THE PARTITION FUNCTION MODULO PRIME POWERS.

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Abstract. Let $\ell \geq 5$ be prime, let $m \geq 1$ be an integer, and let $p(n)$ denote the partition function. Folsom, Kent, and Ono recently proved that there exists a positive integer $b_\ell(m)$ of size roughly m^2 such that the module formed from the $\mathbb{Z}/\ell^m\mathbb{Z}$-span of generating functions for $p\left(\frac{\ell^b n+1}{24}\right)$ with odd $b \geq b_\ell(m)$ has finite rank. The same result holds with "odd" b replaced by "even" b. Furthermore, they proved an upper bound on the ranks of these modules. This upper bound is independent of m; it is $\lfloor \frac{\ell+12}{24} \rfloor$.

In this paper, we prove, with a mild condition on ℓ, that $b_\ell(m) \leq 2m-1$. Our bound is sharp in all computed cases with $\ell \geq 29$. To deduce it, we prove structure theorems for the relevant $\mathbb{Z}/\ell^m\mathbb{Z}$-modules of modular forms. This work sheds further light on a question of Mazur posed to Folsom, Kent, and Ono.

1. Introduction and statement of results.

Let n be a positive integer. A partition of n is a non-increasing sequence of positive integers whose sum is n. The ordinary partition function, $p(n)$, counts the number of partitions of n. By convention, we set $p(0) := 1$; for $\alpha \notin \mathbb{N} \cup \{0\}$, we set $p(\alpha) := 0$.

Some of the most fundamental and elegant arithmetic properties of $p(n)$ are the Ramanujan congruences and their prime power extensions proved by Atkin [5], Ramanujan [19], and Watson [25]. Let $\ell \geq 5$ be prime, and let $b \geq 0$ be an integer. With $1 \leq \delta_\ell(b) \leq \ell^b - 1$ and $24\delta_\ell(b) \equiv 1 \pmod{\ell^b}$, the extensions are, for all $n \geq 0$,

$$\begin{aligned} p(5^b n + \delta_5(b)) &\equiv 0 \pmod{5^b}, \\ p(7^b n + \delta_7(b)) &\equiv 0 \pmod{7^{\lfloor b/2 \rfloor + 1}}, \\ p(11^b n + \delta_{11}(b)) &\equiv 0 \pmod{11^b}. \end{aligned} \tag{1.1}$$

These congruences have inspired a terrific amount of interest in the study of $p(n)$, its generating function, and allied functions. Landmark works include, for example, the papers of Andrews and Garvan [4] and of Atkin and Swinnerton-Dyer [8] on the rank and crank partition statistics. They also include papers of Ahlgren and Ono [1], [2], [16] which prove, for fixed M coprime to 6, that there are infinitely many non-nested arithmetic progressions $An + B$ such that $p(An + B) \equiv 0 \pmod{M}$. For further examples, see [3], [17], and [18] and the references therein.

1.1. **Main theorems.** We now focus on recent work of Folsom, Kent, and Ono [12] related to (1.1). Both this work and the works proving (1.1) result from using the theory of modular forms to study generating functions of type

$$P_\ell(b; z) := \sum_{n=0}^{\infty} p\left(\frac{\ell^b n + 1}{24}\right) q^{n/24}. \tag{1.2}$$

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The principal result in [12] is the following.

Theorem 1.1 ([12], Theorem 1.2)**.** *Let $\ell \geq 5$ be prime, and let $m \geq 1$. Then there is an integer*

$$b_\ell(m) \leq 2\left(\left\lfloor \frac{\ell-1}{12} \right\rfloor + 2\right) m - 3$$

such that the $\mathbb{Z}/\ell^m\mathbb{Z}$-module

$$\mathrm{Span}_{\mathbb{Z}/\ell^m\mathbb{Z}}\{P_\ell(b;z) : b \geq b_\ell(m),\ b \text{ odd}\}$$

has finite rank

$$r_\ell(m) \leq \left\lfloor \frac{\ell-1}{12} \right\rfloor - \left\lfloor \frac{\ell^2-1}{24\ell} \right\rfloor = \left\lfloor \frac{\ell+12}{24} \right\rfloor := R_\ell. \tag{1.3}$$

Similarly, the "even" $\mathbb{Z}/\ell^m\mathbb{Z}$-module $\mathrm{Span}_{\mathbb{Z}/\ell^m\mathbb{Z}}\{P_\ell(b;z) : b \geq b_\ell(m),\ b \text{ even}\}$ *has finite rank bounded by R_ℓ.*

Remarks.

(1) From (1.3), primes $\ell \in \{5, 7, 11\}$ have $R_\ell = 0$, which explains (1.1). For primes $13 \leq \ell \leq 31$, (1.3) gives $R_\ell = 1$. It follows for all integers b_1, b_2 with $b_1 \equiv b_2 \pmod 2$ and $b_2 > b_1 \geq b_\ell(m)$, that there exists an integer $A_\ell(b_1, b_2, m)$ with
$$p\left(\frac{\ell^{b_2}n+1}{24}\right) \equiv A_\ell(b_1, b_2, m) \cdot p\left(\frac{\ell^{b_1}n+1}{24}\right) \pmod{\ell^m} \text{ for all } n.$$
This is Theorem 1.1 of [12].

(2) The authors of [12] use Theorem 1.1 to settle a conjecture of Atkin from [6]. For primes $5 \leq \ell \leq 31$, $m \geq 1$, and $b \geq b_\ell(m)$, they prove (see Theorem 1.3 of [12]) that $P_\ell(b; 24z)$ is an eigenform modulo ℓ^m for all of the Hecke operators in half-integral weight $\ell^{m-1}(\ell-1) - 1/2$ on the group $\Gamma_0(576)$.

(3) For all $m \geq 1$, we have $r_\ell(m) \leq r_\ell(m+1)$ and $b_\ell(m) \leq b_\ell(m+1)$.

The bound on $b_\ell(m)$ in Theorem 1.1 is not sharp since, for example, (1.1) implies that one may take $b_5(m) = b_{11}(m) = m$, while one may take $b_7(m) = 2m - 2$. Our main result is a sharpened bound on $b_\ell(m)$. In general, $b_\ell(m)$ depends on an explicitly calculable constant d_ℓ related to the nullity of an operator $D(\ell)$ (see (1.7) below) on cusp forms of weight $\ell - 1$ on $\mathrm{SL}_2(\mathbb{Z})$ with ℓ-integral coefficients reduced modulo ℓ. In Section 5, we define d_ℓ. Calculations reveal, for all primes $5 \leq \ell \leq 1300$ (the primes we considered), that $d_\ell = 0$.

Theorem 1.2. *Let $\ell \geq 5$ be prime, and suppose that $d_\ell = 0$. Then for all $m \geq 1$, we have*

$$b_\ell(m) \leq 2m - 1.$$

Remarks.

(1) The bound on $b_\ell(m)$ is sharp for all computed cases with $\ell \geq 29$.

(2) In Section 5, we modify the bound on $b_\ell(m)$ in the theorem for primes ℓ with $d_\ell > 0$.

We exhibit an example of the type of congruence predicted by Theorems 1.1 and 1.2. See Section 6.1 for further examples and Section 6.2 comments on how these examples were computed.

Example. Let $\ell = 53$. Our calculations show that $r_{53}(m) = R_{53} = 2$ for all $m \geq 1$, that $b_{53}(1) = 1$, and that $b_{53}(2) = 3$. The following congruences hold for all $n \geq 0$:

$$p(53n + 42) \equiv 22p(53^3 n + 117861) + 25p(53^5 n + 331071432) \pmod{53},$$
$$p(53^3 n + 117861) \equiv 2672p(53^5 n + 331071432) + 2304p(53^7 n + 929979652371) \pmod{53^2}.$$

Next, we give a consequence of Theorems 1.1 and 1.2.

Corollary 1.3. *Let $\ell \geq 5$ be prime, let $m \geq 1$, and let $b_\ell(m)$ be as in Theorem 1.2. Then there exists an integer $c_\ell \geq 1$ such that for all $b \geq b_\ell(m)$ and all $n \geq 0$, we have*

$$p\left(\frac{\ell^b n + 1}{24}\right) \equiv p\left(\frac{\ell^{b+2c_\ell \ell^{m-1}} n + 1}{24}\right) \pmod{\ell^m}.$$

We illustrate the corollary with an example.

Example. Let $\ell = 41$. We find that $c_\ell = 10$, and thus for all $n \geq 0$, we have

$$p(41n + 12) \equiv p(41^{21} n + 215 \ldots 4912) \pmod{41}.$$

We refer to Section 6 for more examples of the corollary.

With $r_\ell(m)$ as in (1.3), the proof in Section 4 shows that c_ℓ is the order of a matrix in $\mathrm{GL}_{r_\ell(1)}(\mathbb{Z}/\ell\mathbb{Z})$. We note that the corollary is similar to the following result of Y. Yang (Theorem 6.7 of [26]). Let $m \neq \ell$ be primes with $m \geq 13$ and $\ell \geq 5$, and let $i \geq 1$. Then for all n, $r \geq 0$, we have

$$p\left(\frac{m^i \ell^r n + 1}{24}\right) \equiv p\left(\frac{m^i \ell^{M+r} n + 1}{24}\right) \pmod{m^i}.$$

Yang's proof uses the existence of a non-trivial Hecke-invariant subspace of half-integral weight cusp forms, and it reveals that M is the order of a matrix in $\mathrm{PGL}_{\lfloor \frac{m}{12} \rfloor}\left(\mathbb{Z}/m^i\mathbb{Z}\right)$.

1.2. **Reformulation of main results.** The work of Folsom, Kent, and Ono introduces a new framework for studying the generating functions $P_\ell(b; z)$ modulo powers of ℓ. The central objects in this framework are certain submodules $\Omega_\ell(m)$ of the $\mathbb{Z}/\ell^m\mathbb{Z}$-module of cusp forms of weight $\ell^{m-1}(\ell - 1)$ on $\mathrm{SL}_2(\mathbb{Z})$ with ℓ-integral coefficients reduced modulo ℓ^m. We define $\Omega_\ell(m)$ in Theorem 1.4 and (1.14) below. Furthermore, the authors in [12] define an operator $D(\ell)$ (see (1.7)) which acts on these submodules and plays an important role in their study. The submodules $\Omega_\ell(m)$ are objects of interest independent of their connection to partitions. Our work in this paper uncovers some of their fine structure properties, thereby addressing a question of Mazur from the appendix to [12], which we restate here.

Question (Mazur). Do the spaces $\Omega_\ell(m)$ "compile well" to produce a clean free $\mathbb{Z}_\ell$-module? Do the Hecke operators work well on these spaces?

We therefore reframe Theorems 1.1 and 1.2 in the abstract context of the submodules $\Omega_\ell(m)$. Let $N \geq 1$ and k be integers. We denote the space of weakly holomorphic modular forms of weight k on $\Gamma_0(N)$ by $M_k^!(\Gamma_0(N))$. A form $f(z) \in M_k^!(\Gamma_0(N))$ has poles, if any, supported at cusps, and it has a Fourier expansion

$$f(z) = \sum_{n=n_0}^{\infty} a(n) q^n \quad (q := e^{2\pi i z})$$

with $n_0 \gg -\infty$. We denote by $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ the subspaces of holomorphic modular forms and cusp forms, respectively. When $N = 1$, we omit reference to the group. For details on modular forms, see Section 2.

Remark. An alternative and more general context for our work arises from viewing modular forms geometrically in the sense of Katz [14]. In this setting, weakly holomorphic modular forms correspond to rational sections of line bundles on modular curves with prescribed divisors corresponding to poles at cusps. One may identify such forms as rules on elliptic curves with level structure. Further, one can use the Tate curve to identify a modular form with its q-expansion. In this way, the technical q-expansion manipulations we require in Section 3, 4, and 5 may be viewed as "mod-ing" out classical moduli problems.

Some of the modular forms we require arise as quotients of

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{1.4}$$

the Dedekind eta-function. An important example is

$$\Phi_\ell(z) := \frac{\eta(\ell^2 z)}{\eta(z)} = q^{\frac{\ell^2-1}{24}} + \cdots \in M_0^!(\Gamma_0(\ell^2)) \cap \mathbb{Z}[[q]]. \tag{1.5}$$

We also define certain operators on spaces of modular forms. For primes $\ell \geq 5$, we define Atkin's $U(\ell)$-operator and Folsom-Kent-Ono's $D(\ell)$-operator on $f(z) \in M_k^!(\Gamma_0(N))$ by

$$f(z) \mid U(\ell) := \sum_{\ell n = n_0}^{\infty} a(\ell n) q^n, \tag{1.6}$$

$$f(z) \mid D(\ell) := (\Phi_\ell(z) f(z)) \mid U(\ell). \tag{1.7}$$

It is useful to package the operators $U(\ell)$ and $D(\ell)$ together as $X(\ell)$ and $Y(\ell)$:

$$f(z) \mid X(\ell) := f(z) \mid U(\ell) \mid D(\ell), \tag{1.8}$$

$$f(z) \mid Y(\ell) := f(z) \mid D(\ell) \mid U(\ell). \tag{1.9}$$

We continue to follow [12] by defining, for all integers $b \geq 0$, a sequence of functions $\{L_\ell(b; z)\}$. We set $L_\ell(0; z) := 1$, and for all $b \geq 1$, we set

$$L_\ell(b; z) := \begin{cases} L_\ell(b-1; z) \mid D(\ell) & \text{if } b \text{ is odd,} \\ L_\ell(b-1; z) \mid U(\ell) & \text{if } b \text{ is even.} \end{cases} \tag{1.10}$$

Euler's infinite product generating function for the partition function,

$$\sum_{m=0}^{\infty} p(m) q^m = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)},$$

is a natural starting point for connecting partitions and modular forms. Using it together with (1.2) and (1.4) - (1.10), one can show as in Lemma 2.1 of [12] that

$$L_\ell(b; z) = \begin{cases} \eta(\ell z) P_\ell(b; z) & \text{if } b \text{ is odd,} \\ \eta(z) P_\ell(b; z) & \text{if } b \text{ is even.} \end{cases}$$

We now fix integers $b \geq 0$ and $m \geq 1$. We study the $\mathbb{Z}/\ell^m\mathbb{Z}$-modules

$$\text{(1.11)} \quad \operatorname{Span}_{\mathbb{Z}/\ell^m\mathbb{Z}}\{L_\ell(\beta; z) \bmod \ell^m : \beta \geq b,\ \beta \equiv b \pmod 2\} =: \begin{cases} \Lambda_\ell^{\text{odd}}(b, m) & \text{if } b \text{ is odd,} \\ \Lambda_\ell^{\text{even}}(b, m) & \text{if } b \text{ is even.} \end{cases}$$

It follows from (1.8), (1.9), and (1.10) that

$$\Lambda_\ell^{\text{odd}}(b, m) = \operatorname{Span}_{\mathbb{Z}/\ell^m\mathbb{Z}}\{L_\ell(b; z) \mid X(\ell)^s : s \geq 0\};$$
$$\Lambda_\ell^{\text{even}}(b, m) = \operatorname{Span}_{\mathbb{Z}/\ell^m\mathbb{Z}}\{L_\ell(b; z) \mid Y(\ell)^t : t \geq 0\}.$$

Moreover, we have the following commutative diagram of $\mathbb{Z}/\ell^m\mathbb{Z}$-module homomorphisms:

$$\text{(1.12)} \quad \begin{array}{ccc} \Lambda_\ell^{\text{odd}}(b, m) & \xrightarrow{U(\ell)} & \Lambda_\ell^{\text{even}}(b+1, m) \\ \Big\downarrow X(\ell) & \swarrow D(\ell) & \Big\downarrow Y(\ell) \\ \Lambda_\ell^{\text{odd}}(b+2, m) & \xrightarrow[U(\ell)]{} & \Lambda_\ell^{\text{even}}(b+3, m) \end{array}$$

In the foregoing context, we recast Theorems 1.1 and 1.2 in a unified form.

Theorem 1.4. *Let $\ell \geq 5$ be prime, let $m \geq 1$, and suppose that $d_\ell = 0$. Then there is an integer*

$$b_\ell(m) \leq 2m - 1$$

such that the nested sequence of $\mathbb{Z}/\ell^m\mathbb{Z}$-modules

$$\Lambda_\ell^{odd}(1, m) \supseteq \Lambda_\ell^{odd}(3, m) \supseteq \cdots \supseteq \Lambda_\ell^{odd}(2b+1, m) \supseteq \cdots$$

is constant for all b with $2b + 1 \geq b_\ell(m)$. Moreover, if one denotes the stabilized $\mathbb{Z}/\ell^m\mathbb{Z}$-module by $\Omega_\ell^{odd}(m)$, then we have

$$\text{(1.13)} \qquad r_\ell(m) := \operatorname{rank}_{\mathbb{Z}/\ell^m\mathbb{Z}}(\Omega_\ell^{odd}(m)) \leq \left\lfloor \frac{\ell - 1}{12} \right\rfloor - \left\lfloor \frac{\ell^2 - 1}{24\ell} \right\rfloor = \left\lfloor \frac{\ell + 12}{24} \right\rfloor := R_\ell.$$

Similarly, the sequence of "even" $\mathbb{Z}/\ell^m\mathbb{Z}$-modules $\{\Lambda_\ell^{even}(b, m) : b \geq b_\ell(m)\}$ is stable. If we denote the stable module by $\Omega_\ell^{even}(m)$, then an upper bound on its rank is R_ℓ.

Remark 1. In view of commutative diagram (1.12), we see that $b_\ell(m)$ is the smallest positive integer b for which $X(\ell) : \Lambda_\ell^{\text{odd}}(b, m) \to \Lambda_\ell^{\text{odd}}(b + 2, m)$ is an isomorphism. Moreover, the theorem implies that the following maps are isomorphisms:

$$U(\ell) : \Omega_\ell^{\text{odd}}(m) \to \Omega_\ell^{\text{even}}(m), \qquad D(\ell) : \Omega_\ell^{\text{even}}(m) \to \Omega_\ell^{\text{odd}}(m),$$
$$X(\ell) : \Omega_\ell^{\text{odd}}(m) \to \Omega_\ell^{\text{odd}}(m), \qquad Y(\ell) : \Omega_\ell^{\text{even}}(m) \to \Omega_\ell^{\text{even}}(m).$$

Remark 2. Theorem 7.1 of the appendix to [12] describes work of Calegari [10] on how the stability and finiteness results in Theorem 1.4 can be generalized using aspects of the theory of half-integral weight overconvergent p-adic modular forms developed by Ramsey [20], [21]. However, bounds on the stability threshold, $b_\ell(m)$, and on the rank, $r_\ell(m)$, require explicit analysis specific to the inputs (1.5) - (1.10). In what follows, we provide such an analysis.

Remark 3. Recent work of Belmont, Lee, Musat, and Trebat-Leder [9] adapts Theorems 1.1, 1.2, and 1.4, to Andrews' smallest parts partition function, $spt(n)$, and to the rth power partition function, $p_r(n)$.

