INFINITE FAMILIES OF INFINITE FAMILIES OF CONGRUENCES FOR k-REGULAR PARTITIONS

ROWLAND CARLSON AND JOHN J. WEBB

ABSTRACT. Let $k \in \{10, 15, 20\}$, and let $b_k(n)$ denote the number k-regular partitions of n. We prove for half of all primes p and any $t \geq 1$ that there exists $p - 1$ arithmetic progressions modulo p^{2t} such that $b_k(n)$ is a multiple of 5 for each n in one of these progressions.

1. INTRODUCTION AND STATEMENT OF RESULTS

For a natural number n, a partition of n is a non-increasing sequence of natural numbers, called parts, whose sum is n. Let $k > 1$ be an integer. We say that a partition is k-regular if none of its parts are divisible by k. We define $b_k(n)$ to be the number of k-regular partitions of n, and let $b_k(0) := 1$ and $b_k(\alpha) := 0$ if $\alpha \notin \mathbb{N} \cup \{0\}$.

Euler made many of the most important contributions to the study of partitions. In particular, Euler discovered the generating functions for many types of partition functions. For k-regular partitions, he showed

$$\sum_{n=0}^{\infty} b_k(n) q^n = \prod_{m=1}^{\infty} \frac{1 - q^{km}}{1 - q^m}\,. \tag{1.1}$$

For $k = 2$, Euler used this generating function to prove that the number of 2-regular partitions of n, in other words partitions of n into odd parts, is equal to the number of partitions of n where each part is distinct. Over the past 100 years, there has been significant interest in infinite families of congruences for partition functions beginning with Ramanujan's congruences for the unrestricted partition function $p(n)$. For all non-negatve integers n we have

$$\begin{aligned} p(5n + 4) &\equiv 0 \pmod{5}\,, \\ p(7n + 5) &\equiv 0 \pmod{7}\,, \text{ and} \\ p(11n + 6) &\equiv 0 \pmod{11}\,. \end{aligned}$$

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Work of Ahlgren and Ono [1], [3], [10] shows that for any moduli m coprime to 6 that there are infinitely many pairs (A, B), such that

$$p(An + B) \equiv 0 \pmod{m};$$

however, Ahlgren and Boylan [2] proved that Ramanujan's congruences are very special in the sense that the only pairs (ℓ, δ) where ℓ is prime and $0 \le \delta \le \ell - 1$ such that for all non-negative integers n,

$$p(\ell n + \delta) \equiv 0 \pmod{\ell}$$

are precisely $(5, 4)$, $(7, 5)$, and $(11, 6)$.

There has also been significant interest in the arithmetic of regular partition functions. Gordon and Ono [8] proved that for any prime ℓ, the density of values of $b_\ell(n)$ divisible by ℓ^m for any $m \ge 1$ is 1 as n goes to infinity. Precise conditions for the values n when ℓ divides $b_\ell(n)$ were proved by Lovejoy and Penniston [9] for $\ell = 3$, and Dandurand and Penniston [5] for $\ell \in \{5, 7, 11\}$. Congruences for regular partitions modulo 2 and 3 were proven in [4], [6], [7], and [13].

In this paper we prove families of congruences modulo 5 for the k-regular partition functions when $k \in \{10, 15, 20\}$.

Theorem 1.1. *Let $n \ge 1$, p be an odd prime, and λ an integer with $p \nmid \lambda$.*

- *If $p \equiv 5$ or $7 \bmod 8$, then for all non-negative integers t we have*

$$b_{10}\left(p^{2n}t + p^{2n-1}\lambda + 3\left(\frac{p^{2n}-1}{8}\right)\right) \equiv 0 \pmod{5}.$$

- *If $p \equiv 5 \bmod 6$, then for all non-negative integers t we have*

$$\begin{aligned} &b_{15}\left(p^{2n}t + p^{2n-1}\lambda + 7\left(\frac{p^{2n}-1}{12}\right)\right) \\ &\qquad \equiv b_{20}\left(p^{2n}t + p^{2n-1}\lambda + 19\left(\frac{p^{2n}-1}{24}\right)\right) \equiv 0 \pmod{5}. \end{aligned}$$

