B.3 Unit Circle Trigonometry

B.3.1 Unit Circle Trigonometry

When we first introduced the trigonometric functions of an angle, we did it for acute angles. What will we do for larger angles, or negative angles? We will use the unit circle standard position of an angle and choose a method that agrees with what we would expect for acute angles.

So consider a positive, acute angle $0 < \theta < \frac{\pi}{2}$ (notice that we continue to use radians) that has been placed in standard position on the unit circle. We have also drawn the corresponding right triangle in standard position. Because the unit circle has radius $r = 1$, the hypotenuse of our triangle has length 1, so that the legs are the cosine (adjacent) and the sine (opposite) of the angle.

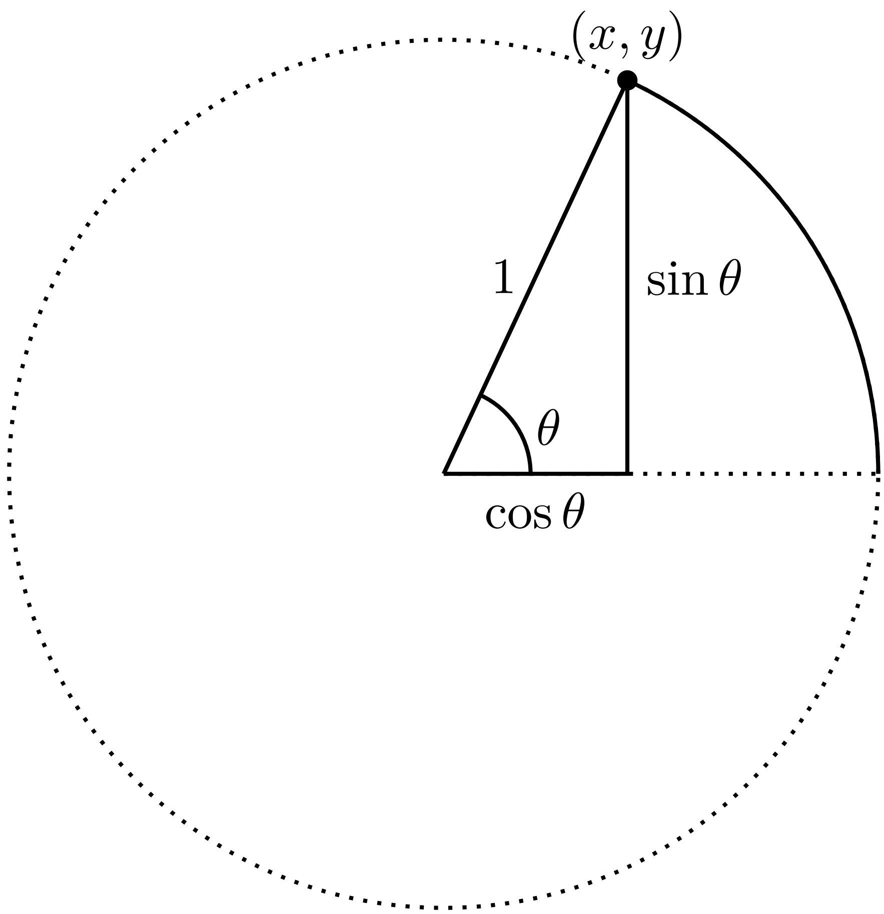

Figure B.3.1 An acute angle on the unit circle and with an associated right triangle in standard position.

By noticing that the length of the legs also corresponds to the x- and y-coordinates, this gives us our desired generalization. For any angle θ, we will define the cosine and sine of the angle as the x- and y-coordinates of the point on the unit circle for the terminal edge of the arc in standard position.

Definition B.3.2 If an angle θ is put in standard position and terminates at a point (x, y) on the unit circle, then the trigonometric functions of θ are defined by

$$\begin{aligned}
\cos\theta &= x \\
\sin\theta &= y \\
\tan\theta &= \frac{\sin\theta}{\cos\theta} = \frac{y}{x} \\
\sec\theta &= \frac{1}{\cos\theta} = \frac{1}{x} \\
\cot\theta &= \frac{\cos\theta}{\sin\theta} = \frac{x}{y} \\
\csc\theta &= \frac{1}{\sin\theta} = \frac{1}{y}
\end{aligned}$$

$\diamondsuit$

I suggest remembering the definitions of cosine and sine in terms of the x- and y-coordinates, and the other functions in terms of the ratios involving sine and cosine. One exception to this general principle is that it can sometimes be helpful to think of the tangent, which is the ratio y/x, as the slope of the terminal edge.