The paper is organized as follows. In Section 2, we state facts we need on modular forms. In Section 3, we prove Lemma 3.1 and Lemma 3.6. These lemmas underpin the facts we prove in Sections 4 and 5 on the algebraic structure of the $\mathbb{Z}/\ell^m\mathbb{Z}$-modules $\Lambda_\ell^{\mathrm{odd}}(b,m)$, $\Lambda_\ell^{\mathrm{even}}(b,m)$, and

$$\Omega_\ell(m) := \Omega_\ell^{\mathrm{odd}}(m) + \Omega_\ell^{\mathrm{even}}(m). \tag{1.14}$$

Stability and finiteness of rank in Theorem 1.4 follow directly from Lemma 3.1 via Corollary 3.5. In Section 4, we use this corollary to exhibit explicit injections from $\Omega_\ell^{\mathrm{odd}}(m)$ and $\Omega_\ell^{\mathrm{even}}(m)$ into $S_{\ell-1}$, thereby proving the upper bound (1.13) and reproving Theorem 1.1. We also prove Corollary 1.3 in Section 4. Theorem 1.2 follows from Lemma 3.6 and the structure developed in Section 4, as we show in Section 5. In Section 6, we give more examples of Theorem 1.2 and Corollary 1.3. We also thoroughly explain how we computed examples for all primes $13 \le \ell \le 1297$.

Acknowledgments. To be entered later.

2. Preliminary facts on modular forms.

The proofs of our results require certain facts from the theory of modular forms. For details, see for example, [11] or [13].

2.1. **Modular forms.** We first discuss operators on spaces of modular forms. One may consult [7] and [23] in addition to the references above. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, and let $N \ge 1$ and k be integers. We define the slash operator on $f(z) \in M_k^!(\Gamma_0(N))$ by

$$(f \mid_k \gamma)(z) := (\det \gamma)^{k/2}(cz+d)^{-k} f(\gamma z). \tag{2.1}$$

Let $\ell \ge 5$ be prime. We define the operator $V(\ell)$ on $f(z) \in M_k^!(\Gamma_0(N))$ by

$$f(z) \mid V(\ell) := \sum_{n=\ell n_0}^{\infty} a(n) q^{\ell n}.$$

With $U(\ell)$ as in (1.6), one finds that

$$f(z) \mid U(\ell) = \ell^{k/2-1} \sum_{j=0}^{\ell-1} f \mid_k \begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix} \tag{2.2}$$

and that

$$f(z) \mid V(\ell) = \ell^{-k/2} f \mid_k \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} = f(\ell z). \tag{2.3}$$

Next, for primes $\ell \nmid N$, we define the Hecke operator $T(\ell, k)$ on $f(z) \in M_k^!(\Gamma_0(N))$ by

$$f(z) \mid T(\ell,k) = f(z) \mid U(\ell) + \ell^{k-1} f(z) \mid V(\ell). \tag{2.4}$$

If $f(z) \in M_k^!(\Gamma_0(\ell))$, then we define the trace of f by

$$\mathrm{Tr}(f) := f + \ell^{1-\frac{k}{2}} \left(f \mid_k W(\ell)\right) \mid U(\ell), \tag{2.5}$$

where $W(\ell) := \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}$. We record basic properties of the operators under consideration. See for example, [7] and [23]. We refer to (1.8) and (1.9) for definitions of the operators $X(\ell)$ and $Y(\ell)$.

Lemma 2.1. *Let $\ell \geq 5$ be prime, and let $k \in \mathbb{Z}$.*

(1) *Let $j \geq 0$, and suppose that $f(z) \in M_k^!(\Gamma_0(\ell^j))$. Then we have*

$$f(z) \mid U(\ell), \quad f(z) \mid D(\ell) \in \begin{cases} M_k^!(\Gamma_0(\ell)), & j \in \{0,1\}, \\ M_k^!(\Gamma_0(\ell^{j-1})), & j \geq 2; \end{cases}$$

$$f(z) \mid X(\ell), \quad f(z) \mid Y(\ell) \in \begin{cases} M_k^!(\Gamma_0(\ell)), & j \in \{0,1,2\}, \\ M_k^!(\Gamma_0(\ell^{j-2})), & j \geq 3; \end{cases}$$

$$f(z) \mid V(\ell) \in M_k^!(\Gamma_0(\ell^{j+1})).$$

(2) *Let $N \geq 1$, let $f(z) \in M_k^!(\Gamma_0(N))$, and suppose that $\ell \nmid N$. Then we have $f(z) \mid \ T(\ell, k) \in \ M_k^!(\Gamma_0(N))$.*

(3) *Suppose that $f(z) \in M_k^!(\Gamma_0(\ell))$. Then we have $f \mid_k W(\ell) \in M_k^!(\Gamma_0(\ell))$ and $Tr(f) \in \ M_k^!$.*

(4) *Suppose that $f(z) \in M_k^!$. Then we have $f \mid_k W(\ell) = \ell^{k/2} f \mid V(\ell) \in M_k^!(\Gamma_0(\ell))$.*

Next, we state the modular transformation law for the eta-function (1.4). For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, there exists a 24-th root of unity $\epsilon_{a,b,c,d}$ for which

$$\eta\left(\frac{az+b}{cz+d}\right) = \epsilon_{a,b,c,d}(cz+d)^{1/2}\eta(z). \tag{2.6}$$

We always take the branch of the square root having non-negative real part. Let $\zeta_{24} = e^{2\pi i/24}$. Special cases of the transformation law include

$$\eta(z+1) = \zeta_{24}\eta(z), \quad \eta\left(\frac{-1}{z}\right) = \sqrt{\frac{z}{i}} \cdot \eta(z). \tag{2.7}$$

Further modular forms which play a central role in our work are given as follows: For $k \geq 4$ and even, we have

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sum_{d \mid n} d^{k-1} q^n \in M_k,$$

where B_k is the kth Bernoulli number. The Ramanujan Delta-function is given by

$$\Delta(z) := \eta(z)^{24} \in S_{12}.$$

We also define, for primes $\ell \geq 5$,

$$A_\ell(z) := \frac{\eta(z)^\ell}{\eta(\ell z)} = 1 + \cdots \in M_{\frac{\ell-1}{2}}(\Gamma_1(\ell)) \cap \mathbb{Z}[[q]]. \tag{2.8}$$

We note that $A_\ell(z)^2 \in M_{\ell-1}(\Gamma_0(\ell)) \cap \mathbb{Z}[[q]]$.

2.2. **Modular forms modulo prime powers.** Let $\ell \geq 5$ be prime, and let $\mathbb{Z}_{(\ell)}$ denote the localization of $\mathbb{Z}$ at ℓ. We first consider modular forms with coefficients in $\mathbb{Z}_{(\ell)}$ reduced modulo ℓ. For details, see [22] and [24]. Let $k \geq 4$ be even, and let $f(z) \in M_k \cap \mathbb{Z}_{(\ell)}[[q]]$. The filtration of $f(z) \in M_k \cap \mathbb{Z}_{(\ell)}[[q]]$ with $f(z) \not\equiv 0 \pmod{\ell}$ is defined by

$$w_\ell(f) := \inf\{k' : \text{there exists } g \in M_{k'} \cap \mathbb{Z}[[q]] \text{ with } f \equiv g \pmod{\ell}\}.$$

If $f(z) \equiv 0 \pmod{\ell}$, then we set $w_\ell(f) := -\infty$. For $f_1(z) \in M_{k_1}$ and $f_2(z) \in M_{k_2}$ with $k_1 \equiv k_2 \pmod{\ell - 1}$ and ℓ-integral coefficients, we have

$$w_\ell(f_1 + f_2) \leq \max\{w_\ell(f_1), w_\ell(f_2)\}; \tag{2.9}$$

equality holds if $w_\ell(f_1) \neq w_\ell(f_2)$. A lemma of Serre (Lemme 2 of [23]) describes how $U(\ell)$ affects filtration.

Lemma 2.2. *Let $\ell \geq 5$ be prime, and let $f(z) \in M_k \cap \mathbb{Z}_{(\ell)}[[q]]$.*

(1) *We have*

$$w_\ell(f \mid U(\ell)) \leq \ell + \frac{w_\ell(f) - 1}{\ell}.$$

(2) *Suppose that $w_\ell(f) = \ell - 1$. Then we have $w_\ell(f \mid U(\ell)) = \ell - 1$.*

We also observe that

$$\Phi_\ell(z) \equiv \Delta(z)^{\frac{\ell^2-1}{24}} \pmod{\ell}. \tag{2.10}$$

We now turn to facts on modular forms with coefficients in $\mathbb{Z}_{(\ell)}$ reduced modulo ℓ^j with $j \geq 1$. For details, see [23]. In view of (2.4) and Lemma 2.1 (2), we have the following.

Proposition 2.3. *Let $k \geq 1$, and let $f(z) \in M_k^! \cap \mathbb{Z}_{(\ell)}((q))$.*

(1) *We have $f(z) \mid T(\ell, k) \equiv f(z) \mid U(\ell) \pmod{\ell^{k-1}}$.*
(2) *The operator $U(\ell)$ stabilizes $M_k^! \cap \mathbb{Z}_{(\ell)}((q))$ modulo ℓ^{k-1}.*

We next give a useful fact on congruences for power series modulo powers of ℓ which follows by induction using Fermat's Little Theorem.

Lemma 2.4. *Suppose that $f(z) \in \mathbb{Z}_{(\ell)}[[q]]$ has $f(z) \equiv 1 \pmod{\ell}$. Then for all $j \geq 1$, we have $f(z)^{\ell^{j-1}} \equiv 1 \pmod{\ell^j}$.*

When $\ell \geq 5$ is prime, properties of Bernoulli numbers imply that $E_{\ell-1}(z) \in \mathbb{Z}_{(\ell)}[[q]]$ and that $E_{\ell-1}(z) \equiv 1 \pmod{\ell}$; Fermat's Little Theorem implies that $A_\ell(z)^2 \equiv 1 \pmod{\ell}$. Therefore, we may apply Lemma 2.4 to these forms.

Proposition 2.5. *For all $j \geq 1$, we have*

$$E_{\ell-1}(z)^{\ell^{j-1}} \equiv 1 \pmod{\ell^j}, \quad A_\ell(z)^{2\ell^{j-1}} \equiv 1 \pmod{\ell^j}.$$

To prove our results, we carefully keep track of the largest power of ℓ dividing all coefficients of series in $\mathbb{Z}_{(\ell)}((q))$. For this purpose, we define v_ℓ on $\mathbb{Q}$ by

$$v_\ell\left(\frac{m}{n}\right) := \mathrm{ord}_\ell(m) - \mathrm{ord}_\ell(n),$$

and we set $v_\ell(0) := \infty$. Our definition extends to $f(z) = \sum a(n)q^n \in \mathbb{Z}_{(\ell)}((q))$ by

$$v_\ell(f) := \inf\{n : v_\ell(a(n))\}. \tag{2.11}$$

With $f(z),\, g(z) \in \mathbb{Z}_{(\ell)}((q))$, we have

$$v_\ell(f+g) \geq \min\{v_\ell(f), v_\ell(g)\}\,; \tag{2.12}$$

equality holds if $v_\ell(f) \neq v_\ell(g)$.

3. Two lemmas.

The proofs of our results rest on two lemmas which we prove in this section. The first lemma asserts, subject to certain hypotheses, that the operator $D(\ell)$ (as in 1.7) stabilizes the space $M_{\ell^{k-1}(\ell-1)} \cap \mathbb{Z}[[q]]$ with coefficients reduced modulo ℓ^k.

Lemma 3.1. *Let $\ell \geq 5$ be prime, let $n \geq 1$, and let $\Psi(z) \in \mathbb{Z}_{(\ell)}[[q]]$. Suppose, for all $1 \leq k \leq n$ that there exists $g_k(z) \in M_{\ell^{k-1}(\ell-1)} \cap \mathbb{Z}[[q]]$ with $\Psi(z) \equiv g_k(z) \pmod{\ell^k}$. Then for all $1 \leq k \leq n$, there exists $h_k(z) \in S_{\ell^{k-1}(\ell-1)} \cap \mathbb{Z}[[q]]$ with $\Psi(z) \mid D(\ell) \equiv h_k(z) \pmod{\ell^k}$.*

Proof. Suppose that $k = 1$ and $\ell \geq 7$. Using (1.7) and (2.10), we compute

$$\Psi(z) \mid D(\ell) \equiv g_1(z) \mid D(\ell) \equiv (g_1(z)\Phi_\ell(z)) \mid U(\ell) \equiv \left(g_1(z)\Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell) \pmod{\ell}.$$

Since $\ell \geq 7$, $g_1(z) \in M_{\ell-1}$, and $\Delta(z)^{\frac{\ell^2-1}{24}} \in S_{\frac{\ell^2-1}{2}}$, an application of Lemma 2.2 gives

$$\begin{aligned} w_\ell\left(\left(g_1(z)\Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell)\right) &\leq \ell + \frac{\ell - 1 + \frac{\ell^2-1}{2} - 1}{\ell} \\ &= (\ell-1)\left(1 + \frac{\ell+5}{2\ell}\right) < 2(\ell-1). \end{aligned} \tag{3.1}$$

It follows that there exists $h_1(z) \in S_{\ell-1}$ with $\Psi(z) \mid D(\ell) \equiv h_1(z) \pmod{\ell}$. If $k = 1$ and $\ell = 5$, Proposition 2.5 implies that the form $g_1(z) \in M_4 \cap \mathbb{Z}[[q]] \subseteq \mathbb{C}E_4(z)$ is congruent modulo 5 to a constant $c \in \mathbb{Z}$. From (1.7) and (2.10), we compute $g_1(z) \mid D(5) \equiv c\Delta(z) \mid U(5) \equiv 0 \pmod{5}$.

Now, we suppose that $k \geq 2$. We have the following congruence of modular forms in $M_{\ell^{k-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ modulo ℓ^k:

$$\begin{aligned} \Psi(z) \mid D(\ell) \equiv g_k(z) \mid D(\ell) &\equiv \left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \cdot E_{\ell-1}(z)^{\ell^{k-1}}\right) \mid D(\ell) \\ &+ \left(g_{k-1}(z)E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)} - \frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \cdot E_{\ell-1}(z)^{\ell^{k-1}}\right) \mid D(\ell) \\ &+ \left(g_k(z) - g_{k-1}(z)E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)}\right) \mid D(\ell) \pmod{\ell^k}. \end{aligned} \tag{3.2}$$

We closely examine each summand.

In view of Proposition 2.5, the first summand simplifies as

$$\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \cdot E_{\ell-1}(z)^{\ell^{k-1}}\right) \mid D(\ell) \equiv \frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \pmod{\ell^k}.$$

From (2.8) and Lemma 2.1, we observe that

$$\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \in M_0^!(\Gamma_0(\ell)) \cap \mathbb{Z}[[q]]. \tag{3.3}$$

We prove the following proposition.

Proposition 3.2. *The form* (3.3) *is congruent modulo* ℓ^k *to a form in* $S_{\ell^{k-1}(\ell-1)} \cap \mathbb{Z}[[q]]$.

Proof. The proof follows Serre's argument for Théorème 10 in [23]. To begin, we define

$$h(z) := E_{\ell-1}(z) - \ell^{\ell-1} E_{\ell-1}(z) \mid V(\ell) \in M_{\ell-1}(\Gamma_0(\ell)) \cap \mathbb{Z}_{(\ell)}[[q]]. \tag{3.4}$$

By Proposition 2.5, we see that $h(z) \equiv 1 \pmod{\ell}$, and hence, from Lemma 2.4 that $h(z)^{\ell^{k-1}} \equiv 1 \pmod{\ell^k}$. Therefore, we have

$$\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \equiv \left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell)\right) h(z)^{\ell^{k-1}} \pmod{\ell^k};$$

(3.3) and (3.4) imply that the form on the right side is in $M^!_{\ell^{k-1}(\ell-1)}(\Gamma_0(\ell)) \cap \mathbb{Z}_{(\ell)}[[q]]$. Using Lemma 2.1 (3), we note that

$$\mathrm{Tr}\left(\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell)\right) h(z)^{\ell^{k-1}}\right)$$

is on $\mathrm{SL}_2(\mathbb{Z})$ with weight $\ell^{k-1}(\ell-1)$. Hence, it suffices to show that

$$\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \equiv \mathrm{Tr}\left(\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell)\right) h(z)^{\ell^{k-1}}\right) \pmod{\ell^k}.$$

With v_ℓ as in (2.11), Lemme 9 of [23] implies that

$$v_\ell\left(\mathrm{Tr}\left(\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell)\right) h(z)^{\ell^{k-1}}\right) - \frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell)\right) \geq$$
$$\min\left(k + v_\ell\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell)\right),\ \ell^{k-1} + 1 + v_\ell\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \mid_0 W(\ell)\right)\right).$$

Since $\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell)$ has integer coefficients, we have $v_\ell\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell)\right) \geq 0$; as such, we show that

$$\ell^{k-1} + 1 + v_\ell\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \mid_0 W(\ell)\right) \geq k. \tag{3.5}$$

We turn to the computation of $\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \mid_0 W(\ell)$. Let $1 \leq j \leq \ell - 1$. We first observe that there exists $-(\ell-1) \leq j' \leq -1$ with $jj' \equiv 1 \pmod{\ell}$. Let $b_j := \frac{jj'-1}{\ell}$. Then we have $jj' - b_j\ell = 1$, so $\begin{pmatrix} j & b_j \\ \ell & j' \end{pmatrix} \in \Gamma_0(\ell)$. Furthermore, we have

$$\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix}\begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} = \begin{pmatrix} j\ell & -1 \\ \ell^2 & 0 \end{pmatrix} = \begin{pmatrix} j & b_j \\ \ell & j' \end{pmatrix}\begin{pmatrix} \ell & -j' \\ 0 & \ell \end{pmatrix}. \tag{3.6}$$

For $1 \leq j \leq \ell - 1$, we use (2.1) and (3.6) to obtain

$$g_{k-1} \mid_{\ell^{k-2}(\ell-1)} \left(\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix}\begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}\right) = g_{k-1} \mid_{\ell^{k-2}(\ell-1)} \left(\begin{pmatrix} j & b_j \\ \ell & j' \end{pmatrix}\begin{pmatrix} \ell & -j' \\ 0 & \ell \end{pmatrix}\right)$$
$$= g_{k-1}(z) \mid_{\ell^{k-2}(\ell-1)} \begin{pmatrix} \ell & -j' \\ 0 & \ell \end{pmatrix} = (\ell^2)^{\ell^{k-2}(\ell-1)/2} \ell^{-\ell^{k-2}(\ell-1)} g_{k-1}\left(z - \frac{j'}{\ell}\right) = g_{k-1}\left(z - \frac{j'}{\ell}\right).$$