Example. Let $p = 23$. By Theorem 1.1, for each $k \in \{10, 15, 20\}$ and any $n \ge 1$, there are 22 separate arithmetic progressions modulo 23^{2n} on which $b_k(n)$ vanishes modulo 5. To be precise, for all $t \ge 0$ and all integers λ with $23 \nmid \lambda$, we have

$$\begin{aligned} b_{10}\bigg(23^{2n}t &+ 23^{2n-1}\lambda + 3\left(\frac{23^{2n}-1}{8}\right)\bigg) \\ &\equiv b_{15}\left(23^{2n}t + 23^{2n-1}\lambda + 7\left(\frac{23^{2n}-1}{12}\right)\right) \\ &\equiv b_{20}\left(23^{2n}t + 23^{2n-1}\lambda + 19\left(\frac{23^{2n}-1}{24}\right)\right) \equiv 0 \pmod{5}. \end{aligned}$$

As with most of the preceding work on congruences for partitions, our proof relies on the theory of modular forms. The generating functions for each of these regular partition functions are congruent to eta-quotients that are weight 2 cusp forms. Whereas Dandurand and Penniston in [5] use the theory of complex multiplication to prove their divisibility results, our approach is a bit more elementary. While each of these cusp forms can be expressed as a sum of eigenforms, it turns out that each of these eigenforms is congruent to a sum of eta-quotients. These eta-quotients indicate infinite families of Hecke operators which annihilate the eigenforms modulo 5.

2. Connection to modular forms

For any complex number z in the upper-half plane, we let $q := e^{2\pi i z}$. For integers $\kappa \geq 0$, $N \geq 1$, and χ a Dirichlet character modulo N, we let $S_\kappa(\Gamma_0(N), \chi)$ denote the space of weight κ holomorphic cusp forms on $\Gamma_0(N)$ with Nebentypus χ. Let p be prime and $g(z) := \sum c(n)q^n \in S_\kappa(\Gamma_0(N), \chi)$. We define the p-th weight κ Hecke operator by

$$g(z) \mid T_{\kappa,p,\chi} := \sum_{n=1}^{\infty} (c(pn) + p^{\kappa-1}\chi(p)c(n/p))q^n\,,$$

where $c(n/p) := 0$ if $p \nmid n$. We note that $T_{\kappa,p,\chi}$ is an endomorphism of $S_\kappa(\Gamma_0(N), \chi)$.

We require Dedekind's eta-function, defined as

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)\,. \tag{2.1}$$

Fix an integer $k > 1$ which is not a power of 2 and ℓ_k, an odd prime which divides k. Let $\alpha(k) := \dfrac{24}{\gcd(k-1, 24)}$. Define

$$F_k(z) := \frac{\eta\left(\frac{k\alpha(k)z}{\ell_k}\right)^{\ell_k}}{\eta(\alpha(k)z)}\,.$$

Let $\beta(k) := \dfrac{k-1}{\gcd(k-1,24)}$. Using (1.1), (2.1), and Fermat's little theorem, it is a simple exercise to show $F_k(z) \in \mathbb{Z}[[q]]$ and

$$\sum_{n=0}^{\infty} b_k(n) q^{\alpha(k)n+\beta(k)} \equiv F_k(z) \pmod{\ell_k}, \tag{2.2}$$

where we say $\sum c(n)q^n \equiv \sum d(n)q^n \pmod{\ell}$ if $c(n) \equiv d(n) \pmod{\ell}$ for all n.

In the following two lemmas we show how $F_k(z)$ being annihilated by a Hecke operator modulo ℓ_k translates into congruences for $b_k(n)$. In Lemma 2.1, we obtain an important relation that arises straight from the definition of the Hecke operator.

Lemma 2.1. *Let $p \geq 5$ be a prime such that $F_k(z)|T_{\kappa,p,\chi} \equiv 0 \pmod{\ell_k}$, and let $1 \leq \gamma \leq \alpha(k)$ satisfy $p\gamma \equiv \beta(k) \pmod{\alpha(k)}$. Then for any non-negative integer j we have*

$$b_k\left(pj + \frac{\gamma p - \beta(k)}{\alpha(k)}\right) \equiv -p^{\kappa-1}\chi(p) b_k\left(\frac{j - \frac{\beta(k)p-\gamma}{\alpha(k)}}{p}\right) \pmod{\ell_k}.$$

Remark. Since $\alpha(k)|24$, we have $(\beta(k)p - \gamma)/\alpha(k) \in \mathbb{Z}$.

