The time value of money: Part I
A reading prepared by Pamela Peterson Drake

OUTLINE

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1. Introduction

A. The basics

Evaluating financial transactions requires valuing uncertain future cash flows; that is, determining what uncertain cash flows are worth at different points in time. We are often concerned about what a future cash flow or a set of future cash flows are worth today, though there are applications in which we are concerned about the value of a cash flow at a future point in time. One complication is that there is a time value of money: a dollar today is not worth a dollar tomorrow or next year. Another complication is that any amount of money promised in the future is uncertain, some riskier than others.

The purpose of this reading is to develop the concepts and mathematical tools that will enable you to move money through time, both backwards and forwards. Once we master these tools, we'll apply them to more realistic transactions.

Moving money through time - that is, finding the equivalent value to money at different points in time - involves translating values from one period to another. Translating a value to the present is referred to as **discounting**. Translating a value to the future is referred to as **compounding**. Translating money from one period involves **interest**, which is how the time value of money and risk enter into process.

The **principal** is the amount borrowed. **Interest** is the compensation for the opportunity cost of funds and the uncertainty of repayment of the amount borrowed; that is, it represents both the price of time and the price of risk. The **price of time** is compensation for the opportunity cost of funds and the **price of risk** is compensation for bearing risk.

Interest is **compound interest** if interest is paid on both the principal and any accumulated interest. Most financial transactions involve compound interest, though there are a few consumer transactions that use **simple interest** (that is, interest paid only on the principal or amount borrowed).

The future value is the sum of the present value and interest:

\[ \text{Future value} = \text{Present value} + \text{interest} \]

The **basic valuation equation** that is the foundation of all the financial mathematics is:
FV = PV \((1 + i)^n\)

Where FV is the future value, PV is the present value, i is the rate of interest, and n is the number of compounding periods. The term \([(1 + i)^n]\) is the **compound factor**.

We can rearrange the basic valuation equation to solve for the present value, PV:

\[
PV = FV \left(\frac{1}{(1 + i)^n}\right) = \frac{FV}{(1 + i)^n},
\]

where \(1/(1+i)^t\) is the **discount factor**.

2. **Compounding**

We begin with compounding because this is the most straightforward way of demonstrating the effects of interest. Consider the following example: Suppose you invest $1,000 in an account that pays 6 percent interest, compounded annually. How much will you have in the account at the end of 1 year if you make no withdrawals? After 2 years? After 5 years? After 10 years? Using the subscript to indicate the year,

- \(FV_1 = \$1,000 \times (1 + 0.06) = \$1,060\)
- \(FV_2 = \$1,000 \times (1 + 0.06 \times 1 + 0.06) = \$1,000 \times (1 + 0.06)^2 = \$1,000 \times (1.1236) = \$1,123.60\)
- \(FV_5 = \$1,000 \times (1 + 0.06)^5 = \$1,000 \times (1.3382) = \$1,338.23\)
- \(FV_{10} = \$1,000 \times (1 + 0.06)^{10} = \$1,000 \times (1.7908) = \$1,790.85\)

What if interest was not compounded interest (that is, if interest is *simple* interest)? Then we would have a lower balance in the account:

- \(FV_1 = \$1,000 + [\$1,000 \times (0.06)] = \$1,060\)
- \(FV_2 = \$1,000 + [\$1,000 \times (0.06)] + [\$1,000 \times (0.06) \times 1] = \$1,000 + [\$1,000 \times (0.06)(2)] = \$1,120\)
- \(FV_5 = \$1,000 + [\$1,000 \times (0.06) \times 5] = \$1,300\)
- \(FV_{10} = \$1,000 + [\$1,000 \times (0.06) \times 10] = \$1,600\)

You can see the difference between compounded and simple interest in Exhibit 1, which shows the growth of $1,000 at 6 percent using both types of interest.

The difference between the future value with compounded interest and that with simple interest is the interest-on-interest. For example, at the end of 10 years the interest on interest is $1,790.85 - $1,600.00 = $190.85.