Using (1.5), (2.1), (2.6), (2.7), (2.8), and (3.6), we find that

$$A_\ell^{2\ell^{k-2}} \mid_{\ell^{k-2}(\ell-1)} \left(\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} \right) = A_\ell \left(z - \frac{j'}{\ell} \right)^{2\ell^{k-2}}$$

$$= \left(\frac{\eta\left(z - \frac{j'}{\ell}\right)^\ell}{\eta(\ell z - j')} \right)^{2\ell^{k-2}} = \left(\zeta_{24}^{j'} \cdot \frac{\eta\left(z - \frac{j'}{\ell}\right)^\ell}{\eta(\ell z)} \right)^{2\ell^{k-2}} \tag{3.7}$$

and that

$$\Phi_\ell \mid_0 \left(\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} \right) = \Phi_\ell \left(\frac{j\ell z - 1}{\ell^2 z} \right) = \frac{\eta\left(\ell^2 \left(\frac{j\ell z - 1}{\ell^2 z}\right)\right)}{\eta\left(\frac{j\ell z - 1}{\ell^2 z}\right)} = \frac{\eta\left(-\frac{1}{z} + j\ell\right)}{\eta\left(\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} z \right)}$$

$$= \frac{\zeta_{24}^{j\ell} \cdot \eta\left(-\frac{1}{z}\right)}{\eta\left(\begin{pmatrix} j & b_j \\ \ell & j' \end{pmatrix} \begin{pmatrix} \ell & -j' \\ 0 & \ell \end{pmatrix} z \right)} = \frac{\zeta_{24}^{j\ell} \left(\frac{z}{i}\right)^{1/2} \eta(z)}{\epsilon_{j,b_j,\ell,j'} \left(\ell \begin{pmatrix} \ell & -j' \\ 0 & \ell \end{pmatrix} z + j' \right)^{1/2} \eta\left(\begin{pmatrix} \ell & -j' \\ 0 & \ell \end{pmatrix} z \right)}$$

$$= \frac{\zeta_{24}^{j\ell} \left(\frac{z}{i}\right)^{1/2} \eta(z)}{\epsilon_{j,b_j,\ell,j'} (\ell z)^{1/2} \eta\left(z - \frac{j'}{\ell}\right)} = \frac{\zeta_{24}^{\ell j} (-i)^{1/2}}{\epsilon_{j,b_j,\ell,j'}} \cdot \frac{\eta(z)}{\ell^{1/2} \eta\left(z - \frac{j'}{\ell}\right)}. \tag{3.8}$$

Next, we observe that

$$\begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \ell^2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix}. \tag{3.9}$$

Using (2.1), (2.3), Lemma 2.1 (4), and (3.9), we obtain

$$g_{k-1} \mid_{\ell^{k-2}(\ell-1)} \left(\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} \right) = g_{k-1} \mid_{\ell^{k-2}(\ell-1)} \left(\begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= g_{k-1} \mid_{\ell^{k-2}(\ell-1)} W(\ell) \mid_{\ell^{k-2}(\ell-1)} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} = \ell^{\ell^{k-2}(\ell-1)/2} g_{k-1} \mid_{\ell^{k-2}(\ell-1)} W(\ell) \mid V(\ell)$$

$$= \ell^{\ell^{k-2}(\ell-1)} g_{k-1}(z) \mid V(\ell^2) = \ell^{\ell^{k-2}(\ell-1)} g_{k-1}(\ell^2 z); \tag{3.10}$$

using (2.1), (2.7), (2.8), and (3.9), we obtain

$$A_\ell^{2\ell^{k-2}} \mid_{\ell^{k-2}(\ell-1)} \left(\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} \right) = A_\ell^{2\ell^{k-2}} \mid_{\ell^{k-2}(\ell-1)} \begin{pmatrix} 0 & -1 \\ \ell^2 & 0 \end{pmatrix}$$

$$= \ell^{-\ell^{k-2}(\ell-1)} z^{-\ell^{k-2}(\ell-1)} A_\ell \left(-\frac{1}{\ell^2 z} \right)^{2\ell^{k-2}} = \ell^{-\ell^{k-2}(\ell-1)} z^{-\ell^{k-2}(\ell-1)} \left(\frac{\eta\left(\frac{-1}{\ell^2 z}\right)^\ell}{\eta\left(\frac{-1}{\ell z}\right)} \right)^{2\ell^{k-2}}$$

$$= \ell^{-\ell^{k-2}(\ell-1)} z^{-\ell^{k-2}(\ell-1)} \left(\frac{\ell^\ell (\frac{z}{i})^{\ell/2} \eta(\ell^2 z)^\ell}{\ell^{1/2} (\frac{z}{i})^{1/2} \eta(\ell z)} \right)^{2\ell^{k-2}} = \ell^{\ell^{k-1}} i^{-\ell^{k-2}(\ell-1)} \left(\frac{\eta(\ell^2 z)^\ell}{\eta(\ell z)} \right)^{2\ell^{k-2}}. \tag{3.11}$$

We deduce from (1.5), (2.1), (2.7), and (3.9) that

$$\Phi_\ell(z) \mid_0 \left(\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} \right) = \Phi_\ell(z) \mid_0 \begin{pmatrix} 0 & -1 \\ \ell^2 & 0 \end{pmatrix} = \Phi_\ell\left(-\frac{1}{\ell^2 z}\right) = \frac{\eta\left(-\frac{1}{z}\right)}{\eta\left(-\frac{1}{\ell^2 z}\right)}$$

$$= \frac{\left(\frac{z}{i}\right)^{1/2} \eta(z)}{\left(\frac{\ell^2 z}{i}\right)^{1/2} \eta(\ell^2 z)} = \frac{1}{\ell \Phi_\ell(z)}. \tag{3.12}$$

Inserting (3.6) - (3.12) into (2.2), we obtain:

$$\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \cdot \Phi_\ell(z)\right) \mid U(\ell) \mid_0 W(\ell) = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \left(\frac{g_{k-1}}{A_\ell^{2\ell^{k-2}}} \cdot \Phi_\ell\right) \mid_0 \left(\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} \right)$$

$$= \frac{1}{\ell} \left(\frac{g_{k-1}}{A_\ell^{2\ell^{k-2}}} \cdot \Phi_\ell\right) \mid_0 \left(\begin{pmatrix} 1 & 0 \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} \right) + \frac{1}{\ell} \sum_{j=1}^{\ell-1} \left(\frac{g_{k-1}}{A_\ell^{2\ell^{k-2}}} \cdot \Phi_\ell\right) \mid_0 \left(\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} \right)$$

$$= \frac{1}{\ell} \left(g_{k-1} \mid_{\ell^{k-2}(\ell-1)} \begin{pmatrix} 0 & -1 \\ \ell^2 & 0 \end{pmatrix} \right) \cdot \left(A_\ell^{2\ell^{k-2}} \mid_{\ell^{k-2}(\ell-1)} \begin{pmatrix} 0 & -1 \\ \ell^2 & 0 \end{pmatrix} \right)^{-1} \cdot \left(\Phi_\ell \mid_0 \begin{pmatrix} 0 & -1 \\ \ell^2 & 0 \end{pmatrix} \right)$$

$$+ \frac{1}{\ell} \sum_{j=1}^{\ell-1} \left(g_{k-1}\left(z - \frac{j'}{\ell}\right) \cdot \left(\zeta_{24}^{j'} \frac{\eta(\ell z)}{\eta\left(z - \frac{j'}{\ell}\right)^\ell} \right)^{2\ell^{k-2}} \cdot \left(\frac{\zeta_{24}^{\ell j}(-i)^{1/2}}{\epsilon_{j,b_j,\ell,j'}} \right) \cdot \frac{\eta(z)}{\ell^{1/2} \eta\left(z - \frac{j'}{\ell}\right)} \right)$$

$$= \frac{1}{\ell} \left(\ell^{\ell^{k-2}(\ell-1)} g_{k-1}(\ell^2 z) \cdot \ell^{-\ell^{k-1}} i^{\ell^{k-2}(\ell-1)} \left(\frac{\eta(\ell z)}{\eta(\ell^2 z)^\ell} \right)^{2\ell^{k-2}} \cdot \frac{1}{\ell \Phi_\ell(z)} \right)$$

$$+ \frac{1}{\ell} \sum_{j=1}^{\ell-1} \left(g_{k-1}\left(z - \frac{j'}{\ell}\right) \cdot \left(\zeta_{24}^{j'} \frac{\eta(\ell z)}{\eta\left(z - \frac{j'}{\ell}\right)^\ell} \right)^{2\ell^{k-2}} \cdot \left(\frac{\zeta_{24}^{\ell j}(-i)^{1/2}}{\epsilon_{j,b_j,\ell,j'}} \right) \cdot \frac{\eta(z)}{\ell^{1/2} \eta\left(z - \frac{j'}{\ell}\right)} \right)$$

$$= \frac{i^{\ell^{k-2}(\ell-1)}}{\ell^{\ell^{k-2}+2}} \cdot \frac{g_{k-1}(\ell^2 z) \eta(\ell z)}{\eta(\ell^2 z)^\ell \Phi_\ell(z)} + \frac{(-i)^{1/2} \eta(\ell z)^{2\ell^{k-2}} \eta(z)}{\ell^{3/2}} \cdot \sum_{j=1}^{\ell-1} \left(\frac{\zeta_{24}^{2j'\ell^{k-2}+\ell j}}{\epsilon_{j,b_j,\ell,j'}} \left(\frac{g_{k-1}\left(z - \frac{j'}{\ell}\right)}{\eta\left(z - \frac{j'}{\ell}\right)^{2\ell^{k-1}+1}} \right) \right).$$

Thus, we conclude that

$$v_\ell\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \mid_0 W(\ell) \right) \geq -\ell^{k-2} - 2 \tag{3.13}$$

from which it follows, for $\ell \geq 5$ and $k \geq 2$, that

$$\ell^{k-1} + 1 + v_\ell\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \mid_0 W(\ell) \right) \geq \ell^{k-1} - \ell^{k-2} - 1 = \ell^{k-2}(\ell - 1) - 1 \geq k.$$

Hence, we complete the verification of (3.5):

$$\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \equiv \mathrm{Tr}\left(\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \right) \cdot h(z)^{\ell^{k-1}} \right) \pmod{\ell^k}.$$

By examining the q-expansion (we omit the details), we see that

$$r_{k,\ell}(z) := \mathrm{Tr}\left(\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \mid D(\ell) \right) \cdot h(z)^{\ell^{k-1}} \right) \in S_{\ell^{k-1}(\ell-1)}$$

satisfies the conclusion of the proposition. □

We simplify the second summand in (3.2) as follows:

$$\left(g_{k-1}(z)E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)} - \frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \cdot E_{\ell-1}(z)^{\ell^{k-1}}\right) \mid D(\ell)$$
$$= \left(E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)} \cdot \left(g_{k-1}(z) - \frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \cdot E_{\ell-1}(z)^{\ell^{k-2}}\right)\right) \mid D(\ell).$$

Using Proposition 2.5, we observe that

$$B_{k,\ell}(z) := g_{k-1}(z) - \frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \cdot E_{\ell-1}(z)^{\ell^{k-2}} \equiv 0 \pmod{\ell^{k-1}} \tag{3.14}$$

and that $E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)} \equiv 1 \pmod{\ell}$. We conclude that

$$E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)} \cdot \frac{B_{k,\ell}(z)}{\ell^{k-1}} \equiv \frac{B_{k,\ell}(z)}{\ell^{k-1}} \pmod{\ell}.$$

Multiplying by ℓ^{k-1} gives

$$E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)} \cdot B_{k,\ell}(z) \equiv B_{k,\ell}(z) \pmod{\ell^k}.$$

Therefore, the second summand in (3.2) modulo ℓ^k is

$$B_{k,\ell}(z) \mid D(\ell) \in M^!_{\ell^{k-2}(\ell-1)}(\Gamma_0(\ell)) \cap \mathbb{Z}_{(\ell)}[[q]]. \tag{3.15}$$

Proposition 3.3. *The form* (3.15) *is congruent modulo* ℓ^k *to a form in* $S_{\ell^{k-1}(\ell-1)} \cap \mathbb{Z}[[q]]$.

Remark. The proof shows that the weight can be taken to be $\ell^{k-2}(\ell-1) + \frac{\ell^2-1}{2}$.

Proof. In view of (1.7), (2.10), (3.14), and Proposition 2.5, we have

$$\frac{B_{k,\ell}(z)}{\ell^{k-1}} \mid D(\ell) = \left(\frac{B_{k,\ell}(z)}{\ell^{k-1}} \cdot \Phi_\ell(z)\right) \mid U(\ell) \equiv \left(\frac{B_{k,\ell}(z)}{\ell^{k-1}} \cdot \Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell)$$
$$\equiv \left(\frac{B_{k,\ell}(z)}{\ell^{k-1}} \cdot \Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell) \cdot E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)-\frac{\ell+1}{2}} \pmod{\ell}.$$

Multiplying by ℓ^{k-1} yields

$$B_{k,\ell}(z) \mid D(\ell) \equiv \left(B_{k,\ell}(z) \cdot \Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell) \cdot E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)-\frac{\ell+1}{2}} \pmod{\ell^k}.$$

This form lies in $M^!_{\ell^{k-1}(\ell-1)}(\Gamma_0(\ell))$. It remains to show that

$$\left(B_{k,\ell}(z) \cdot \Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell) = \left(\left(g_{k-1}(z) - \frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \cdot E_{\ell-1}(z)^{\ell^{k-2}}\right) \cdot \Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell)$$

is congruent modulo ℓ^k to a cusp form on $\mathrm{SL}_2(\mathbb{Z})$. Since $g_{k-1}(z)\Delta(z)^{\frac{\ell^2-1}{24}} \in S_{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}}$ and $\ell^{k-2}(\ell-1) + \frac{\ell^2-1}{2} - 1 \geq k$, we see from Proposition 2.3 that $g_{k-1}(z)\Delta(z)^{\frac{\ell^2-1}{24}} \mid U(\ell)$ is congruent modulo ℓ^k to a form in the same space. Therefore, it suffices to show that

$$\left(\frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \cdot E_{\ell-1}(z)^{\ell^{k-2}}\Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell)$$

is congruent modulo ℓ^k to a form in $S_{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}}$. For convenience, we define

$$C_{k,\ell}(z) := \frac{g_{k-1}(z)}{A_\ell(z)^{2\ell^{k-2}}} \cdot E_{\ell-1}(z)^{\ell^{k-2}}\Delta(z)^{\frac{\ell^2-1}{24}} \in M^!_{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}}(\Gamma_0(\ell)).$$

Using the trace (2.5), we study

$$\ell^{\frac{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}}{2}-1}\mathrm{Tr}\left(C_{k,\ell}\mid_{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}} W(\ell)\right) = C_{k,\ell}(z)\mid U(\ell) + \ell^{\frac{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}}{2}-1}C_{k,\ell}\mid_{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}} W(\ell).$$

By Lemma 2.1 (3), this form is on $\mathrm{SL}_2(\mathbb{Z})$. Therefore, we show that

$$\ell^{\frac{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}}{2}-1}C_{k,\ell}\mid_{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}} W(\ell)\equiv 0 \pmod{\ell^k}. \tag{3.16}$$

Employing (2.3) and Lemma 2.1 (4), we obtain

$$\begin{aligned}\left(g_{k-1}E_{\ell-1}^{\ell^{k-2}}\Delta^{\frac{\ell^2-1}{24}}\right)\mid_{2\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}} W(\ell) &= \ell^{\frac{2\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}}{2}}\left(g_{k-1}(z)E_{\ell-1}(z)^{\ell^{k-2}}\Delta(z)^{\frac{\ell^2-1}{24}}\right)\mid V(\ell)\\ &= \ell^{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{4}}g_{k-1}(\ell z)E_{\ell-1}(\ell z)^{\ell^{k-2}}\Delta(\ell z)^{\frac{\ell^2-1}{24}}.\end{aligned} \tag{3.17}$$

Next, we use (2.1), (2.7), and (2.8) to compute $A_\ell^{2\ell^{k-2}}\mid_{\ell^{k-2}(\ell-1)} W(\ell)$:

$$A_\ell^{2\ell^{k-2}}\mid_{\ell^{k-2}(\ell-1)} W(\ell) = \ell^{-\frac{\ell^{k-2}(\ell-1)}{2}}z^{-\ell^{k-2}(\ell-1)}\left(\frac{\eta\left(\frac{-1}{\ell z}\right)^\ell}{\eta\left(-\frac{1}{z}\right)}\right)^{2\ell^{k-2}}$$

$$= \ell^{-\frac{\ell^{k-2}(\ell-1)}{2}}z^{-\ell^{k-2}(\ell-1)}\left(\frac{\left(\frac{\ell z}{i}\right)^{\frac{\ell}{2}}}{\left(\frac{z}{i}\right)^{\frac12}}\cdot\frac{\eta(\ell z)^\ell}{\eta(z)}\right)^{2\ell^{k-2}} = \ell^{\frac{\ell^{k-1}+\ell^{k-2}}{2}}i^{\ell^{k-2}(\ell-1)}\left(\frac{\eta(\ell z)^\ell}{\eta(z)}\right)^{2\ell^{k-2}}. \tag{3.18}$$

We substitute (3.17) and (3.18) to show that

$$\begin{aligned}&\ell^{\frac{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}}{2}-1}C_{k,\ell}\mid_{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}} W(\ell)\\ &= \frac{\ell^{\frac{\ell^{k-2}(\ell-1)}{2}+\frac{\ell^2-1}{4}-1+\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{4}}g_{k-1}(\ell z)E_{\ell-1}(\ell z)^{\ell^{k-2}}\Delta(z)^{\frac{\ell^2-1}{24}}}{\ell^{\frac{\ell^{k-1}+\ell^{k-2}}{2}}i^{\ell^{k-2}(\ell-1)}\left(\frac{\eta(\ell z)^\ell}{\eta(z)}\right)^{2\ell^{k-1}}}\\ &= \ell^{\ell^{k-2}(\ell-2)+\frac{\ell^2-1}{2}-1}i^{\ell^{k-2}(\ell-1)}\cdot\frac{g_{k-1}(\ell z)E_{\ell-1}(\ell z)^{\ell^{k-2}}\Delta(\ell z)^{\frac{\ell^2-1}{24}}}{\left(\frac{\eta(\ell z)^\ell}{\eta(z)}\right)^{2\ell^{k-1}}}.\end{aligned}$$

We recall that $g_{k-1}(z)$, $E_{\ell-1}(z)$, $\Delta(z)$, and $\frac{\eta(\ell z)^\ell}{\eta(z)}\in\mathbb{Z}_{(\ell)}[[q]]$. It follows, for all $k\geq 2$ and all primes $\ell\geq 5$, that

$$v_\ell\left(\ell^{\frac{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}}{2}-1}C_{k,\ell}\mid_{\ell^{k-2}(\ell-1)+\frac{\ell^2-1}{2}} W(\ell)\right)\geq \ell^{k-2}(\ell-2)+\frac{\ell^2-1}{2}-1\geq 3\ell^{k-2}\geq k,$$

which verifies (3.16), and with it, the proposition. □

We now turn to the third summand in (3.2). Using Proposition 2.5 and the hypothesis that $g_k(z) \equiv g_{k-1}(z) \pmod{\ell^{k-1}}$, we find that

$$F_{k,\ell}(z) := g_k(z) - g_{k-1}(z) E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)} \equiv 0 \pmod{\ell^{k-1}}. \tag{3.19}$$

We prove the following proposition.

Proposition 3.4. *The form $F_{k,\ell}(z) \mid D(\ell)$ is congruent modulo ℓ^k to a form in $S_{\ell^{k-1}(\ell-1)} \cap \mathbb{Z}[[q]]$.*

Remark. The proof shows that the weight can be taken to be $(\ell^{k-2}+1)(\ell-1)$.