Proof. Let $F_k(z) := \sum_{n=1}^{\infty} c(n)q^n$. Using (2.2) we see

$$b_k\left(\frac{n-\beta(k)}{\alpha(k)}\right) \equiv c(n) \pmod{\ell_k}. \tag{2.3}$$

By our assumption, $F_k(z)|T_{\kappa,p,\chi} \equiv 0 \pmod{\ell_k}$, the definition of the Hecke operator implies for all n that

$$c(pn) \equiv \left(-p^{\kappa-1}\chi(p)\right) c\left(\frac{n}{p}\right) \pmod{\ell_k}.$$

Note that if $\frac{n}{p} \notin \mathbb{Z}$ then the righthand side above is 0.

Using (2.3) this means that:

$$b_k\left(\frac{pn-\beta(k)}{\alpha(k)}\right) \equiv \left(-p^{\kappa-1}\chi(p)\right) b_k\left(\frac{\frac{n}{p}-\beta(k)}{\alpha(k)}\right) \pmod{\ell_k}. \tag{2.4}$$

This congruence contains non-trivial information only if on at least one side of the congruence an integer is being plugged into $b_k(n)$. Thus we identify for which n we have $\frac{pn-\beta(k)}{\alpha(k)} \in \mathbb{Z}$. By definition of γ, we have that $\frac{pn-\beta(k)}{\alpha(k)} \in \mathbb{Z}$ if and only if $n \equiv \gamma \pmod{\alpha(k)}$. Thus for any nonnegative integer j, we write $n = \alpha(k)j + \gamma$ and use (2.4) to see

$$b_k\left(\frac{p\left(\alpha(k)j+\gamma\right)-\beta(k)}{\alpha(k)}\right) \equiv \left(-p^{\kappa-1}\chi(p)\right) b_k\left(\frac{\frac{(\alpha(k)j+\gamma)}{p}-\beta(k)}{\alpha(k)}\right) \pmod{\ell_k},$$

which simplifies to

$$b_k\left(pj+\frac{\gamma p-\beta(k)}{\alpha(k)}\right) \equiv \left(-p^{\kappa-1}\chi(p)\right) b_k\left(\frac{j-\frac{\beta(k)p-\gamma}{\alpha(k)}}{p}\right) \pmod{\ell_k}. \quad \square$$

The next lemma shows how Lemma 2.1 translates into $p-1$ separate congruences modulo p^{2n} such that $b_k(n)$ vanishes modulo ℓ_k.

Lemma 2.2. *Let $p \geq 5$ be a prime such that $F_k(z)|T_{\kappa,p,\chi} \equiv 0 \pmod{\ell_k}$, and let t be a non-negative integer. Then the following statements are true.*

(1) *For all $n \geq 1$*

$$b_k\left(p^{2n}t+\beta(k)\left(\frac{p^{2n}-1}{\alpha(k)}\right)\right) \equiv \left(-p^{\kappa-1}\chi(p)\right)^n b_k(t) \pmod{\ell_k}.$$

(2) *If $\lambda \geq \dfrac{\gamma-\beta(k)p}{\alpha(k)}$ is an integer with $p \nmid \lambda$, and $n \geq 1$*

$$b_k\left(p^{2n}t+p^{2n-1}\lambda+\beta(k)\left(\frac{p^{2n}-1}{\alpha(k)}\right)\right) \equiv 0 \pmod{\ell_k}.$$

Remark. Statement (2) actually holds for all integers λ such that $p \nmid \lambda$. If λ is sufficiently negative such that $p^{2n}t + p^{2n-1}\lambda + \beta(k)(p^{2n}-1)/\alpha(k) < 0$, then the congruence trivially holds since there are no partitions of a negative number. On the other hand, if $p^{2n}t+p^{2n-1}\lambda+\beta(k)(p^{2n}-1)/\alpha(k) > 0$, then we could choose $t' \geq 0$ and $\lambda' \geq \dfrac{\gamma-\beta(k)p}{\alpha(k)}$ such that $p^{2n}t+p^{2n-1}\lambda = p^{2n}t'+p^{2n-1}\lambda'$.

