 Where does the angle $\theta = \frac{28\pi}{3}$ terminate?

Solution. This is the same as dealing with improper fractions or division with remainders. First, recognize that $2\pi = \frac{6\pi}{3}$. Second, determine how many times 6 goes into 28 using division, to find 4 with a remainder of 4 $(28 = 4 \cdot 6 + 4)$. Rewrite the fraction:

$$\theta = \frac{28\pi}{3} = 4 \cdot 2\pi + \frac{4\pi}{3}.$$

To interpret this angle, recognize that the integer multiple of 2π corresponds to 4 complete rotations. Next, determine what fraction of a rotation corresponds to $\frac{4\pi}{3}$:

$$\frac{4\pi}{3} = \frac{2}{3}(2\pi).$$

So we continue another two-thirds of a rotation, which is 60 degrees past a half rotation and 30 degrees short of vertical.

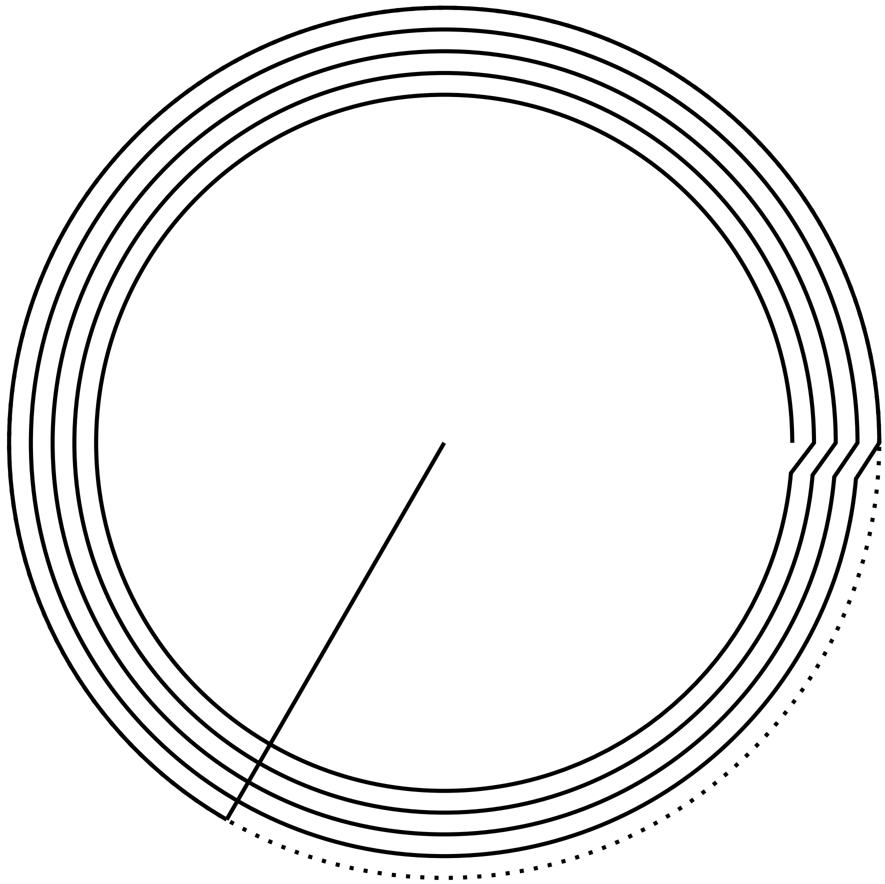

Figure B.2.4 Terminal position of $\theta = \frac{28\pi}{3}$.

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