Proof. From (1.7), (2.10), and (3.19), we deduce that

$$\frac{F_{k,\ell}(z)}{\ell^{k-1}} \mid D(\ell) \equiv \left(\frac{F_{k,\ell}(z)}{\ell^{k-1}} \cdot \Delta(z)^{\frac{\ell^2-1}{24}} \right) \mid U(\ell) \pmod{\ell},$$

and we observe that $\frac{F_{k,\ell}(z)}{\ell^{k-1}} \cdot \Delta(z)^{\frac{\ell^2-1}{24}} \in S_{\ell^{k-1}(\ell-1)+\frac{\ell^2-1}{2}} \cap \mathbb{Z}_{(\ell)}[[q]]$. Since $\ell \geq 5$, an application of Lemma 2.2 (1) yields

$$w_\ell \left(\left(\frac{F_{k,\ell}(z)}{\ell^{k-1}} \cdot \Delta(z)^{\frac{\ell^2-1}{24}} \right) \mid U(\ell) \right) \leq \ell + \frac{\ell^{k-1}(\ell-1) + \frac{\ell^2-1}{2} - 1}{\ell}$$

$$= (\ell-1) \left(\ell^{k-2} + 1 + \frac{\ell+3}{2\ell} \right) < (\ell-1)(\ell^{k-2}+2). \tag{3.20}$$

From (3.20) and Proposition 2.5, we see that there exists $f_{k,\ell}(z) \in S_{(\ell^{k-2}+1)(\ell-1)}$ with

$$\left(\frac{F_{k,\ell}(z)}{\ell^{k-1}} \cdot \Delta(z)^{\frac{\ell^2-1}{24}} \right) \mid U(\ell) \equiv f_{k,\ell}(z) \equiv f_{k,\ell}(z) E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)-1} \pmod{\ell}. \tag{3.21}$$

Multiplying by ℓ^{k-1} produces

$$\left(F_{k,\ell}(z) \cdot \Delta(z)^{\frac{\ell^2-1}{24}} \right) \mid U(\ell) \equiv \ell^{k-1} f_{k,\ell}(z) E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)-1} \pmod{\ell^k},$$

and we note that $f_{k,\ell}(z) E_{\ell-1}(z)^{\ell^{k-2}(\ell-1)-1} \in S_{\ell^{k-1}(\ell-1)} \cap \mathbb{Z}[[q]]$. □

Lemma 3.1 now follows from Propositions 3.2, 3.3, and 3.4. □

Corollary 3.5. *Let $\ell \geq 5$ be prime, let $k \geq 1$, and let $b \geq 0$. Then there exists $f_k(b;z) \in M_{\ell^{k-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ with $L_\ell(b;z) \equiv f_k(b;z) \pmod{\ell^k}$.*

Proof. We proceed by induction on b. Let $b = 0$; for all $k \geq 1$, Proposition 2.5 gives

$$L_\ell(0;z) = 1 \equiv E_{\ell-1}(z)^{\ell^{k-1}} \pmod{\ell^k}.$$

For the induction step, let $b \geq 0$ be a fixed even integer, and suppose, for all $k \geq 1$, that there exists $f_k(b;z) \in M_{\ell^{k-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ with $L_\ell(b;z) \equiv f_k(b;z) \pmod{\ell^k}$. In particular, the form $L_\ell(b;z)$ satisfies the hypotheses of Lemma 3.1. For all $k \geq 1$, the lemma now implies that there exists $h_k(b;z) \in M_{\ell^{k-1}(\ell-1)}$ with

$$L_\ell(b+1;z) = L_\ell(b;z) \mid D(\ell) \equiv h_k(b;z) \pmod{\ell^k}.$$

In this way, we satisfy the conclusion of the corollary for index $b+1$ with $f_k(b+1;z) := h_k(b;z)$. Next, we observe, for all $k \geq 1$ and primes $\ell \geq 5$, that $\ell^{k-1}(\ell-1) - 1 \geq k$. It follows from Proposition 2.3 that

$$L_\ell(b+2;z) = L_\ell(b+1;z) \mid U(\ell) \equiv f_k(b+1;z) \mid U(\ell) \equiv f_k(b+1;z) \mid T(\ell, \ell^{k-1}(\ell-1)) \pmod{\ell^k}.$$

Part (2) of Lemma 2.1 implies that $f_k(b+2;z) := f_k(b+1;z) \mid T(\ell, \ell^{k-1}(\ell-1))$ satisfies the conclusion of the corollary for index $b+2$. $\square$

Remark: Let $\ell \geq 5$ be prime, let $m \geq 1$, and let $M(\ell, m)$ denote the $\mathbb{Z}/\ell^m\mathbb{Z}$-module of modular forms in $M_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ with coefficients reduced modulo ℓ^m. Corollary 3.5 implies the following nesting of $\mathbb{Z}/\ell^m\mathbb{Z}$-modules:

$$M(\ell,m) \supseteq \Lambda_\ell^{\mathrm{even}}(0,m) \supseteq \Lambda_\ell^{\mathrm{even}}(2,m) \supseteq \cdots \supseteq \Lambda_\ell^{\mathrm{even}}(2b,m) \supseteq \cdots,$$
$$M(\ell,m) \supseteq \Lambda_\ell^{\mathrm{odd}}(1,m) \supseteq \Lambda_\ell^{\mathrm{odd}}(3,m) \supseteq \cdots \supseteq \Lambda_\ell^{\mathrm{odd}}(2b+1,m) \supseteq \cdots.$$

Since $M(\ell, m)$ has finite size, these sequences must stabilize as the finite-rank modules $\Omega_\ell^{\mathrm{even}}(m)$ (respectively, $\Omega_\ell^{\mathrm{odd}}(m)$), as asserted in Theorem 1.4. In Section 4, we exhibit injections into $S_{\ell-1}$ to show that the ranks are bounded independently of m.

Our second main lemma of this section asserts, subject to certain hypotheses, that the operator $Y(\ell)$ (as in 1.9) contracts the weight of a form in $M_{\ell^{j-1}(\ell-1)} \cap \mathbb{Z}[[q]]$ with coefficients reduced modulo ℓ^j.

Lemma 3.6. *Let $\ell \geq 5$ be prime, let $n \geq 1$, and let $\Upsilon(z) \in \mathbb{Z}_{(\ell)}[[q]]$. Suppose, for all $1 \leq j \leq n$ that there exists $g_j(z) \in M_{\ell^{j-1}(\ell-1)} \cap \mathbb{Z}[[q]]$ with $\Upsilon(z) \equiv g_j(z) \pmod{\ell^j}$. Suppose further that $\Upsilon(z) \mid D(\ell) \equiv 0 \pmod{\ell}$. Then for all $2 \leq j \leq n$, there exists $h_j(z) \in S_{\ell^{j-2}(\ell-1)} \cap \mathbb{Z}[[q]]$ with $\Upsilon(z) \mid Y(\ell) \equiv h_j(z) \pmod{\ell^j}$.*

Proof. Suppose that $j = 2$ and $\ell = 5$. Theorem 3.1 implies that $g_2(z) \mid D(5)$ is congruent modulo 25 to a form in $S_{20} \cap \mathbb{Z}_{(5)}[[q]] \subseteq \mathbb{C}\Delta(z)E_4(z)^2$. By hypothesis, we have $g_2(z) \mid D(5) \equiv g_1(z) \mid D(5) \equiv 0 \pmod{5}$. From these facts, we find that there exists $c \in \mathbb{Z}$ with

$$g_2(z) \mid D(5) \equiv 5c\Delta(z)E_4(z)^2 \pmod{25}.$$

Dividing by 5 yields

$$\frac{g_2(z) \mid D(5)}{5} \equiv c\Delta(z)E_4(z)^2 \equiv c\Delta(z) \pmod{5}.$$

We apply $U(5)$ and observe that $\Delta(z) \mid U(5) \equiv 0 \pmod{5}$ to obtain

$$\left(\frac{g_2(z) \mid D(5)}{5}\right) \mid U(5) \equiv c\Delta(z) \mid U(5) \equiv 0 \pmod{5}.$$

To conclude, we multiply by 5, giving

$$g_2(z) \mid Y(5) = g_2(z) \mid D(5) \mid U(5) \equiv 0 \pmod{25}.$$

Now, we suppose that $j \geq 2$ and $\ell \geq 5$ excepting $(j, \ell) = (2, 5)$. We decompose $\Upsilon(z) \mid Y(\ell)$ using modular forms in $M_{\ell^{j-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$, as in the proof of Lemma 3.1:

$$\begin{aligned}\Upsilon(z) \mid Y(\ell) \equiv& \left(\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}} \cdot E_{\ell-1}(z)^{\ell^{j-1}}\right) \mid Y(\ell) \\ &+ \left(g_{j-1}(z)E_{\ell-1}(z)^{\ell^{j-2}(\ell-1)} - \frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}} \cdot E_{\ell-1}(z)^{\ell^{j-1}}\right) \mid Y(\ell) \\ &+ \left(g_j(z) - g_{j-1}(z)E_{\ell-1}(z)^{\ell^{j-2}(\ell-1)}\right) \mid Y(\ell) \pmod{\ell^j}.\end{aligned} \tag{3.22}$$

We study each summand in turn.

In view of Proposition 2.5 and (1.9), the first summand is

$$\left(\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}}\cdot E_{\ell-1}(z)^{\ell^{j-1}}\right)\mid Y(\ell)\equiv\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}}\mid Y(\ell)\equiv\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}}\mid D(\ell)\mid U(\ell)\pmod{\ell^j}.$$

Since $j\geq 2$, Proposition 2.5 and the hypotheses of the lemma imply that

$$\Upsilon(z)\mid D(\ell)\equiv\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}}\mid D(\ell)\equiv 0\pmod{\ell}.$$

Using Proposition 2.5 again, we have

$$\frac{1}{\ell}\left(\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}}\mid D(\ell)\right)\equiv\frac{1}{\ell}\left(\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}}\mid D(\ell)\right)E_{\ell-1}(z)^{\ell^{j-2}}\pmod{\ell^{j-1}}.\tag{3.23}$$

We define

$$G_{j,\ell}(z):=\left(\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}}\mid D(\ell)\right)E_{\ell-1}(z)^{\ell^{j-2}}\in M^!_{\ell^{j-2}(\ell-1)}(\Gamma_0(\ell))\cap\mathbb{Z}_{(\ell)}[[q]].$$

Multiplying by ℓ and applying $U(\ell)$ in (3.23) gives

$$\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}}\mid Y(\ell)\equiv G_{j,\ell}(z)\mid U(\ell)\pmod{\ell^j}.\tag{3.24}$$

Proposition 3.7. *The form* (3.24) *is congruent modulo ℓ^j to a form in $S_{\ell^{j-2}(\ell-1)}\cap\mathbb{Z}[[q]]$.*

Proof. Using (2.5), we consider

$$\ell^{\frac{\ell^{j-2}(\ell-1)}{2}-1}\mathrm{Tr}\left(G_{j,\ell}\mid_{\ell^{j-2}(\ell-1)}W(\ell)\right)=G_{j,\ell}(z)\mid U(\ell)+\ell^{\frac{\ell^{j-2}(\ell-1)}{2}-1}G_{j,\ell}\mid_{\ell^{j-2}(\ell-1)}W(\ell).$$

From Lemma 2.1 (3), we see that this form is on $\mathrm{SL}_2(\mathbb{Z})$. Therefore, we show that

$$\ell^{\frac{\ell^{j-2}(\ell-1)}{2}-1}G_{j,\ell}\mid_{\ell^{j-2}(\ell-1)}W(\ell)\equiv 0\pmod{\ell^j}.\tag{3.25}$$

Since $E_{\ell-1}(z)$ is on $\mathrm{SL}_2(\mathbb{Z})$, Lemma 2.1 (4) yields

$$E_{\ell-1}^{\ell^{j-2}}\mid_{\ell^{j-2}(\ell-1)}W(\ell)=\ell^{\frac{\ell^{j-2}(\ell-1)}{2}}E_{\ell-1}(z)^{\ell^{j-2}}\mid V(\ell).$$

Using (3.13), we conclude for all $j\geq 2$ and all primes $\ell\geq 5$ excepting $(j,\ell)=(2,5)$ that

$$v_\ell\left(\ell^{\frac{\ell^{j-2}(\ell-1)}{2}-1}G_{j,\ell}\mid_{\ell^{j-2}(\ell-1)}W(\ell)\right)=\frac{\ell^{j-2}(\ell-1)}{2}-1+v_\ell\left(\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}}\mid D(\ell)\mid_0 W(\ell)\right)$$
$$+v_\ell\left(E_{\ell-1}^{\ell^{j-2}}\mid_{\ell^{j-2}(\ell-1)}W(\ell)\right)\geq\frac{\ell^{j-2}(\ell-1)}{2}-1-\ell^{j-2}-2+\frac{\ell^{j-2}(\ell-1)}{2}=\ell^{j-2}(\ell-2)-3\geq j.$$

Therefore, (3.25) holds, and the first summand in (3.22) is congruent to a form on $\mathrm{SL}_2(\mathbb{Z})$ of weight $\ell^{j-2}(\ell-1)$. An examination of its q-series reveals that it is a cusp form (we omit the details). □

The second summand simplifies as

$$\left(g_{j-1}(z)E_{\ell-1}(z)^{\ell^{j-2}(\ell-1)}-\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}}\cdot E_{\ell-1}(z)^{\ell^{j-1}}\right)\mid Y(\ell)$$
$$\equiv\left(g_{j-1}(z)-\frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}}\cdot E_{\ell-1}(z)^{\ell^{j-2}}\right)\mid Y(\ell)\pmod{\ell^j}.$$

As in (3.14), we have

$$B_{j,\ell}(z) := g_{j-1}(z) - \frac{g_{j-1}(z)}{A_\ell(z)^{2\ell^{j-2}}} \cdot E_{\ell-1}(z)^{\ell^{j-2}} \equiv 0 \pmod{\ell^{j-1}}. \tag{3.26}$$

Proposition 3.8. *The form $B_{j,\ell}(z) \mid Y(\ell)$ is congruent modulo ℓ^j to a form in $S_{\ell^{j-2}(\ell-1)} \cap \mathbb{Z}[[q]]$.*

Remark. The weight can be taken to be $(\ell^{j-3}+1)(\ell-1)$ if $j \geq 3$.

Proof. From Proposition 3.3, we find that there exists $f_{j,\ell}(z) \in S_{\ell^{j-2}(\ell-1)+\frac{\ell^2-1}{2}} \cap \mathbb{Z}[[q]]$ with

$$\left(B_{j,\ell}(z) \cdot \Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell) \equiv f_{j,\ell}(z) \pmod{\ell^j}.$$

Using (1.9), (2.10), and (3.26) we deduce that

$$\frac{B_{j,\ell}(z)}{\ell^{j-1}} \mid Y(\ell) \equiv \left(\frac{B_{j,\ell}(z)}{\ell^{j-1}} \cdot \Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell) \mid U(\ell) \equiv \frac{f_{j,\ell}(z)}{\ell^{j-1}} \mid U(\ell) \pmod{\ell}.$$

For $j \geq 3$, Lemma 2.2 yields

$$\begin{aligned} w_\ell\left(\frac{f_{j,\ell}(z)}{\ell^{j-1}} \mid U(\ell)\right) &\leq \ell + \frac{\ell^{j-2}(\ell-1) + \frac{\ell^2-1}{2} - 1}{\ell} \\ &= (\ell-1)\left(\ell^{j-3} + 1 + \frac{\ell+3}{2\ell}\right) < (\ell-1)(\ell^{j-3}+2). \end{aligned}$$

In view of the calculation (3.1) for $j=2$, we find for $\ell \geq 5$ and $j \geq 2$, that there exists

$$t_{j,\ell}(z) \in \begin{cases} S_{\ell-1}, & j=2; \\ S_{(\ell^{j-3}+1)(\ell-1)}, & j \geq 3 \end{cases}$$

for which

$$\frac{B_{j,\ell}(z)}{\ell^{j-1}} \mid Y(\ell) \equiv \frac{f_{j,\ell}(z)}{\ell^{j-1}} \mid U(\ell) \equiv \begin{cases} t_{j,\ell}(z), & j=2; \\ t_{j,\ell}(z) E_{\ell-1}(z)^{\ell^{j-3}(\ell-1)-1}, & j \geq 3 \end{cases} \pmod{\ell}.$$

The forms on the right side lie in $S_{\ell^{j-2}(\ell-1)}$. Multiplying by ℓ^{j-1}, we have

$$B_{j,\ell}(z) \mid Y(\ell) \equiv \begin{cases} \ell^{j-1} t_{j,\ell}(z), & j=2; \\ \ell^{j-1} t_{j,\ell}(z) E_{\ell-1}(z)^{\ell^{j-3}(\ell-1)-1}, & j \geq 3 \end{cases} \pmod{\ell^j}.$$

Therefore, the second summand in (3.22) is congruent modulo ℓ^j to a form in $S_{\ell^{j-2}(\ell-1)}$. □

For the third summand, we use work from the proof of Proposition 3.4. As in (3.19), we have

$$F_{j,\ell}(z) := g_j(z) - g_{j-1}(z) E_{\ell-1}(z)^{\ell^{j-2}(\ell-1)} \equiv 0 \pmod{\ell^{j-1}}. \tag{3.27}$$

Proposition 3.9. *The form $F_{j,\ell}(z) \mid Y(\ell)$ is congruent modulo ℓ^j to a form in $S_{\ell^{j-2}(\ell-1)} \cap \mathbb{Z}[[q]]$.*

Remark. The weight can be taken to be $(\ell^{j-3}+1)(\ell-1)$ if $j \geq 3$.

Proof. From (1.7), (2.10), (3.21), and (3.27), we see that

$$\frac{F_{j,\ell}(z)}{\ell^{j-1}} \mid D(\ell) \equiv \left(\frac{F_{j,\ell}(z)}{\ell^{j-1}} \cdot \Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell) \pmod{\ell}$$

is congruent modulo ℓ to a form in $S_{(\ell^{j-2}+1)(\ell-1)}$. With $\ell \geq 5$ and $j = 2$, Lemma 2.2 gives

$$w_\ell\left(\left(\frac{F_{2,\ell}(z)}{\ell}\cdot \Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell) \mid U(\ell)\right) \leq \ell + \frac{2(\ell-1)-1}{\ell} = (\ell-1)\left(1+\frac{3}{\ell}\right) < 2(\ell-1).$$

For $j \geq 3$, Lemma 2.2 implies that

$$\begin{aligned} w_\ell\left(\left(\frac{F_{j,\ell}(z)}{\ell^{j-1}}\cdot \Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell) \mid U(\ell)\right) &\leq \ell + \frac{\left(\ell^{j-2}+1\right)(\ell-1)-1}{\ell} \\ &= (\ell-1)\left(\ell^{j-3}+1+\frac{2}{\ell}\right) < (\ell-1)\left(\ell^{j-3}+2\right). \end{aligned}$$

Hence, for $\ell \geq 5$ and $j \geq 2$, there exists

$$s_{j,\ell}(z) \in \begin{cases} S_{\ell-1}, & j=2; \\ S_{(\ell^{j-3}+1)(\ell-1)}, & j \geq 3 \end{cases}$$

for which

$$\begin{aligned} \frac{F_{j,\ell}(z)}{\ell^{j-1}} \mid Y(\ell) &\equiv \left(\frac{F_{j,\ell}(z)}{\ell^{j-1}}\cdot \Delta(z)^{\frac{\ell^2-1}{24}}\right) \mid U(\ell) \mid U(\ell) \\ &\equiv \begin{cases} s_{j,\ell}(z), & j=2; \\ s_{j,\ell}(z)E_{\ell-1}(z)^{\ell^{j-3}(\ell-1)-1}, & j\geq 3 \end{cases} \pmod{\ell}. \end{aligned}$$

The forms on the right side lie in $S_{\ell^{j-2}(\ell-1)}$. Multiplying by ℓ^{j-1}, we obtain

$$F_{j,\ell}(z) \mid Y(\ell) \equiv \begin{cases} \ell^{j-1}s_{j,\ell}(z), & j=2; \\ \ell^{j-1}s_{j,\ell}(z)E_{\ell-1}(z)^{\ell^{j-3}(\ell-1)-1}, & j\geq 3 \end{cases} \pmod{\ell^j}.$$

We deduce that the third summand in (3.22) is congruent modulo ℓ^j to a form in $S_{\ell^{j-2}(\ell-1)}$. □

Lemma 3.6 follows from Proposition 3.7, Proposition 3.8, and Proposition 3.9. □

4. The modules $\Lambda_\ell(b,m)$ and $\Omega_\ell(m)$.

4.1. **Module Structure of $\Omega_\ell^{\mathrm{odd}}(m)$ and $\Omega_\ell^{\mathrm{even}}(m)$.** In this section, we examine the relationship between $\Omega_\ell^{\mathrm{odd}}(m)$, $\Omega_\ell^{\mathrm{even}}(m)$, and $S_{\ell-1}$. Let $b \geq 1$ be odd. We recall the commutative diagram (1.12) of $\mathbb{Z}/\ell^m\mathbb{Z}$-module homomorphisms:

$$\begin{array}{ccc} \Lambda_\ell^{\mathrm{odd}}(b,m) & \xrightarrow{U(\ell)} & \Lambda_\ell^{\mathrm{even}}(b+1,m) \\ {\scriptstyle X(\ell)}\downarrow & \swarrow{\scriptstyle D(\ell)} & \downarrow{\scriptstyle Y(\ell)} \\ \Lambda_\ell^{\mathrm{odd}}(b+2,m) & \xrightarrow[U(\ell)]{} & \Lambda_\ell^{\mathrm{even}}(b+3,m) \end{array}$$

The remark following Corollary 3.5 implies, for all odd $b \geq b_\ell(m)$, that $\Lambda_\ell^{\mathrm{odd}}(b,m) = \Omega_\ell^{\mathrm{odd}}(m)$ and that $\Lambda_\ell^{\mathrm{even}}(b+1,m) = \Omega_\ell^{\mathrm{even}}(m)$. Hence, for all such b, the homomorphisms $U(\ell)$ and $D(\ell)$ are isomorphisms between $\Omega_\ell^{\mathrm{odd}}(m)$ and $\Omega_\ell^{\mathrm{even}}(m)$. It follows that $X(\ell)$ and $Y(\ell)$ are automorphisms on $\Omega_\ell^{\mathrm{odd}}(m)$ and $\Omega_\ell^{\mathrm{even}}(m)$, respectively.