Proof of Lemma 2.2. We are going to prove this using induction on n. Suppose that $n=1$. For any integer λ satisfying $\lambda \geq \dfrac{\gamma-\beta(k)p}{\alpha(k)}$, it follows that $pt+\lambda+\dfrac{\beta(k)p-\gamma}{\alpha(k)}$ is a non-negative integer. We use

Lemma 2.1 replacing $pt+\lambda+\frac{\beta(k)p-\gamma}{\alpha(k)}$ in for j to see

$$b_k\left(p\left(pt+\frac{\beta(k)p-\gamma}{\alpha(k)}+\lambda\right)+\frac{\gamma p-\beta(k)}{\alpha(k)}\right)\equiv$$
$$\left(-p^{\kappa-1}\chi(p)\right)b_k\left(\frac{\left(pt+\frac{\beta(k)p-\gamma}{\alpha(k)}+\lambda\right)-\frac{\beta(k)p-\gamma}{\alpha(k)}}{p}\right)\pmod{\ell_k}.$$

Simplifying both sides, we get
(2.5)
$$b_k\left(p^2t+p\lambda+\beta(k)\left(\frac{p^2-1}{\alpha(k)}\right)\right)\equiv\left(-p^{\kappa-1}\chi(p)\right)b_k\left(t+\frac{\lambda}{p}\right)\pmod{\ell_k}.$$

We note that if $\lambda=0$ then (2.5) is equivalent to statement (1) for $n=1$ of the lemma. If $p\nmid\lambda$ then the righthand side of (2.5) is 0 by definition, which establishes statement (2) for $n=1$ of the lemma.

Next we prove the induction step. Assume the statements of the lemma are true for all $1\le m\le n$. For any integer $\lambda\ge\dfrac{\gamma-\beta(k)p}{\alpha(k)}$, we note that

$$\begin{aligned}p^{2n+2}t+&p^{2n+1}\lambda+\beta(k)\left(\frac{p^{2n+2}-1}{\alpha(k)}\right)\\&=p^{2n+2}t+p^{2n+1}\lambda+\beta(k)\left(\frac{p^{2n+2}-p^2}{\alpha(k)}+\frac{p^2-1}{\alpha(k)}\right)\\&=p^2\left(p^{2n}t+p^{2n-1}\lambda+\beta(k)\left(\frac{p^{2n}-1}{\alpha(k)}\right)\right)+\beta(k)\left(\frac{p^2-1}{\alpha(k)}\right).\end{aligned}$$

By our assumption with $m=1$ and replacing t with $p^{2n}t+p^{2n-1}\lambda+\beta(k)(p^{2n}-1)/\alpha(k)$, we have

$$\begin{aligned}&b_k\left(p^2\left(p^{2n}t+p^{2n-1}\lambda+\beta(k)\left(\frac{p^{2n}-1}{\alpha(k)}\right)\right)+\beta(k)\left(\frac{p^2-1}{\alpha(k)}\right)\right)\\&\equiv\left(-p^{\kappa-1}\chi(p)\right)b_k\left(p^{2n}t+p^{2n-1}\lambda+\beta(k)\left(\frac{p^{2n}-1}{\alpha(k)}\right)\right)\pmod{\ell_k}.\end{aligned}$$

If $p\nmid\lambda$ by the induction hypothesis with $m=n$ the righthand side is congruent to 0, which establishes the statement (2) of the lemma. On the other hand, if $\lambda=0$ by the induction hypothesis with $m=n$, we have

$$\begin{aligned}b_k\left(p^{2n+2}t+\beta(k)\left(\frac{p^{2n+2}-1}{\alpha(k)}\right)\right)\\&\equiv\left(-p^{\kappa-1}\chi(p)\right)b_k\left(p^{2n}t+\beta(k)\left(\frac{p^{2n}-1}{\alpha(k)}\right)\right)\\&\equiv\left(-p^{\kappa-1}\chi(p)\right)^{n+1}b_k(t)\pmod{\ell_k},\end{aligned}$$

which establishes statement (1) of the lemma. $\square$

We emphasize that Lemma 2.2 applies in great generality. For example let $k=\ell_k=13$. We see that $F_{13}(z)=\eta(2z)^{12}\in S_6(\Gamma_0(144),\chi_0)$, where χ_0 denotes the trivial character. Since $F_{13}(z)|T_{6,151,\chi_0}\equiv 0 \pmod{13}$, we have for all $n\geq 1$, $t\geq 0$, and $\lambda\geq -75$ with $151\nmid\lambda$ that

$$b_{13}\left(151^{2n}t+151^{2n-1}\lambda+\frac{151^{2n}-1}{2}\right)\equiv 0\pmod{13}.$$