B.3.2 Periodic Behavior of Trigonometric Functions

Because the terminal edge of an arc repeatedly passes through the same points, the unit circle definitions of trigonometric functions create periodic functions. Most of the functions have a period of 2π corresponding to the rotation necessary to return to the same point. However, the tangent and cotangent functions each have a period of π. This is a consequence of the definition involving both x and y in their definitions. The ratio will be the same if both values change sign, which is precisely what happens for a half-rotation in the angle.

How can you remember the graphs of the functions? You can do this while also reinforcing your understanding of the unit-circle definitions by imagining rotating around the unit circle in a counter-clockwise direction. Draw the cosine (x-coordinate), sine (y-coordinate) and tangent (slope) functions as you go.

- Start at angle $\theta = 0$ with a terminal point $(1, 0)$. Using the coordinates and slope of the terminal edge gives:

$$\cos 0 = 1 \qquad \sin 0 = 0 \qquad \tan 0 = 0$$

- Go half a right angle to $\theta = \frac{\pi}{4}$ with a terminal point where $x = y$ at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Using the coordinates and slope of the terminal edge gives:

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \tan\frac{\pi}{4} = 1$$

- Finish the right angle at $\theta = \frac{\pi}{2}$ with a terminal point at $(0, 1)$. Using the coordinates and slope of the terminal edge gives:

$$\cos\frac{\pi}{2} = 0 \qquad \sin\frac{\pi}{2} = 1 \qquad \tan\frac{\pi}{4} = \text{undef.}$$

In general, you should remember that cosine and sine oscillate between peak values of -1 and 1. Paying attention to the unit circle, you will be able to identify the actual points where these are reached. The tangent, which measures a slope, goes through all possible values with negative values when the angle is in the second or fourth quadrants and positive when the angle is in the first or third quadrants.

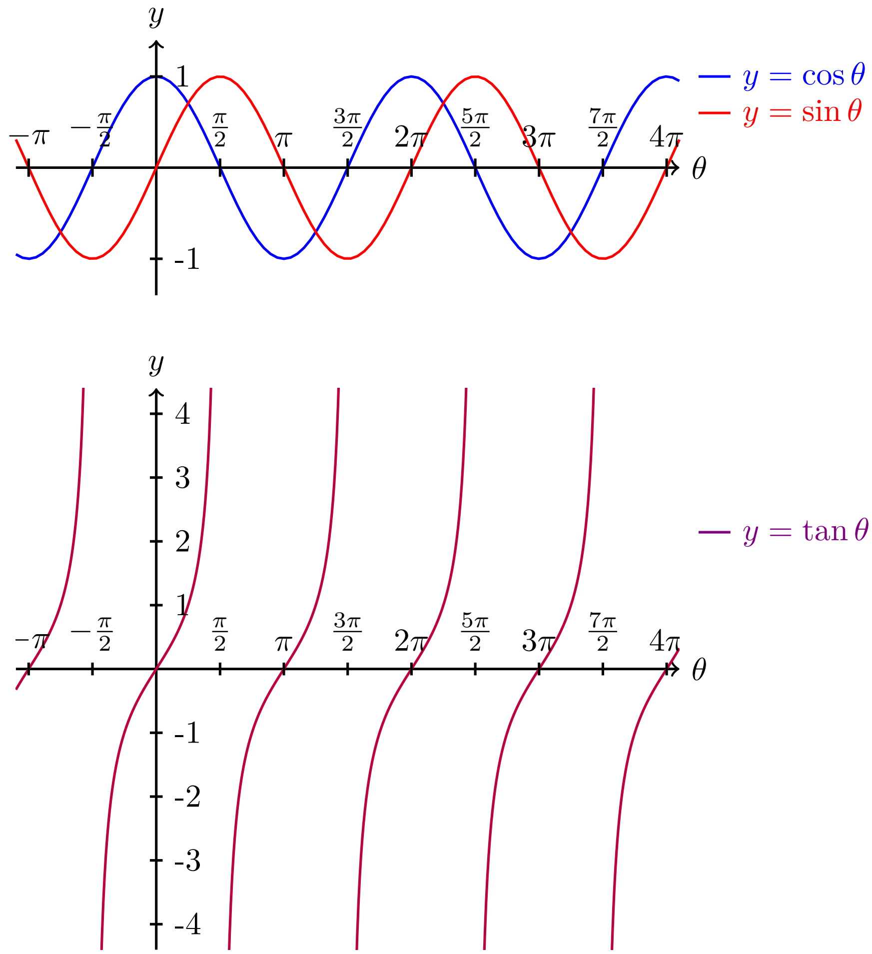

Figure B.3.3 Periodic graphs of the cosine, sine and tangent functions.

When thinking about the tangent function, if you will remember that it involves division by the cosine, then the breaks (vertical asymptotes) occur at every point where the cosine is zero. This exactly corresponds to where the terminal edge of the angle on the unit circle is vertical.