We now study the structure that these maps impose on the modules $\Omega_\ell^{\mathrm{odd}}(m)$ and $\Omega_\ell^{\mathrm{even}}(m)$. We recall two elementary results from commutative algebra.

Lemma 4.1. *Let A be a finite local ring, let M be an A-module of finite rank r, and let $T : M \to M$ be an A-isomorphism. Then there exists an integer $n > 0$ such that T^n is the identity map on M.*

Proof. By Nakayama's Lemma (see [15], for example), an A-isomorphism $T : M \to M$ is representable by a matrix in $\mathrm{GL}_r(A)$. Since $\mathrm{GL}_r(A)$ is a finite group, the isomorphism T must have finite order. □

Lemma 4.2. *Let A be a local ring, let M be a finitely generated A-module, and let $T : M \to M$ be an A-isomorphism. Then for all $m \in M$ and $n \geq 0$, we have*

$$m \in A[T^n(m),\ T^{n+1}(m),\ T^{n+2}(m), \ldots].$$

Proof. If A is finite, Lemma 4.1 implies the result. Now, suppose that A is infinite, and let $m \in M$ and $n \geq 1$. Since $T^{n-1}(m)$ satisfies the characteristic polynomial for T, it is expressible in terms of $T^n(m),\ T^{n+1}(m), \ldots$. Similarly, if $n \geq 2$, then $T^{n-2}(m)$ is expressible in terms of $T^{n-1}(m),\ T^n(m), \ldots$, and hence, in terms of $T^n(m), \ldots$. The result follows from iterating this process. □

Next, we give explicit injective $\mathbb{Z}/\ell^m\mathbb{Z}$-module homorphisms on $\Omega_\ell^{\mathrm{odd}}(m)$ and $\Omega_\ell^{\mathrm{even}}(m)$ into $S_{\ell-1}$.

Theorem 4.3. *Let $\ell \geq 5$ be prime, and let $m \geq 1$. Then there exist injective $\mathbb{Z}/\ell^m\mathbb{Z}$-module homomorphisms*

$$\Pi_o : \Omega_\ell^{odd}(m) \hookrightarrow S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]],$$
$$\Pi_e : \Omega_\ell^{even}(m) \hookrightarrow S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$$

which satisfy the following property. For all $\mu \in \Omega_\ell^{odd}(m)$ and $\nu \in \Omega_\ell^{even}(m)$ with $v_\ell(\mu) = i < m$ and $v_\ell(\nu) = j < m$, we have

$$\Pi_o(\mu) \equiv \mu \pmod{\ell^{i+1}}, \quad \Pi_e(\nu) \equiv \nu \pmod{\ell^{j+1}}.$$

Proof. We consider the following two submodules of $S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$:

$$\mathcal{S}_0 := \left\{ f(z) E_{\ell-1}(z)^{\ell^{m-1}-1} : f(z) \in S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]] \right\},$$
$$\mathcal{S}_1 := \left\{ g(z) : g(z) = \sum_{m=m_0}^{\infty} a_g(m) q^m \in S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]] \text{ with } m_0 > \left\lfloor \tfrac{\ell-1}{12} \right\rfloor \right\}.$$

We can construct a basis $\{f_1 = q + \cdots, \ldots, f_n = q^n + \cdots\}$ for $S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ with $f_k(z) \in \mathcal{S}_0$ for $k \leq \lfloor \frac{\ell-1}{12} \rfloor$ and $f_k(z) \in \mathcal{S}_1$ otherwise. It follows that $S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]] = \mathcal{S}_0 \oplus \mathcal{S}_1$. Hence, $g(z) \in S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ is uniquely expressible as $g(z) = g_0(z) + g_1(z)$ with $g_i(z) \in \mathcal{S}_i$. Next, we reduce coefficients of the forms in these spaces modulo ℓ^m, and we define $\mathcal{S}^* \subset S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ to be the largest $\mathbb{Z}/\ell^m\mathbb{Z}$-submodule such that $X(\ell)$ is an isomorphism on $\mathcal{S}^*$ modulo ℓ^m.

Lemma 4.4. *Suppose that $f(z) \in \mathcal{S}^*$ has $v_\ell(f) = i < m$, and that $f(z) = f_0(z) + f_1(z)$ with $f_w(z) \in \mathcal{S}_w$. Then we have $v_\ell(f_1) > i$.*

Remark. Using (2.12), we see that since $v_\ell(f) = i \geq \min\{v_\ell(f_0), v_\ell(f_1)\}$ and $v_\ell(f_1) > i$, we must have $v_\ell(f_0) = i$.

Proof. We first assert that

$$v_\ell(f_0) \geq i. \tag{4.1}$$

By definition of $\mathcal{S}_0$ and $\mathcal{S}_1$, we may write

$$f_0(z) = \sum_{n=n_0}^{\infty} a_0(n)q^n, \quad 0 \leq n_0 \leq \left\lfloor \frac{\ell-1}{12} \right\rfloor; \quad f_1(z) = \sum_{n=n_1}^{\infty} a_1(n)q^n, \quad n_1 > \left\lfloor \frac{\ell-1}{12} \right\rfloor.$$

We also write $f(z) = \sum a_f(n)q^n$, and we note that $v_\ell(f) = i = \min\{v_\ell(a_f(n))\}$. It follows for all $n_0 \leq n \leq n_1 - 1$, that $v_\ell(a_f(n)) = v_\ell(a_0(n)) \geq i$. Hence, we must have $v_\ell(f_0) \geq i$.

We now suppose that $v_\ell(f_1) \leq i$ and argue by contradiction. We require two claims.

Claim 4.5. *If $v_\ell(f_1) \leq i$, then we have $v_\ell(f_0) \geq v_\ell(f_1) = i$.*

Proof of Claim 4.5. If we suppose that $v_\ell(f_0) < v_\ell(f_1)$, then it follows by (2.12) that

$$i = v_\ell(f) = \min\{v_\ell(f_0), v_\ell(f_1)\} = v_\ell(f_0) < v_\ell(f_1) \leq i,$$

a contradiction. Therefore, we have $v_\ell(f_0) \geq v_\ell(f_1)$. Next, if we suppose that $v_\ell(f_0) > v_\ell(f_1)$, we find from (2.12) that

$$i = v_\ell(f) = \min\{v_\ell(f_0), v_\ell(f_1)\} = v_\ell(f_1).$$

If we suppose that $v_\ell(f_0) = v_\ell(f_1)$, then the hypothesis together with (4.1) give

$$i \leq v_\ell(f_0) = v_\ell(f_1) \leq i.$$

□

Claim 4.6. *Let $\ell \geq 5$ be prime, and suppose that $f(z) \not\equiv 0 \pmod{\ell}$ is in $M_k \cap \mathbb{Z}_{(\ell)}[[q]]$ with $k \equiv 0 \pmod{\ell-1}$.*

1. *Suppose that $w_\ell(f) = \ell - 1$. Then we have $w_\ell(f \mid X(\ell)) \leq w_\ell(f)$.*
2. *Suppose that $w_\ell(f) > \ell - 1$. Then we have $w_\ell(f \mid X(\ell)) < w_\ell(f)$.*

Proof of Claim 4.6. We first suppose that $w_\ell(f) = \ell - 1$. When $\ell \in \{5, 7, 11\}$, we note that $M_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]] \subseteq \mathbb{C}E_{\ell-1}$. Hence, from Proposition 2.5, we see that there are no forms $f(z)$ with $w_\ell(f) = \ell - 1$. Moreover, a form $f(z) \in M_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$ is congruent modulo ℓ to a constant. Therefore, we have $w_\ell(f) = 0$.

For $\ell \geq 13$, we apply Lemma 2.2, (1.7), (1.8), and (2.10) to obtain

$$\begin{aligned} w_\ell(f \mid X(\ell)) &= w_\ell\left(f \mid U(\ell) \mid D(\ell)\right) = w_\ell\left(\left(f \mid U(\ell) \cdot \Delta^{\frac{\ell^2-1}{24}}\right) \mid U(\ell)\right) \\ &\leq \ell + \frac{w_\ell\left(f \mid U(\ell) \cdot \Delta^{\frac{\ell^2-1}{24}}\right) - 1}{\ell} \leq \ell + \frac{\ell - 1 + \frac{\ell^2-1}{2} - 1}{\ell} = (\ell-1)\left(\frac{3\ell+5}{2\ell}\right) < 2(\ell-1). \end{aligned}$$

Part (1) of the claim now follows for $\ell \geq 13$.

When $w_\ell(f) > \ell - 1$, we first observe that the result holds if $f(z) \mid U(\ell) \equiv 0 \pmod{\ell}$. Therefore, we suppose that $f(z) \mid U(\ell) \not\equiv 0 \pmod{\ell}$. As above, we apply Lemma 2.2, (1.7), (1.8), and (2.10) to deduce the conclusion of the lemma. To begin, we we find that

$$\ell - 1 < w_\ell\left(f \mid U(\ell) \cdot \Delta^{\frac{\ell^2-1}{24}}\right) \leq \ell + \frac{w_\ell(f) - 1}{\ell} + \frac{\ell^2 - 1}{2}.$$

Hence, we compute

$$w_\ell\left(f \mid X(\ell)\right) = w_\ell\left(f \mid U(\ell) \mid D(\ell)\right) = w_\ell\left(\left(f \mid U(\ell) \cdot \Delta^{\frac{\ell^2-1}{24}}\right) \mid U(\ell)\right)$$

$$\leq \ell + \frac{\left(\ell + \frac{w_\ell(f)-1}{\ell} + \frac{\ell^2-1}{2}\right) - 1}{\ell} = \frac{1}{\ell^2}\left((\ell^2-1)\left(\frac{3\ell+2}{2}\right) + w_\ell(f)\right).$$

We conclude that $\frac{3\ell+2}{2} < w_\ell(f)$ implies $w_\ell(f \mid X(\ell)) < w_\ell(f)$. For $\ell \geq 7$, the conditions $k \equiv 0 \pmod{\ell}$ and $w_\ell(f) > \ell - 1$ give $w_\ell(f) \geq 2(\ell-1) > \frac{3\ell+2}{2}$. Part (2) of the claim follows for $\ell \geq 7$.

For $\ell = 5$, it suffices to show that $\frac{3\ell+2}{2} = 17/2 < w_5(f)$. The hypotheses $w_5(f) > 4$ and $k \geq 0 \pmod 4$ imply that $w_5(f) \in \{8, 12, \dots\}$. Since $M_8 \cap \mathbb{Z}_{(5)}[[q]] \subseteq \mathbb{C}E_4^2$, we see from Proposition 2.5 that there are no forms $f(z)$ with $w_5(f) = 8$. Hence, we have $17/2 < 12 \leq w_5(f)$ which gives $w_5(f(z) \mid X(5)) \leq w_5(f)$. □

Returning to the proof of Lemma 4.4, we consider the following sequence of modular forms in $S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$:

$$h_0(z) := f(z),\ h_1(z) := h_0(z) \mid X(\ell),\ h_2(z) := h_1(z) \mid X(\ell), \dots.$$

Since $f(z) \in \mathcal{S}^*$, Lemma 4.1 implies that there exists $n \geq 1$ such that $h_0(z) \equiv h_n(z) \pmod{\ell^m}$. We recall that $v_\ell(f) = i < m$ to see that $\ell^{-i}h_0(z) \equiv \ell^{-i}h_n(z) \pmod{\ell}$. Hence, we have $w_\ell(\ell^{-i}h_0) = w_\ell(\ell^{-i}h_n)$. Supposing, by way of contradiction, that $v_\ell(f_1) \leq v_\ell(f) = i$, Claim 4.5 gives $v_\ell(f_0) \geq v_\ell(f_1) = i$. Since $f_0(z) \in \mathcal{S}_0$ and $f_1(z) \in \mathcal{S}_1$, we observe that

$$w_\ell(\ell^{-i}f_0) \leq \ell - 1 < 2(\ell-1) \leq w_\ell(\ell^{-i}f_1).$$

Using this fact together with (2.9), we deduce that

$$w_\ell(\ell^{-i}h_0) = w_\ell(\ell^{-i}(f_0+f_1)) = \max\{w_\ell(\ell^{-i}f_0), w_\ell(\ell^{-i}f_1)\} = w_\ell(\ell^{-i}f_1) > \ell - 1.$$

Therefore, Claim 4.6 gives

$$w_\ell(\ell^{-i}h_0) > w_\ell(\ell^{-i}h_1) \geq w_\ell(\ell^{-i}h_2) \geq \cdots.$$

In particular, we have $w_\ell(\ell^{-i}h_0) > w_\ell(\ell^{-i}h_n)$, a contradiction. □

Theorem 4.3 depends on the following corollary to Lemma 4.4

Corollary 4.7. *Let $f(z)$, $g(z) \in \mathcal{S}^*$, and suppose that $f(z) = f_0(z) + f_1(z)$ and $g(z) = g_0(z) + g_1(z)$ with $f_0(z)$, $g_0(z) \in \mathcal{S}_0$ and $f_1(z)$, $g_1(z) \in \mathcal{S}_1$. Suppose further that $f_0(z) \equiv g_0(z) \pmod{\ell^m}$. Then we have $f(z) \equiv g(z) \pmod{\ell^m}$.*

Proof. Suppose on the contrary that $v_\ell(f-g) = j < m$. Then we have

$$f - g = (f_0 - g_0) + (f_1 - g_1) \in \mathcal{S}^*, \quad f_0 - g_0 \in \mathcal{S}_0, \quad f_1 - g_1 \in \mathcal{S}_1.$$

We apply Lemma 4.4 to deduce that $v_\ell(f_1 - g_1) > j$; the hypothesis gives $v_\ell(f_0 - g_0) \geq m > j$. Hence, we find from (2.12) that

$$v_\ell(f-g) = j \geq \min\{v_\ell(f_0-g_0), v_\ell(f_1-g_1)\} > j,$$

a contradiction. □

We now construct the injection $\Pi_{\mathrm{o}} : \Omega_\ell^{\mathrm{odd}}(m) \hookrightarrow S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$ as a composition of $\mathbb{Z}/\ell^m\mathbb{Z}$-module homomorphisms Φ_1, Φ_2, and Φ_3.

Corollary 3.5 and the remark following it imply that $X(\ell)$ is an isomorphism on $\Omega_\ell^{\mathrm{odd}}(m)$. Since $\mathcal{S}^*$ is the largest $\mathbb{Z}/\ell^m\mathbb{Z}$-submodule of $S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ with this property, we see that $\Omega_\ell^{\mathrm{odd}}(m) \subseteq \mathcal{S}^*$; we define Φ_1 to be the inclusion $\Omega_\ell^{\mathrm{odd}}(m) \hookrightarrow \mathcal{S}^*$. To define Φ_2, we let $f(z) = f_0(z) + f_1(z) \in \mathcal{S}^*$ with $f_0 \in \mathcal{S}_0$ and $f_1 \in \mathcal{S}_1$, and we suppose that $v_\ell(f) = i < m$. Lemma 4.4 implies that $f(z) \equiv f_0(z) \pmod{\ell^{v_\ell(f)+1}}$. Therefore, the map $\Phi_2 : f(z) \mapsto f(z) \pmod{\ell^{v_\ell(f)+1}}$ has $\Phi_2 : \mathcal{S}^* \to \mathcal{S}_0$. Furthermore, Φ_2 is injective by Corollary 4.7. We next define the map Φ_3 on $\mathcal{S}_0$. Suppose that $f(z) \in \mathcal{S}_0$. By definition of $\mathcal{S}_0$, there exists $g(z) \in S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$ with $f(z) = g(z)E_{\ell-1}(z)^{\ell^{m-1}-1}$. We define $\Phi_3 : \mathcal{S}_0 \to S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$ to be the isomorphism that maps $f(z)$ to $g(z)$. To summarize, we have

$$\Pi_{\mathrm{o}} : \Omega_\ell^{\mathrm{odd}}(m) \xrightarrow{\Phi_1} \mathcal{S}^* \xrightarrow{\Phi_2} \mathcal{S}_0 \xrightarrow{\Phi_3} S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]];$$

the first two maps are injections, while the third is an isomorphism. Moreover, if we suppose that $f(z) \in \Omega_\ell^{\mathrm{odd}}(m)$ has $v_\ell(f) < m$, then we have

$$\Pi_{\mathrm{o}}(f(z)) \equiv f(z) \pmod{\ell^{v_\ell(f)+1}}.$$

One can similarly construct $\Pi_{\mathrm{e}} : \Omega_\ell^{\mathrm{even}}(m) \hookrightarrow S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$. Since $\mathcal{S}^* \mid U(\ell) \mid D(\ell) = \mathcal{S}^* \mid X(\ell) = \mathcal{S}^*$, we observe that $\mathcal{S}^* \mid U(\ell) \cong \mathcal{S}^*$. An argument similar to the above shows that $\mathcal{S}^* \mid U(\ell)$ is the largest submodule of $S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ on which $Y(\ell)$ is an isomorphism. In this setting, one can prove facts analogous to Lemma 4.4 and Corollary 4.7. One can also define injective homomorphisms Φ_1' and Φ_2', and an isomorphism Φ_3' whose composition

$$\Pi_{\mathrm{e}} : \Omega_\ell^{\mathrm{even}}(m) \xrightarrow{\Phi_1'} \mathcal{S}^* \mid U(\ell) \xrightarrow{\Phi_2'} \mathcal{S}_0 \xrightarrow{\Phi_3'} S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$$

is the desired map. □

Remark. The injections Π_{o} and Π_{e} preserve order of vanishing. From the definition (1.7) of $D(\ell)$, we find that $f(z) \in \Omega_\ell^{\mathrm{odd}}(m)$ has order of vanish at infinity $> \left\lfloor \frac{\ell^2-1}{24\ell} \right\rfloor$. Hence, we recover the bound R_ℓ on the $\mathbb{Z}/\ell^m\mathbb{Z}$-ranks of $\Omega_\ell^{\mathrm{odd}}(m)$ and $\Omega_\ell^{\mathrm{even}}(m)$ as in (1.13) of Theorem 1.4.