3. Congruences between eigenforms and eta-quotients

In this section we indentify infinite families of Hecke operators which annihilate $F_k(z)$ mod 5 for $k\in\{10,15,20\}$. Applying Lemma 2.2 to these families will complete the proof of Theorem 1.1.

For non-zero integers d, let χ_d denote the Kronecker character $\chi_d(n)=\left(\frac{d}{n}\right)$. Theorems of Gordon, Hughes, Newman, and Ligozat (see, for example, [11]) show that

$$\begin{aligned}F_{10}(z)&=\frac{\eta(16z)^5}{\eta(8z)}\in S_2(\Gamma_0(128),\chi_2),\\F_{15}(z)&=\frac{\eta(36z)^5}{\eta(12z)}\in S_2(\Gamma_0(432),\chi_3),\\F_{20}(z)&=\frac{\eta(96z)^5}{\eta(24z)}\in S_2(\Gamma_0(2304),\chi_0).\end{aligned}$$

We also require the following powerful theorem of Sturm.

Theorem 3.1 (Sturm, [11]). *Suppose that $\mathcal{O}_K$ is the ring of integers for a number field K, $N\geq 1$, and $f(z)=\sum c(n)q^n$, $g(z)=\sum d(n)q^n$ such that $f(z),g(z)\in S_k(\Gamma_0(N),\chi)\cap\mathcal{O}_K[[q]]$. Let $\mathcal{I}\subseteq\mathcal{O}_K$ be an ideal. If for all $n\leq\frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(N)]$ we have*

$$c(n)\equiv d(n)\pmod{\mathcal{I}},$$

then $f(z)\equiv g(z)\pmod{\mathcal{I}}$.

The value $\frac{k}{12}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$ is often called the Sturm bound for $S_k(\Gamma_0(N), \chi)$. We note that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{\ell \text{ prime}, \ell | N} \left(1 + \frac{1}{\ell}\right).$$

We say that $h(z) \in S_k(\Gamma_0(N), \chi)$ is a newform if for each prime p there exists some μ_p such that $h(z) \mid T_{k,p,\chi} = \mu_p h(z)$, in other words $h(z)$ is a Hecke eigenform, and $h(z) \notin S_k(\Gamma_0(N'), \chi)$ for any $N' < N$. In $S_2(\Gamma_0(128), \chi_2)$, there exist unique newforms $h_{10}(z) = \sum_{n=1}^{\infty} \lambda(n) q^n$ and $\overline{h_{10}(z)} = \sum_{n=0}^{\infty} \overline{\lambda(n)} q^n$ with q-series

$$h_{10}(z) = q + 2\sqrt{-2}\, q^3 + O(q^9), \quad \overline{h_{10}(z)} = q - 2\sqrt{-2}\, q^3 + O(q^9).$$

Using Theorem 3.1, we verify that

$$h_{10}(z) \equiv \frac{\eta\,(8z)^5}{\eta\,(16z)} + 2\sqrt{-2}\, F_{10}(z) \pmod 5 \tag{3.1}$$

and

$$\overline{h_{10}(z)} \equiv \frac{\eta\,(8z)^5}{\eta\,(16z)} - 2\sqrt{-2}\, F_{10}(z) \pmod 5. \tag{3.2}$$

Note that $F_{10}(z)$ is only supported on powers of q that are 3 mod 8 and $\dfrac{\eta\,(8z)^5}{\eta\,(16z)}$ is only supported on powers of q that are 1 mod 8. This implies that the coefficients for $h_{10}(z)$ and $\overline{h_{10}(z)}$ are 0 mod 5 whenever the power of q is not 1 or 3 mod 8. Since $h_{10}(z)$ and $\overline{h_{10}(z)}$ are Hecke eigenforms, we have for all primes p which are 5 or 7 mod 8 that $h_{10}(z)|T_{2,p,\chi_2} \equiv \overline{h_{10}(z)}|T_{2,p,\chi_2} \equiv 0$ mod 5.