4.2. **Proof of Corollary 1.3.** We suppose that $b \geq 1$ is odd. The proof holds with suitable modifications for even b. Let $\mathcal{S}$ be the largest subspace of $S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$ over $\mathbb{Z}/\ell\mathbb{Z}$ on which $X(\ell)$ is an isomorphism. We proceed by induction on m. For $m = 1$ and $b \geq b_\ell(1)$, we have $L_\ell(b; z) \in \Omega_\ell^{\mathrm{odd}}(1) \subseteq \mathcal{S}$. By Lemma 4.1, $X(\ell)$ has finite order, c_ℓ, on $\mathcal{S}$. The $m = 1$ case of Corollary 1.3 follows from

$$L_\ell(b; z) \equiv L_\ell(b; z) \mid X(\ell)^{c_\ell} \pmod{\ell}.$$

Moreover, we note, for all $F(z) \in \mathcal{S}$, that

$$F(z) \mid X(\ell)^{c_\ell} \equiv F(z) \pmod{\ell}. \tag{4.2}$$

Next, we fix $m \geq 2$, we set

$$\mathcal{X}(\ell, m) := X(\ell)^{c_\ell \ell^{m-2}},$$

and we suppose, for $b' \geq b_\ell(m-1)$, that

$$L_\ell(b'; z) \equiv L_\ell(b'; z) \mid \mathcal{X}(\ell, m) \equiv L_\ell(b' + 2c_\ell\ell^{m-2}; z) \pmod{\ell^{m-1}}.$$

Let $b \geq b_\ell(m) \geq b_\ell(m-1)$. Since $b \geq b_\ell(m-1)$, the inductive hypothesis gives

$$L_\ell(b;z) \equiv L_\ell(b;z) \mid \mathcal{X}(\ell,m) \pmod{\ell^{m-1}};$$

since $b \geq b_\ell(m)$, we find that

$$L_\ell(b;z), \quad L_\ell(b;z) \mid \mathcal{X}(\ell,m) \in \Omega_\ell^{\mathrm{odd}}(m).$$

It follows that there exists $f(z) \in \Omega_\ell^{\mathrm{odd}}(m)$ with $f(z) \equiv 0 \pmod{\ell^{m-1}}$ for which

$$L_\ell(b;z) \equiv L_\ell(b;z) \mid \mathcal{X}(\ell,m) + f(z) \equiv L_\ell(b + 2c_\ell \ell^{m-2}; z) + f(z) \pmod{\ell^m}. \tag{4.3}$$

We next recall that $\mathcal{S}^*$ is the largest $\mathbb{Z}/\ell^m\mathbb{Z}$-submodule of $S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ on which $X(\ell)$ is an isomorphism modulo ℓ^m. We let μ be the rank of $\mathcal{S}^*$, and we let

$$\{g_1(z), \ldots, g_\mu(z)\} \tag{4.4}$$

be a basis for $\mathcal{S}^*$. There exists a submodule $\mathcal{N}^* \subseteq S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ such that

$$S_{\ell^{m-1}(\ell-1)} = \mathcal{S}^* \oplus \mathcal{N}^*. \tag{4.5}$$

We observe, for all $f(z) \in \mathcal{N}^*$, that there exists $t_f \geq 1$ with

$$f(z) \mid X(\ell)^{t_f} \equiv 0 \pmod{\ell^m}. \tag{4.6}$$

We now give lemmas necessary for the conclusion of Corollary 1.3.

Lemma 4.8. *For $1 \leq i \leq \mu$, let $g_i(z)$ be as in* (4.4). *We have $v_\ell(g_i) = 0$.*

Proof. We suppose on the contrary that, for example, $v_\ell(g_1) \geq 1$. It follows that there exists $h(z) \in S_{\ell^{m-1}(\ell-1)}$ with

$$g_1(z) \equiv \ell^{v_\ell(g_1)} h(z) \pmod{\ell^m} \tag{4.7}$$

and $v_\ell(h) = 0$. Using (4.5), we see that there exists $h_{\mathcal{S}^*}(z) \in \mathcal{S}^*$ and $h_{\mathcal{N}^*} \in \mathcal{N}^*$ with

$$h(z) \equiv h_{\mathcal{S}^*}(z) + h_{\mathcal{N}^*}(z) \pmod{\ell^m}. \tag{4.8}$$

Now, since $h_{\mathcal{N}^*}(z) \in \mathcal{N}^*$, (4.6) implies that there exists $t \geq 1$ such that

$$h_{\mathcal{N}^*}(z) \mid X(\ell)^t \equiv 0 \pmod{\ell^m}. \tag{4.9}$$

We also note by Lemma 4.1 that there exists $n \geq 1$ with

$$X(\ell)^n = 1_{\mathcal{S}^*}, \tag{4.10}$$

the identity on $\mathcal{S}^*$. We let $k \geq 1$ have $nk \geq t$, and we use (4.9) and (4.10) to conclude that

$$h_{\mathcal{N}^*}(z) \mid X(\ell)^{nk} \equiv 0, \ h_{\mathcal{S}^*}(z) \mid X(\ell)^{nk} \equiv h_{\mathcal{S}^*}(z), \ g_1(z) \mid X(\ell)^{nk} \equiv g_1(z) \pmod{\ell^m}. \tag{4.11}$$

From (4.8) and (4.11) we obtain

$$h(z) \mid X(\ell)^{nk} \equiv h_{\mathcal{S}^*}(z) \pmod{\ell^m}. \tag{4.12}$$

Applying $X(\ell)^{nk}$ in (4.7) and using (4.11), we deduce that

$$\ell^{v_\ell(g_1)} h(z) \mid X(\ell)^{nk} \equiv g_1(z) \mid X(\ell)^{nk} \equiv g_1(z) \quad (\mathrm{mod}\ \ell)^m. \tag{4.13}$$

We multiply by $\ell^{v_\ell(g_1)}$ in (4.12); substituting the result in (4.13) gives

$$g_1(z) \equiv \ell^{v_\ell(g_1)} h_{\mathcal{S}^*}(z) \pmod{\ell^m}.$$

Since $h_{\mathcal{S}^*}(z) \in \mathcal{S}^*$, we find $\alpha_1, \ldots, \alpha_\mu \in \mathbb{Z}/\ell^m\mathbb{Z}$ with

$$h_{\mathcal{S}^*}(z) \equiv \alpha_1 g_1(z) + \cdots + \alpha_\mu g_\mu(z) \pmod{\ell^m}.$$

Multiplying by $\ell^{v_\ell(g_1)}$ and using (4.7) yields

$$0 \equiv (\ell^{v_\ell(g_1)}\alpha_1 - 1)g_1(z) + \ell^{v_\ell(g_1)}(\alpha_2 g_1(z) + \cdots + \alpha_\mu g_\mu(z)) \pmod{\ell^m}.$$

Assuming that $v_\ell(g_1) \geq 1$, we have $\ell^{v_\ell(g_1)}\alpha_1 - 1 \not\equiv 0 \pmod{\ell^m}$, which contradicts the linear independence modulo ℓ^m of $\{g_1, \dots, g_\mu\}$. Hence, we have $v_\ell(g_1) = 0$. □

For the next lemmas, let $f(z) \in \Omega_\ell^{\mathrm{odd}}(m) \subseteq \mathcal{S}^*$ be as in (4.3). There exists $a_1, \dots, a_\mu \in \mathbb{Z}/\ell^m\mathbb{Z}$ with

$$f(z) \equiv a_1 g_1 + \cdots + a_\mu g_\mu \pmod{\ell^m}. \tag{4.14}$$

Lemma 4.9. *For $1 \leq j \leq \mu$, let a_j be as in* (4.14)*. We have $a_j \equiv 0 \pmod{\ell^{m-1}}$.*

Proof. If $f(z) \equiv 0 \pmod{\ell^m}$, then the result holds with $a_i \equiv 0 \pmod{\ell^m}$ by (4.14) since $\{g_1, \dots, g_\mu\}$ is a basis for $\mathcal{S}^*$. Recalling that $f(z) \equiv 0 \pmod{\ell^{m-1}}$, it suffices to consider $v_\ell(f) = m-1$. If the statement of the lemma is false, then, for example, we have $v_\ell(a_1) < m-1$. Using (4.14) and $v_\ell(f) = m-1$, we find that

$$0 \equiv a_1 g_1 + \cdots + a_\mu g_\mu \pmod{\ell^{v_\ell(a_1)+1}};$$

multiplying by $\ell^{m-(v_\ell(a_1)+1)}$ gives

$$0 \equiv \ell^{m-(v_\ell(a_1)+1)}(a_1 g_1 + \cdots + a_\mu g_\mu) \pmod{\ell^m}.$$

We compute $v_\ell\left(\ell^{m-(v_\ell(a_1)+1)}a_1\right) = m - (v_\ell(a_1)+1) + v_\ell(a_1) = m-1$; it follows that $\ell^{m-(v_\ell(a_1)+1)}a_1 \not\equiv 0 \pmod{\ell^m}$, contradicting the linear independence of $\{g_1, \dots, g_\mu\}$. Hence, we have $a_1 \equiv 0 \pmod{\ell^{m-1}}$. □

Lemma 4.10. *Let c_ℓ be as in* (4.2)*. Then we have*

$$f(z) \mid X(\ell)^{c_\ell} \equiv f(z) \pmod{\ell^m}.$$

Proof. For all $1 \leq j \leq \mu$, we have $g_j(z) \pmod{\ell}$ in $\mathcal{S}$. Hence, from (4.2), we see that $g_j(z) \mid X(\ell)^{c_\ell} \equiv g_j(z) \pmod{\ell^m}$. With a_j as in (4.14), Lemma 4.9 implies $a_j g_j(z) \mid X(\ell)^{c_\ell} \equiv a_j g_j(z) \pmod{\ell^m}$. It follows that

$$f(z) \mid X(\ell)^{c_\ell} \equiv (a_1 g_1 + \cdots + a_\mu g_\mu) \mid X(\ell)^{c_\ell} \equiv a_1 g_1 + \cdots + a_\mu g_\mu \equiv f(z) \pmod{\ell^m}.$$

□

Lemma 4.11. *Let $1 \leq i \leq \ell$. Then we have*

$$L_\ell(b;z) \mid \mathcal{X}(\ell,m)^\ell \equiv L_\ell(b;z) \mid \mathcal{X}(\ell,m)^{\ell-i} - i f(z) \pmod{\ell^m}. \tag{4.15}$$

Proof. We induct on i. From (4.3) and Lemma 4.10, we compute

$$\begin{aligned} L_\ell(b;z) \mid \mathcal{X}(\ell,m)^\ell &\equiv (L_\ell(b;z) \mid \mathcal{X}(\ell,m)) \mid \mathcal{X}(\ell,m)^{\ell-1} \\ &\equiv (L_\ell(b;z) - f(z)) \mid \mathcal{X}(\ell,m)^{\ell-1} \\ &\equiv L_\ell(b;z) \mid \mathcal{X}(\ell,m)^{\ell-1} - f(z) \mid \mathcal{X}(\ell,m)^{\ell-1} \\ &\equiv L_\ell(b;z) \mid \mathcal{X}(\ell,m)^{\ell-1} - f(z) \pmod{\ell^m}. \end{aligned}$$

Next, we fix $1 \leq i \leq \ell - 1$, and use (4.3), Lemma 4.10, and (4.15) to compute

$$\begin{aligned} L_\ell(b;z) \mid \mathcal{X}(\ell,m)^\ell &\equiv L_\ell(b;z) \mid \mathcal{X}(\ell,m)^{\ell-i} - if(z) \\ &\equiv (L_\ell(b;z) \mid \mathcal{X}(\ell,m)) \mid \mathcal{X}(\ell,m)^{\ell-(i+1)} - if(z) \\ &\equiv (L_\ell(b;z) - f(z)) \mid \mathcal{X}(\ell,m)^{\ell-(i+1)} - if(z) \\ &\equiv L_\ell(b;z) \mid \mathcal{X}(\ell,m)^{\ell-(i+1)} - (i+1)f(z) \pmod{\ell^m}. \end{aligned}$$

The result follows. □

To complete the proof of Corollary 1.3, we let $i = \ell$ in the lemma and recall that $f(z) \equiv 0 \pmod{\ell^{m-1}}$ to obtain

$$L_\ell(b + 2c_\ell \ell^{m-1}; z) = L_\ell(b;z) \mid \mathcal{X}(\ell,m)^\ell \equiv L_\ell(b;z) - \ell f(z) \equiv L_\ell(b;z) \pmod{\ell^m}.$$

5. The proof of Theorem 1.2.

5.1. **Preliminary lemmas.** We observe from Lemma 3.1 (resp. Claim 4.6 (1)) that $D(\ell)$ (resp. $X(\ell)$) preserves $S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$ with coefficients reduced modulo ℓ. We recall that $\mathcal{S}$ is the largest subspace of $S_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$ over $\mathbb{Z}/\ell\mathbb{Z}$ on which $X(\ell)$ is an isomorphism. We define

$$d_\ell := \min\left\{t \geq 0 \ : \ \forall f \in M_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]], \ f \mid D(\ell) \mid X(\ell)^t \in \mathcal{S}\right\}. \tag{5.1}$$

It follows that a simple bound on d_ℓ is

$$d_\ell \leq \dim(S_{\ell-1}) = \left\lfloor \frac{\ell-1}{12} \right\rfloor.$$

We note again, for primes $5 \leq \ell < 1300$, that we have $d_\ell = 0$. We prove the following general theorem.

Theorem 5.1. *Let $\ell \geq 5$ be prime, let $m \geq 1$, and let d_ℓ be as in* (5.1). *Then we have*

$$b_\ell(m) \leq 2(d_\ell + 1)m - 1 = 2d_\ell + 2m - 1.$$

Remark. Theorem 1.2 is the case $d_\ell = 0$.

The proof of Theorem 5.1 requires four preliminary lemmas.

Lemma 5.2. *Let $\ell \geq 5$ be prime, let $m \geq 1$, and let d_ℓ be as in* (5.1). *Suppose, for some even $b \geq 0$, that $\lambda(z) \in \Lambda_\ell^{even}(b,m)$ and that $0 \leq v_\ell(\lambda) = i < m$. Suppose further that $f(z) \in M_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$ satisfies*

$$\lambda(z) \equiv \ell^i f(z) \pmod{\ell^{i+1}}.$$

Then there exists $\mu(z) \in \Omega_\ell^{odd}(m)$ such that

$$\lambda(z) \mid D(\ell) \mid X(\ell)^{d_\ell} \equiv \mu(z) \pmod{\ell^{i+1}}.$$

Proof. Since $f(z) \in M_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]]$, (5.1) implies that $f(z) \mid D(\ell) \mid X(\ell)^{d_\ell} \in \mathcal{S}$. The hypotheses on $\lambda(z)$ imply that $\ell^{-i}\lambda(z) \in \mathbb{Z}_{(\ell)}[[q]]$ and that $\ell^{-i}\lambda(z) \equiv f(z) \pmod{\ell}$. We apply Lemma 4.2 with $A=\mathbb{Z}/\ell\mathbb{Z}$, $M=\mathcal{S}$, $T=X(\ell)$, and $m=f(z)\mid D(\ell)\mid X(\ell)^{d_\ell}$ to find, for all $n\geq 0$, that

$$\begin{aligned}(5.2)\qquad &\frac{\lambda(z)}{\ell^i}\mid D(\ell)\mid X(\ell)^{d_\ell}\pmod{\ell}\\ &\in \mathrm{Span}_{\mathbb{Z}/\ell\mathbb{Z}}\left\{\frac{\lambda(z)}{\ell^i}\mid D(\ell)\mid X(\ell)^{d_\ell+n},\ \frac{\lambda(z)}{\ell^i}\mid D(\ell)\mid X(\ell)^{d_\ell+n+1},\dots\right\}.\end{aligned}$$

Observing that $i+1\leq m$ and that $\lambda(z)\in\Lambda_\ell^{\mathrm{even}}(b,m)$, we see that $\lambda(z)\pmod{\ell^{i+1}}\in\Lambda_\ell^{\mathrm{even}}(b,i+1)$. Hence, for all $j\geq 0$, we have

$$(5.3)\qquad \lambda(z)\mid D(\ell)\mid X(\ell)^{d_\ell+j}\pmod{\ell^{i+1}}\in\Lambda_\ell^{\mathrm{odd}}(b+2(d_\ell+j)+1,\ i+1).$$

We multiply (5.2) by ℓ^i and we use (5.3) together with the nesting property of the modules $\Lambda_\ell^{\mathrm{odd}}(b,m)$ as in the remark following Corollary 3.5 to establish for all $n\geq 0$ that

$$\begin{aligned}&\lambda(z)\mid D(\ell)\mid X(\ell)^{d_\ell}\pmod{\ell^{i+1}}\\ &\in\mathrm{Span}_{\mathbb{Z}/\ell^{i+1}\mathbb{Z}}\left\{\lambda(z)\mid D(\ell)\mid X(\ell)^{d_\ell+n},\dots\right\}\subseteq\Lambda_\ell^{\mathrm{odd}}(b+2(d_\ell+n)+1,\ i+1).\end{aligned}$$

In particular, for odd b' large enough with $b'>b_\ell(m)$, we conclude that there exists $\mu(z)\in\Omega_\ell^{\mathrm{odd}}(m)=\Lambda_\ell^{\mathrm{odd}}(b',m)$ with $\lambda(z)\mid D(\ell)\mid X(\ell)^{d_\ell}\equiv\mu(z)\pmod{\ell^{i+1}}$, as required. □

As in the proof of Theorem 4.3, we set $\mathcal{S}^*\subset S_{\ell^{m-1}(\ell-1)}\cap\mathbb{Z}_{(\ell)}[[q]]$ to be the largest $\mathbb{Z}/\ell^m\mathbb{Z}$-submodule such that $X(\ell)$ is an isomorphism on $\mathcal{S}^*$ modulo ℓ^m. We need the following lemma to prove Lemma 5.5 below.

Lemma 5.3. *Let $\ell\geq 5$ be prime, let $m\geq 1$, let $f(z)\in\mathcal{S}^*$, and suppose that $0\leq v_\ell(f)=i<m$. Then for all $1\leq s\leq m-i$, the form $f(z)$ is congruent modulo ℓ^{i+s} to a form in $S_{\ell^{s-1}(\ell-1)}$.*

Proof. We proceed via induction on $s\geq 1$. The case $s=1$ follows from Lemma 4.4. Therefore, we fix $1\leq s_0<m-i$ and suppose, for all $1\leq s\leq s_0$, that $f(z)$ is congruent modulo ℓ^{i+s} to a form in $S_{\ell^{s-1}(\ell-1)}$. Since $v_\ell(f)=i$, we note for all such s that $\ell^{-i}f(z)$ is congruent modulo ℓ^s to a form in $S_{\ell^{s-1}(\ell-1)}$. In particular, with $s=s_0$, we obtain $f_{s_0}(z)\in S_{\ell^{s_0-1}(\ell-1)}$ with

$$(5.4)\qquad \ell^{-i}f(z)\equiv\ell^{-i}f_{s_0}(z)\pmod{\ell^{s_0}}.$$

Next, we observe that

$$(5.5)\qquad \frac{f_{s_0}(z)}{\ell^i}\cdot E_{\ell-1}(z)^{\ell^{s_0-1}(\ell-1)}\in S_{\ell^{s_0}(\ell-1)}.$$

From Proposition 2.5 and (5.4), we see, for all $1\leq s\leq s_0$ that

$$(5.6)\qquad \frac{f_{s_0}(z)}{\ell^i}\cdot E_{\ell-1}(z)^{\ell^{s_0-1}(\ell-1)}\equiv\frac{f_{s_0}(z)}{\ell^i}\equiv\frac{f(z)}{\ell^i}\pmod{\ell^s}.$$

Noting the induction hypothesis on $\ell^{-i}f(z)$, (5.5), and (5.6), we conclude for all $1\leq s\leq s_0+1$ that $\ell^{-i}f_{s_0}(z)E_{\ell-1}(z)^{\ell^{s_0-1}(\ell-1)}$ is congruent modulo ℓ^s to a form in $S_{\ell^{s-1}(\ell-1)}$. We also note by Proposition 2.3 (2), that $\ell^{-i}f_{s_0}(z)E_{\ell-1}(z)^{\ell^{s_0-1}(\ell-1)}\mid U(\ell)$ is congruent modulo ℓ^s to a form in the same space. Hence, we may apply Lemma 3.1 and (1.8) to show, for all $1\leq s\leq s_0+1$ and for all $t\geq 1$, that there exists $F_{s,t}(z)\in S_{\ell^{s-1}(\ell-1)}$ such that

$$(5.7)\qquad \frac{f_{s_0}(z)}{\ell^i}\cdot E_{\ell-1}(z)^{\ell^{s_0-1}(\ell-1)}\mid X(\ell)^t\equiv F_{s,t}(z)\pmod{\ell^s}.$$

Since $f(z) \in \mathcal{S}^*$, Lemma 4.1 implies that there exists $n \geq 1$ with $f(z) \mid X(\ell)^n \equiv f(z) \pmod{\ell^m}$. We use $i + s_0 + 1 \leq m$ and $v_\ell(f(z)) = i$ to conclude that

$$\frac{f(z)}{\ell^i} \mid X(\ell)^n \equiv \frac{f(z)}{\ell^i} \pmod{\ell^{s_0+1}}. \tag{5.8}$$

For convenience, we set

$$f^*(z) := f(z) - f_{s_0}(z) E_{\ell-1}(z)^{\ell^{s_0-1}(\ell-1)}. \tag{5.9}$$

Multiplying by ℓ^i in (5.6) gives

$$f^*(z) \equiv 0 \pmod{\ell^{i+s_0}}.$$

Therefore, we consider the quantity

$$k = w_\ell\left(\frac{f^*}{\ell^{i+s_0}}\right).$$

If $k = -\infty$, then (5.5) and (5.9) imply that $f(z)$ is congruent $\pmod{\ell^{i+s_0+1}}$ to a form in $S_{\ell^{s_0}(\ell-1)}$. Hence, we suppose that $k \neq -\infty$; we have $k \equiv 0 \pmod{\ell - 1}$. With n as in (5.8), we apply Claim 4.6 to find, for $r \geq 1$ large enough, that

$$w_\ell\left(\frac{f^*}{\ell^{i+s_0}} \mid X(\ell)^{nr}\right) \leq \ell^{s_0}(\ell - 1). \tag{5.10}$$

We first suppose that this filtration is not $-\infty$; it must therefore be $j(\ell - 1)$ for some $0 \leq j \leq \ell^{s_0}$. Using Proposition 2.5, It follows that there exists $G_r(z) \in S_{\ell^{s_0}(\ell-1)}$ for which

$$\frac{f^*(z)}{\ell^{i+s_0}} \mid X(\ell)^{nr} \cdot E_{\ell-1}(z)^{\ell^{s_0}-j} \equiv G_r(z) \pmod{\ell}. \tag{5.11}$$

Starting from

$$f(z) = f_{s_0}(z) \cdot E_{\ell-1}(z)^{\ell^{s_0-1}(\ell-1)} + f^*(z),$$

we apply $X(\ell)^{nr}$ and (5.8) (multiplying by ℓ^i) to obtain

$$f(z) \equiv f_{s_0}(z) \cdot E_{\ell-1}(z)^{\ell^{s_0-1}(\ell-1)} \mid X(\ell)^{nr} + f^*(z) \mid X(\ell)^{nr} \pmod{\ell^{i+s_0+1}}. \tag{5.12}$$

In (5.7), we let $t = nr$ and $s = s_0 + 1$, and we multiply by ℓ^i to show that the first summand on the right side of (5.12) is congruent modulo ℓ^{i+s_0+1} to a form in $S_{\ell^{s_0}(\ell-1)}$. Similarly, we multiply by ℓ^{i+s_0} in (5.11) to deduce that the second summand on the right side of (5.12) is congruent modulo ℓ^{i+s_0+1} to a form in $S_{\ell^{s_0}(\ell-1)}$. We now see that the left side of (5.12) must also be in this space modulo ℓ^{i+s_0+1}. Hence, the lemma is proved when (5.10) is not $-\infty$. When (5.10) has value $-\infty$, we deduce that $f^*(z) \mid X(\ell)^{nr} \equiv 0 \pmod{\ell^{i+s_0+1}}$. We insert this into (5.12) and note again that the first summand on the right side of (5.12) is in $S_{\ell^{s_0}(\ell-1)}$ modulo ℓ^{i+s_0+1} to obtain the desired result. □

Remarks.

(1) A modification of the proof using Proposition 2.5 shows that the conclusion continues to hold under the weaker hypothesis that $f(z) \equiv 0 \pmod{\ell^i}$ (i.e., $v_\ell(f) \geq i$).

(2) The lemma also continues to hold for $f(z) \in \mathcal{S}^* \mid U(\ell)$. As discussed in the proof of Theorem 4.3, $\mathcal{S}^* \mid U(\ell)$ is the largest $\mathbb{Z}/\ell^m\mathbb{Z}$-submodule of $S_{\ell^{m-1}(\ell-1)}$ on which $Y(\ell)$ is an isomorphism modulo ℓ^m. Further, we note that $\Omega_\ell^{\text{even}}(m) \subseteq \mathcal{S}^* \mid U(\ell)$ and $\Omega_\ell^{\text{odd}}(m) \subseteq \mathcal{S}^*$.

We next prove the $m = 1$ case of Theorem 5.1.

Lemma 5.4. *Let $\ell \geq 5$ be prime, and let d_ℓ be as in* (5.1). *Then we have*

$$L_\ell(2d_\ell + 1; z) \in \Omega_\ell^{odd}(1).$$

Proof. We apply Lemma 5.2 with

$$\lambda(z) = L_\ell(0; z) = 1 \in \Lambda_\ell^{\text{even}}(0, 1), \quad f(z) = E_{\ell-1}(z) \in M_{\ell-1} \cap \mathbb{Z}_{(\ell)}[[q]];$$

hence, we have $i = 0$. In particular, Lemma 5.2 yields $\mu(z) \in \Omega_\ell^{\text{odd}}(m)$ with

$$L_\ell(2d_\ell + 1; z) = L_\ell(0; z) \mid D(\ell) \mid X(\ell)^{d_\ell} \equiv \mu(z) \pmod{\ell}.$$

Since reduction modulo ℓ maps $\Omega_\ell^{\text{odd}}(m) \to \Omega_\ell^{\text{odd}}(1)$, the lemma follows. □

The final preliminary lemma plays a central role in the proof of Theorem 5.1 for $m \geq 2$.

Lemma 5.5. *Suppose that $\ell \geq 5$ is prime, $m \geq 2$, $1 \leq s \leq m - 1$, and d_ℓ is as in* (5.1). *Then there exists $\nu(2(d_\ell + 1)s; z) \in \Omega_\ell^{even}(m)$ and $\tau(2(d_\ell + 1)s; z) \in S_{\ell^{m-s-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ with the following properties.*

1. *We have $\tau(2(d_\ell + 1)s; z) \equiv 0 \pmod{\ell^s}$.*
2. *For all k with $s + 1 \leq k \leq m$, the form $\tau(2(d_\ell + 1)s; z)$ is congruent modulo ℓ^k to a form in $S_{\ell^{k-s-1}(\ell-1)}$.*
3. *We have $L_\ell(2(d_\ell + 1)s; z) \equiv \nu(2(d_\ell + 1)s; z) + \tau(2(d_\ell + 1)s; z) \pmod{\ell^m}$.*

Proof. The proof proceeds by induction on s. In view of the proof of Lemma 5.4, we see that there exists $\mu(z) \in \Omega_\ell^{\text{odd}}(m)$ with

$$L_\ell(2d_\ell + 1; z) \equiv \mu(z) \pmod{\ell}. \tag{5.13}$$

Since $D(\ell) : \Omega_\ell^{\text{even}}(m) \to \Omega_\ell^{\text{odd}}(m)$ is an isomorphism, there exists $\nu(2d_\ell; z) \in \Omega_\ell^{\text{even}}(m)$ with

$$\nu(2d_\ell; z) \mid D(\ell) \equiv \mu(z) \pmod{\ell^m}. \tag{5.14}$$

We claim that the form $L_\ell(2d_\ell; z) - \nu(2d_\ell; z)$ satisfies the hypotheses of Lemma 3.6. We observe that $\nu(2d_\ell; z) \in \Omega_\ell^{\text{even}}(m)$ implies, for all $1 \leq k \leq m$, that $\nu(2d_\ell; z) \pmod{\ell^k} \in \Omega_\ell^{\text{even}}(k) \subseteq M_{\ell^{k-1}(\ell-1)}$. Similarly, Corollary 3.5 implies that $L_\ell(2d_\ell; z)$ is congruent modulo ℓ^k to a form in $M_{\ell^{k-1}(\ell-1)}$. Therefore, there exists $\alpha(k; z) \in M_{\ell^{k-1}(\ell-1)} \cap \mathbb{Z}[[q]]$ with

$$L_\ell(2d_\ell; z) - \nu(2d_\ell; z) \equiv \alpha(k; z) \pmod{\ell^k}. \tag{5.15}$$

Moreover, (1.7), (5.13), and (5.14) imply that

$$(L_\ell(2d_\ell; z) - \nu(2d_\ell; z)) \mid D(\ell) \equiv L_\ell(2d_\ell + 1; z) - \nu(2d_\ell; z) \mid D(\ell) \equiv 0 \pmod{\ell}. \tag{5.16}$$

Hence, our claim holds. Applying Lemma 3.6 and using (5.15) gives, for $2 \leq k \leq m$, a form $h(k; z) \in S_{\ell^{k-2}(\ell-1)} \cap \mathbb{Z}[[q]]$ with

$$(L_\ell(2d_\ell; z) - \nu(2d_\ell; z)) \mid Y(\ell) \equiv \alpha(k; z) \mid Y(\ell) \equiv h(k; z) \pmod{\ell^k}. \tag{5.17}$$

We next claim that $\tau(2(d_\ell + 1); z) := h(m; z) \in S_{\ell^{m-2}(\ell-1)}$ satisfies the conclusion of the present lemma for $s = 1$. For $2 \leq k \leq m$, (5.17) implies that $h(m; z) \equiv h(k; z) \pmod{\ell^k}$, and (1.9), (5.16), and (5.17) imply that $h(m; z) \equiv 0 \pmod{\ell}$. Now, since $Y(\ell) : \Omega_\ell^{\text{even}}(m) \to \Omega_\ell^{\text{even}}(m)$, we have

$$\nu(2(d_\ell + 1); z) := \nu(2d_\ell; z) \mid Y(\ell) \in \Omega_\ell^{\text{even}}(m). \tag{5.18}$$

It follows from (1.10), (1.9), (5.17), and (5.18) that

$$\begin{aligned} L_\ell(2(d_\ell+1);z) &= L_\ell(2d_\ell;z) \mid Y(\ell) = \nu(2d_\ell;z) \mid Y(\ell) + (L_\ell(2d_\ell;z) - \nu(2d_\ell;z)) \mid Y(\ell) \\ &\equiv \nu(2(d_\ell+1);z) + h(m;z) \equiv \nu(2(d_\ell+1)s;z) + \tau(2(d_\ell+1);z) \pmod{\ell^m}. \end{aligned}$$

We now suppose, for fixed $1 \le s \le m-2$, that there exists $\nu(2(d_\ell+1)s;z)$ and $\tau(2(d_\ell+1)s;z)$ satisfying the conclusion of the lemma. Condition (1) implies that $v_\ell(\tau(2(d_\ell+1)s;z)) \ge s$; we may assume that

$$v_\ell(\tau(2(d_\ell+1)s;z)) = s. \tag{5.19}$$

We first show that $\tau(2(d_\ell+1)s;z)$ satisfies the hypotheses of Lemma 5.2. The hypothesis on $\nu(2(d_\ell+1)s;z)$, the definition of $b_\ell(m)$, and the nesting property of the modules $\Lambda_\ell^{\mathrm{even}}(2(d_\ell+1)s;m)$ give

$$\nu(2(d_\ell+1)s;z) \in \Omega_\ell^{\mathrm{even}}(m) = \Lambda_\ell^{\mathrm{even}}(b_\ell(m),m) \subseteq \Lambda_\ell^{\mathrm{even}}(2(d_\ell+1)s,m). \tag{5.20}$$

From condition (3), we see that $\tau(2(d_\ell+1)s;z) \equiv L_\ell(2(d_\ell+1)s;z) - \nu(2(d_\ell+1)s;z) \pmod{\ell^m}$. In view of (5.20), it follows that $\tau(2(d_\ell+1)s;z) \in \Lambda_\ell^{\mathrm{even}}(2(d_\ell+1)s,m)$. With $k = s+1$ in condition (2), we find that $\tau(2(d_\ell+1)s;z)$ is congruent modulo ℓ^{s+1} to a form in $S_{\ell-1}$. We may now apply Lemma 5.2 to $\tau(2(d_\ell+1)s;z)$ to produce $\gamma(s;z) \in \Omega_\ell^{\mathrm{odd}}(m)$ with

$$\tau(2(d_\ell+1)s;z) \mid D(\ell) \mid X(\ell)^{d_\ell} \equiv \gamma(s;z) \pmod{\ell^{s+1}}. \tag{5.21}$$

From (5.19), we find that

$$\gamma(s;z) \equiv 0 \pmod{\ell^s}. \tag{5.22}$$

Since $D(\ell) : \Omega_\ell^{\mathrm{even}}(m) \to \Omega_\ell^{\mathrm{odd}}(m)$ is an isomorphism, there exists $\beta(s;z) \in \Omega_\ell^{\mathrm{even}}(m)$ with

$$\beta(s;z) \mid D(\ell) \equiv \gamma(s;z) \pmod{\ell^m}. \tag{5.23}$$

Noting (5.22) and that $s < m$, we also have $\beta(s;z) \mid D(\ell) \equiv 0 \pmod{\ell^s}$. Reduction modulo ℓ^s surjects onto $\Omega_\ell^{\mathrm{even}}(s)$, and $D(\ell)$ is an isomorphism on $\Omega_\ell^{\mathrm{even}}(s)$. Hence, we deduce that

$$\beta(s;z) \equiv 0 \pmod{\ell^s}. \tag{5.24}$$

Using (1.8) and (1.9), we observe that

$$\tau(2(d_\ell+1)s;z) \mid D(\ell) \mid X(\ell)^{d_\ell} = \tau(2(d_\ell+1)s;z) \mid Y(\ell)^{d_\ell} \mid D(\ell). \tag{5.25}$$

Since $s+1 < m$, it follows from (5.21), (5.23), and (5.25) that

$$\tau(2(d_\ell+1)s;z) \mid Y(\ell)^{d_\ell} \mid D(\ell) \equiv \beta(s;z) \mid D(\ell) \pmod{\ell^{s+1}}. \tag{5.26}$$

We next show that

$$\Xi(z) := \frac{\tau(2(d_\ell+1)s;z) \mid Y(\ell)^{d_\ell} - \beta(s;z)}{\ell^s} \tag{5.27}$$

satisfies the hypotheses of Lemma 3.6. Dividing by ℓ^s in (5.26) gives

$$\Xi(z) \mid D(\ell) = \frac{\tau(2(d_\ell+1)s;z) \mid Y(\ell)^{d_\ell} - \beta(s;z)}{\ell^s} \mid D(\ell) \equiv 0 \pmod{\ell}. \tag{5.28}$$

Condition (2) and (5.19) imply, for all $s+1 \le k \le m$, that $\ell^{-s}\tau(2(d_\ell+1)s;z)$ is congruent modulo ℓ^{k-s} to a form in $S_{\ell^{k-s-1}(\ell-1)}$. We use Proposition 2.3 and Lemma 3.1 (replacing k with $k-s$ and n with $m-s$) to show that $\ell^{-s}\tau(2(d_\ell+1)s;z) \mid Y(\ell)^{d_\ell}$ remains in the space $S_{\ell^{k-s-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ with coefficients reduced modulo ℓ^{k-s}. In view of (5.24) and the fact that $\beta(s;z) \in \Omega_\ell^{\mathrm{even}}(m)$, we apply Lemma 5.3 and the remarks following it to $\ell^{-s}\beta(s;z)$ to show that the same conclusion holds for this form. From (5.27) we therefore conclude, for all $s+1 \leq k \leq m$, that $\Xi(z)$ modulo ℓ^{k-s} is congruent to a form in $S_{\ell^{k-s-1}(\ell-1)}$. We may now apply Lemma 3.6 (with $j = k - s$ and $n = m - s$) to $\Xi(z)$ to obtain, for all $s+2 \leq k \leq m$, forms $f(k;z) \in S_{\ell^{k-s-2}(\ell-1)}$ with

$$\Xi(z) \mid Y(\ell) \equiv f(k;z) \pmod{\ell^{k-s}}. \tag{5.29}$$

To conclude, we show that

$$\tau(2(d_\ell+1)(s+1);z) := \ell^s f(m;z) \in S_{\ell^{m-s-2}(\ell-1)}, \tag{5.30}$$

$$\nu(2(d_\ell+1)(s+1);z) := (\nu(2(d_\ell+1)s;z) \mid Y(\ell)^{d_\ell} + \beta(s;z)) \mid Y(\ell) \tag{5.31}$$

satisfy the conditions of the lemma. From (5.27), (5.29), and (5.30), we observe that

$$\tau(2(d_\ell+1)(s+1);z) \equiv \big(\tau(2(d_\ell+1)s;z) \mid Y(\ell)^{d_\ell} - \beta(s;z)\big) \mid Y(\ell) \pmod{\ell^m}. \tag{5.32}$$