Using (3.1) and (3.2) we have

$$F_{10}(z) \equiv 2\sqrt{-2}\, \left(\overline{h_{10}(z)} - h_{10}(z)\right) \pmod 5.$$

This immediately implies that for any prime p such that $h_{10}(z)|T_{2,p,\chi_2} \equiv 0$ mod 5 that $F_{10}(z)|T_{2,p,\chi_2} \equiv 0$ mod 5. In fact, (3.1) and (3.2) imply even more. For any prime p, we have

$$F_{10}(z)|T_{2,p,\chi_2} \equiv \begin{cases} \lambda(p) F_{10}(z) & \pmod 5 \quad \text{if } p \equiv 1 \pmod 8, \\ \lambda(p)\sqrt{-2}\dfrac{\eta(8z)^5}{\eta(16z)} & \pmod 5 \quad \text{if } p \equiv 3 \pmod 8, \\ 0 & \pmod 5 \quad \text{if } p \equiv 5, 7 \pmod 8. \end{cases}$$

The situation for $F_{15}(z)$ is very similar. In $S_2(\Gamma_0(432),\chi_3)$, there exist unique newforms $h_{15}(z)$ and $\overline{h_{15}(z)}$ with q-series

$$h_{15}(z) = q + \sqrt{-27}q^7 + O(q^{12}), \quad \overline{h_{15}(z)} = q - \sqrt{-27}q^7 + O(q^{12}).$$

By Theorem 3.1 we see that

$$\begin{aligned} h_{15}(z) &\equiv \frac{\eta(12z)^5}{\eta(36z)} + \sqrt{-27}\,F_{15}(z) \pmod 5 \text{ and} \\ \overline{h_{15}(z)} &\equiv \frac{\eta(12z)^5}{\eta(36z)} - \sqrt{-27}\,F_{15}(z) \pmod 5. \end{aligned}$$

Since $F_{15}(z)$ and $\eta(12z)^5/\eta(36z)$ are both only supported on powers of q which are 1 mod 6, similar arguments show $F_{15}(z)|T_{2,p,\chi_3} \equiv 0$ mod 5 whenever p is a prime which is 5 mod 6.

In $S_2(\Gamma_0(2304),\chi_0)$, there exist unique newforms with the following q-series

$$\begin{aligned} h_{20}(z) =& q - \sqrt{12}q^7 + 2\sqrt{12}q^{13} - 8q^{19} + O(q^{25}), \\ h'_{20}(z) =& q + \sqrt{12}q^7 - 2\sqrt{12}q^{13} - 8q^{19} + O(q^{25}), \\ g_{20}(z) =& q + \sqrt{12}q^7 + 2\sqrt{12}q^{13} + 8q^{19} + O(q^{25}), \text{ and} \\ g'_{20}(z) =& q - \sqrt{12}q^7 - 2\sqrt{12}q^{13} + 8q^{19} + O(q^{25}). \end{aligned}$$

We note that $g_{20}(z)$ and $g'_{20}(z)$ are twists of $h_{20}(z)$ and $h'_{20}(z)$, respectively, by the character χ_{-1}. We find that each of these newforms can be written modulo 5 as a linear combination of $F_{20}(z)$, $\eta(24z)^5/\eta(96z)$, $\eta(24z)^3\eta(96z)$, and $\eta(24z)\eta(96z)^3$. Since none of these eta-quotients are supported on powers of q which are 5 mod 6, we conclude that $F_{20}(z)|T_{2,p,\chi_0} \equiv 0 \pmod 5$ whenever p is a prime which is 5 mod 6.

We conclude by remarking that all of the results in this section could be proved using the theory of complex multiplication. For example, the newforms $h_{15}(z)$ and $\overline{h_{15}(z)}$ have complex multiplication by χ_{-3}, which implies that their q-expansions are only supported on powers congruent to 1 mod 6. We direct the reader interested in more information on this topic to [12].

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Department of Mathematics, Wake Forest University
E-mail address: carlrd11@wfu.edu

Department of Mathematics, Wake Forest University
E-mail address: webbjj@wfu.edu