We first note that since $Y(\ell)$ maps $\Omega_\ell^{\text{even}}(m)$ to itself and since $\nu(2(d_\ell+1)s;z)$ and $\beta(s;z) \in \Omega_\ell^{\text{even}}(m)$, we have $\nu(2(d_\ell+1)(s+1);z) \in \Omega_\ell^{\text{even}}(m)$. To verify condition (1), we multiply by ℓ^s and apply $U(\ell)$ in (5.28) to obtain $\ell^s \cdot \Xi(z) \mid Y(\ell) \equiv 0 \pmod{\ell^{s+1}}$. Similarly, in (5.29), we multiply by ℓ^s and set $k = m$. Noting that $s+1 < m$ and using (5.30), we find that $\ell^s \cdot \Xi(z) \mid Y(\ell) \equiv \tau(2(d_\ell+1)(s+1);z) \pmod{\ell^{s+1}}$. It follows that $\tau(2(d_\ell+1)(s+1);z) \equiv 0 \pmod{\ell^{s+1}}$, as desired. From (5.29) and (5.30), we find, for all $s+2 \leq k \leq m$, that $\tau(2(d_\ell+1)(s+1);z)$ is congruent modulo ℓ^k to a form in $S_{\ell^{k-s-2}(\ell-1)}$, namely $\ell^s f(k;z)$. This is condition (2) of the lemma. Lastly, we verify condition (3). By the induction hypothesis, (1.10), (1.9), (5.31), and (5.32), we have

$$\begin{aligned}
L_\ell(2(d_\ell+1)(s+1);z) &\equiv L_\ell(2(d_\ell+1)s;z) \mid Y(\ell)^{d_\ell+1} \\
&\equiv (\nu(2(d_\ell+1)s;z) + \tau(2(d_\ell+1)s;z)) \mid Y(\ell)^{d_\ell+1} \\
&\equiv (\nu(2(d_\ell+1)s;z) \mid Y(\ell)^{d_\ell} + \beta(s;z)) \mid Y(\ell) \\
&+ (\tau(2(d_\ell+1)s;z) \mid Y(\ell)^{d_\ell} - \beta(s;z)) \mid Y(\ell) \\
&\equiv \nu(2(d_\ell+1)(s+1);z) + \tau(2(d_\ell+1)(s+1);z) \pmod{\ell^m}.
\end{aligned}$$

The lemma is proved. □

5.2. **Proof of Theorem 5.1.** Let $\ell \geq 5$ be prime. To prove Theorem 5.1, it suffices to show for all $m \geq 1$ that

$$L_\ell(2(d_\ell+1)m-1;z) \in \Omega_\ell^{\text{odd}}(m). \tag{5.33}$$

The $m = 1$ case is Lemma 5.4. We now use Lemmas 5.2 and 5.5 to prove the theorem for $m \geq 2$. We let $s = m-1$ in Lemma 5.5 to obtain $\nu(2(d_\ell+1)(m-1);z) \in \Omega_\ell^{\text{even}}(m)$ and $\tau(2(d_\ell+1)(m-1);z) \in S_{\ell-1} \cap \mathbb{Z}[[q]]$ satisfying conditions (1), (2), and (3) of the lemma. Condition (1) states that $\tau(2(d_\ell+1)(m-1);z) \equiv 0 \pmod{\ell^{m-1}}$. If $\tau(2(d_\ell+1)(m-1);z) \equiv 0 \pmod{\ell^m}$, then condition (3) implies that $L_\ell(2(d_\ell+1)(m-1);z) \equiv \nu(2(d_\ell+1)(m-1);z) \pmod{\ell^m}$. It follows that $L_\ell(2(d_\ell+1)(m-1);z) \in \Omega_\ell^{\text{even}}(m)$. Basic properties of $X(\ell)$ and $D(\ell)$ given in the beginning of Section 4.1 imply that

$$D(\ell)X(\ell)^{d_\ell} : \Omega_\ell^{\text{even}}(m) \to \Omega_\ell^{\text{odd}}(m) \tag{5.34}$$

Hence, we find that $L_\ell(2(d_\ell+1)m-1;z) = L_\ell(2(d_\ell+1)(m-1);z) \mid D(\ell) \mid X(\ell)^{d_\ell} \in \Omega_\ell^{\text{odd}}(m)$.

Therefore, we assume that

$$v_\ell(\tau(2(d_\ell+1)(m-1);z)) = m-1. \tag{5.35}$$

In this case, condition (3) gives

$$L_\ell(2(d_\ell+1)(m-1);z) \equiv \nu(2(d_\ell+1)(m-1);z) + \tau(2(d_\ell+1)(m-1);z) \pmod{\ell^m}. \tag{5.36}$$

It follows that $\tau(2(d_\ell+1)(m-1);z) \in \Lambda_\ell^{\text{even}}(2(d_\ell+1)(m-1),m)$. Moreover, (5.35) and the fact that $\tau(2(d_\ell+1)(m-1);z) \in S_{\ell-1}$ imply that this form satisfies the hypotheses of Lemma 5.2. Using this lemma and the isomorphism (5.34), we deduce the existence of $\beta(z) \in \Omega_\ell^{\text{even}}(m)$ with

$$\tau(2(d_\ell+1)(m-1);z) \mid D(\ell) \mid X(\ell)^{d_\ell} \equiv \beta(z) \mid D(\ell) \mid X(\ell)^{d_\ell} \pmod{\ell^m}. \tag{5.37}$$

We now rewrite (5.36) as

$$\begin{aligned} L_\ell(2(d_\ell+1)(m-1);z) \equiv\ & \nu(2(d_\ell+1)(m-1);z) + \beta(z) \\ & + (\tau(2(d_\ell+1)(m-1);z) - \beta(z)) \pmod{\ell^m}. \end{aligned}$$

From (5.37), we find that

$$\begin{aligned} L_\ell(2(d_\ell+1)m-1;z) &= L_\ell(2(d_\ell+1)(m-1);z) \mid D(\ell) \mid X(\ell)^{d_\ell} \\ &\equiv (\nu(2(d_\ell+1)(m-1);z) + \beta(z)) \mid D(\ell) \mid X(\ell)^{d_\ell} \\ &\quad + (\tau(2(d_\ell+1)(m-1);z) - \beta(z)) \mid D(\ell) \mid X(\ell)^{d_\ell} \\ &\equiv (\nu(2(d_\ell+1)(m-1);z) + \beta(z)) \mid D(\ell) \mid X(\ell)^{d_\ell} \pmod{\ell^m}. \end{aligned}$$

Using (5.34) again we conclude that $L_\ell(2(d_\ell+1)m-1;z) \in \Omega_\ell^{\text{odd}}(m)$, completing the proof of the theorem. □

6. Calculations: examples and comments.

6.1. **Examples.** In this section, we give selected examples to illustrate Theorem 1.2 and Corollary 1.3. In the course of our investigation, we calculated bases for the spaces $\Omega_\ell^{\text{odd}}(m)$ and $\Omega_\ell^{\text{even}}(m)$ in the following cases:

- $m = 1$, primes $13 \le \ell < 1300$,
- $m = 2$, primes $13 \le \ell \le 89$,
- $m = 3$, primes $13 \le \ell \le 29$,
- $m = 4$, $\ell = 13$.

We first give examples of Theorem 1.2. We recall from (1.13) that $r_\ell(m)$ is the rank of $\Omega_\ell^{\text{odd}}(m)$ as a $\mathbb{Z}/\ell^m\mathbb{Z}$-module and that $R_\ell = \lfloor \frac{\ell+12}{24} \rfloor$ is the upper bound for this rank.

Example 1. Let $\ell = 29$. We find that $r_{29}(1) = 1 = R_{29}$. By the third remark after Theorem 1.1, we see that $r_{29}(m) = 1$ for all $m \ge 1$. We compute $b_{29}(1) = 1$, $b_{29}(2) = 3$, and $b_{29}(3) = 5$. We use this data and explicit computation to verify, for all $n \ge 0$, that

$$\begin{aligned} p(29n+23) &\equiv 10p(29^3n+806) \pmod{29}, \\ p(29^3n+806) &\equiv 329p(29^5n+19308) \pmod{29^2}, \\ p(29^5n+19308) &\equiv 14706p(29^7n+13656152078) \pmod{29^3}. \end{aligned}$$

Example 2. Let $\ell = 89$. We have $r_{89}(1) = 4 = R_{89}$; therefore, we have $r_{89}(m) = 4$ for all $m \ge 1$. We also compute $b_{89}(1) = 1$ and $b_{89}(2) = 3$. Our computations yield the following congruences for all $n \geq 0$:

$$\begin{aligned} p(89n+26) \equiv\ & 87p(89^3n+7591)+62p(89^5n+1628684006) \\ & +14p(89^7n+12900806011196)+78p(89^9n+102\dots186) \pmod{89}, \\ p(89^3n+7591) \equiv\ & 1244p(89^5n+1628684006)+5135p(89^7n+12900806011196) \\ & +1082p(89^9n+102\dots186)+968p(89^{11}n+809\dots976) \pmod{89^2}. \end{aligned}$$

Example 3. Let $\ell = 1297$. We calculate $r_{1297}(1) = 54$. Hence, for all $m \geq 1$, we have $r_{1297}(m) = 54$. We also have $b_{1297}(1) = 1$. For all $n \geq 0$, we find that

$$\begin{aligned} p(1297n+1243) \equiv\ & 1171p(1297^3n+2090915695)+207p(1297^5n+3517357200300163)+ \\ & \vdots \\ & +1242p(1297^{107}n+116...975)+1108p(1297^{109}n+195...683) \pmod{1297}. \end{aligned}$$

For all primes $\ell \leq 1297$ with the exception of $\ell = 607$, our calculations show that $r_\ell(1) = R_\ell$, and hence, that $r_\ell(m) = R_\ell$ for all $m \geq 1$. For $\ell = 607$, we find that $r_{607}(m) = R_{607} - 1$ for all $m \geq 1$.

We next give examples of Corollary 1.3.

Example 1. Let $\ell = 37$. We find that $c_{37} = 36$, and we discover, for all $n \geq 0$, that

$$p(37n+17) \equiv p(37^{73}n+138...7757) \pmod{37}.$$

Example 2. Let $\ell = 137$. Our computations give $c_{137} = 177423288$. Thus, the following congruence holds for all $n \geq 0$:

$$p(137n+40) \equiv p(137^{354846577}n+531\dots1080) \pmod{137}.$$

6.2. **Comments on computation.** Let $\ell \geq 13$ be prime, and let $m \geq 1$. We describe how to calculate a relation modulo ℓ^m between the $r_\ell(m)+1$ functions $\{L_\ell(b_\ell(m);z), L_\ell(b_\ell(m)+2;z), \dots, L_\ell(b_\ell(m)+2r_\ell(m);z)\} \subseteq \Omega_\ell^{\mathrm{odd}}(m)$. By Corollary 3.5, this calculation takes place in the $\mathbb{Z}/\ell^m\mathbb{Z}$-module $S_{\ell^{m-1}(\ell-1)} \cap \mathbb{Z}_{(\ell)}[[q]]$ with coefficients reduced modulo ℓ^m.

Let

$$t_{\ell,m} = \left\lfloor \frac{\ell^{m-1}(\ell-1)}{12} \right\rfloor.$$

We require the rank of $S_{\ell^{m-1}(\ell-1)}$, given by

$$s_{\ell,m} := \begin{cases} t_{\ell,m}-1 & \text{if } \ell^{m-1}(\ell-1) \equiv 2 \pmod{12}, \\ t_{\ell,m} & \text{otherwise,} \end{cases}$$

and we require the forms

$$F_{\ell,m}(z) := \begin{cases} 1 & \text{if } \ell^{m-1}(\ell-1) \equiv 0 \pmod{12}, \\ E_4(z)^2E_6(z) & \text{if } \ell^{m-1}(\ell-1) \equiv 2 \pmod{12}, \\ E_4(z) & \text{if } \ell^{m-1}(\ell-1) \equiv 4 \pmod{12}, \\ E_6(z) & \text{if } \ell^{m-1}(\ell-1) \equiv 6 \pmod{12}, \\ E_4(z)^2 & \text{if } \ell^{m-1}(\ell-1) \equiv 8 \pmod{12}, \\ E_4(z)E_6(z) & \text{if } \ell^{m-1}(\ell-1) \equiv 10 \pmod{12}. \end{cases}$$

In this notation, a standard upper-triangular basis for the space $S_{\ell^{m-1}(\ell-1)}$ is

$$\left\{\Delta(z)^k E_4(z)^{3(s_{\ell,m}-k)} F_{\ell,m}(z) = q^k + \cdots\right\}_{k=1}^{s_{\ell,m}}.$$

Hence, to distinguish a form in $S_{\ell^{m-1}(\ell-1)}$ it suffices to compute its coefficients to order $O(q^{t_{\ell,m}})$.

We seek to efficiently calculate $\Phi_\ell(z) = \eta(\ell^2 z)/\eta(z) \pmod{\ell^m}$. When $m = 1$, we use (2.10). For $m \geq 2$, we use the following proposition.

Proposition 6.1. *Let $m \geq 2$. Then we have*

$$\Phi_\ell(z) \equiv \frac{\eta(\ell^2 z)\eta(z)^{2\ell^m-1}}{\eta(\ell^{m+1}z)^{2\ell^{m-1}}} \prod_{k=1}^{m} \eta(\ell^k z)^{2\ell^m - 2\ell^{m-1}} \pmod{\ell^m}. \tag{6.1}$$

Proof. By Proposition 2.5, for all $k \geq 0$, we have $A_\ell(\ell^k z)^{2\ell^{m-1}} \equiv 1 \pmod{\ell^m}$. Thus, using definitions (1.5) and (2.8), we have

$$\Phi_\ell(z) \equiv \Phi_\ell(z) \prod_{k=0}^{m} A_\ell(\ell^k z)^{2\ell^{m-1}} \equiv \frac{\eta(\ell^2 z)}{\eta(z)} \prod_{k=0}^{m} \frac{\eta(\ell^k z)^{2\ell^m}}{\eta(\ell^{k+1}z)^{2\ell^{m-1}}} \pmod{\ell^m}.$$

Simplification by grouping factors yields the proposition. □

We note that

$$\left(q^{-\frac{\ell^{m+1}}{24}} \eta(\ell^{m+1}z)\right)^{2\ell^{m-1}} = \prod_{n=1}^{\infty} (1 - q^{\ell^{m+1}n})^{2\ell^{m-1}} = 1 + O(q^{\ell^{m+1}})$$

and that $t_{\ell,m} < \ell^{m+1}$. Therefore, to compute the right side of (6.1) to order $O(q^{t_{\ell,m}})$ we may disregard the contribution from the denominator. Rather, to compute the right side to suitable order, it suffices to compute

$$q^{-\frac{\ell^{2m}}{12}} \eta(\ell^2 z)\eta(z)^{2\ell^m-1} \prod_{k=1}^{m} \eta(\ell^k z)^{2\ell^m-2\ell^{m-1}} = q^{\frac{\ell^2-1}{24}} \prod_{n=1}^{\infty} (1-q^n)^{2\ell^m-1}(1-q^{\ell^2 n}) \prod_{k=1}^{m} (1-q^{\ell^k n})^{2\ell^m-2\ell^{m-1}}.$$

For this purpose, we use Euler's Pentagonal Number Theorem:

$$\prod_{n=1}^{\infty} (1 - q^n) = 1 + \sum_{k=1}^{\infty} (-1)^k \left(q^{\frac{k(3k-1)}{2}} + q^{\frac{k(3k+1)}{2}}\right).$$

We now turn to computation of $r_\ell(m)$ and $b_\ell(m)$. To start, we compute

$$\{L_\ell(2m-1; z),\ L_\ell(2m+1; z), \ldots, L_\ell(2(m + R_\ell - 1) - 1; z)\} \pmod{\ell^m}.$$

A collection $f_1, \ldots, f_n \in \mathbb{Z}_{(\ell)}[[q]]$ is linearly independent over $\mathbb{Z}_{(\ell)}[[q]]$ if and only if the relation $a_1 f_1 + \cdots + a_n f_n \equiv 0 \pmod{\ell^m}$ implies, for all $1 \leq i \leq n$, that $a_i \equiv 0 \pmod{\ell^m}$; if $m \geq 2$, we further require that not all $a_i \equiv 0 \pmod{\ell}$. Next, we determine the largest $0 \leq s \leq R_\ell - 1$ for which

$$I_{\ell,m,s} := \{L_\ell(2m-1; z),\ L_\ell(2m+1; z), \ldots, L_\ell(2(m+s-1) - 1; z)\} \pmod{\ell^m}$$

is linearly independent, and we set $J_{\ell,m,s} := \mathrm{Span}_{\mathbb{Z}/\ell^m\mathbb{Z}}(I_{\ell,m,s})$. In all calculated examples we found that $s = R_\ell - 1$ except for $\ell = 607$, in which case, we computed $s = R_{607} - 2 = 24$.

Continuing our search for relations modulo ℓ^m, we first suppose that

$$L_\ell(2(m+s) - 1; z) \in J_{\ell,m,s}. \tag{6.2}$$

Then there exists $c_0, \ldots, c_{s-1} \in \mathbb{Z}$ for which

$$L_\ell(2(m+s)-1; z) \equiv c_0 L_\ell(2m-1; z) + \ldots + c_{s-1} L_\ell(2(m+s-1)-1; z) \pmod{\ell^m}.$$

As $X(\ell)$ is cyclic on $J_{\ell,m,s}$, its matrix representation in the basis $I_{\ell,m,s}$ is

$$[X(\ell)] = \begin{pmatrix} 0 & 0 & \cdots & 0 & c_0 \\ 1 & 0 & \cdots & 0 & c_1 \\ 0 & 1 & \cdots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & c_{s-1} \end{pmatrix}.$$

We also suppose that

$$\ell \nmid c_0. \tag{6.3}$$

Then $X(\ell)$ is an isomorphism on $J_{\ell,m,s}$ since $c_0 \in (\mathbb{Z}/\ell^m\mathbb{Z})^\times$ is the determinant of the matrix $[X(\ell)]$. We conclude that

$$d_\ell = 0, \quad b_\ell(m) \leq 2m-1, \quad r_\ell(m) = s, \quad \Omega_\ell^{\text{odd}}(m) = J_{\ell,m,s}. \tag{6.4}$$

Conditions (6.2) and (6.3) were met in all calculated examples.

It remains to compute the precise value of $b_\ell(m)$. We recall, for all $b \geq b_\ell(m)$, that

$$\text{Span}_{\mathbb{Z}/\ell^m\mathbb{Z}} \{L_\ell(b; z), L_\ell(b+2; z), \ldots, L_\ell(b+2(r_\ell(m)-1); z)\} = \Omega_\ell^{\text{odd}}(m).$$

Hence, there exists $c_0, \ldots, c_{r_\ell(m)-1} \in \mathbb{Z}$ such that

$$L_\ell(b+2r_\ell; z) \equiv c_0 L_\ell(b; z) + \ldots + c_{r_\ell-1} L_\ell(b+2(r_\ell(m)-1); z) \pmod{\ell^m}. \tag{6.5}$$

Moreover, the coefficients are independent of b. Therefore, we seek b minimal for which a congruence of type (6.5) holds. We use (6.4) to expedite this search.

On the other hand, if either of (6.2) or (6.3) fail to hold, then we have $d_\ell > 0$. In this setting, $X(\ell)$ on $S_{\ell-1}$ has an eigenvalue $\lambda \equiv 0 \pmod{\ell}$. The corresponding eigenspace has dimension $d_\ell + 1$ and is not contained in $\mathcal{S}$ (as in (5.1)). We conclude that $r_\ell(m) \leq R_\ell - d_\ell < R_\ell$. Using these facts, an analysis similar to that for when $d_\ell = 0$ enables calculation of $r_\ell(m)$ and $b_\ell(m)$.

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