Most financial transactions involve compound interest. If the method of calculating interest is not stated, you should assume that the interest is compound interest.

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### Example 1: Compound Interest

**Problem**
Suppose that the Fosters invest $100,000 today in an investment that produces a return of 5 percent per year. What will their investment be worth in two years?

**Solution**
\[
\begin{align*}
FV_2 &= $100,000 (1 + 0.05)^2 \\
FV_2 &= $100,000 (1.1025) \\
FV_2 &= $110,250
\end{align*}
\]

### Example 2: Compound Interest vs. Simple Interest

**Problem**
Suppose you are faced with a choice between two accounts, Account A and Account B. Account A provides 5 percent interest, compounded annually and Account B provides 5.25 percent simple interest. Consider a deposit of $10,000 today. Which account provides the highest balance at the end of 4 years?

**Solution**

**Account A**
\[
FV_A = $10,000 (1 + 0.05)^4 = $12,155.06
\]

**Account B**
\[
FV_B = $10,000 + ($10,000 (0.0525)(4)) = $12,100.00
\]

---

A. Calculator and spreadsheet solutions

The calculations can be made easier with the help of either a financial calculator or a spreadsheet program. The calculator financial functions assume compound interest; if you wish to perform a calculation with simple interest, you must rely on the mathematical programs of your calculator.

Using a calculator, for example, we calculate the future value of $1,000 invested 10 years at 6 percent with the following key strokes:

<table>
<thead>
<tr>
<th>TI 83/84</th>
<th>HP10B</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Using TVM Solver</strong></td>
<td></td>
</tr>
<tr>
<td>N = 10</td>
<td>1000 +/- PV</td>
</tr>
<tr>
<td>I% = 6</td>
<td>10 N</td>
</tr>
<tr>
<td>PV = -1000</td>
<td>6 I/YR</td>
</tr>
<tr>
<td>PMT = 0</td>
<td></td>
</tr>
<tr>
<td>FV = Solve</td>
<td></td>
</tr>
</tbody>
</table>

PV is the present value, N is the number of compound periods, I percent or I/YR is the interest rate per period, and FV is the future value.

You will notice that we changed the sign on the PV. This is because of the way the financial function is programmed, assuming that the present value is the outflow. The changing of the sign for the present value is required in most (but not all) financial calculators.

If we want to use the math functions, instead of the financial program of a calculator, you would need to use a power key, such as $y^x$ or $^\wedge$ and input the interest in decimal form:

\[
FV = \frac{PV(1 + r)^n}{(1 + r)^n}
\]

---

1 In the TI83/84, The TVM Solver is found in APPS or FINANCE, depending on the model. “Solve” is executed on the TI83/84 using the ALPHA and then SOLVE keys.
In Microsoft Excel®, the calculation uses the worksheet function FV:

\[
=\text{FV}(\text{rate}, \text{nper}, \text{pmt}, \text{pv}, \text{type})
\]

Where “type” is 0 (indicating cash flows and values occur at the end of the period). Using notation similar to that found on calculators, this command becomes:

\[
=\text{FV}(i, N, \text{PMT}, \text{PV}, 0)
\]

Because there are no other cash flows in this problem, PMT (which represents periodic cash flows, such as a mortgage payment) is zero. To calculate the FV, the function requires the following inputs:

\[
=\text{FV}(0.06, 10, 0, -1000, 0)
\]

Note that in the financial functions of your calculator, the interest rate is represented as a whole number (that is, 6 for 6 percent), whereas in the math functions of your calculator and in spreadsheet functions, the interest rate is input in decimal form (that is, .06 for 6 percent).

**Why can’t I calculate the future value with simple interest using my calculator functions?**

Calculators’ time value of money programs are set up to perform calculations involving compound interest. If you want to calculate the future value using simple interest, you must resort to old fashion mathematics:

Simple interest = Principal amount \times interest rate per period \times number of periods

Or

Simple interest = \text{PV} \times i \times N

The future value of a lump-sum if interest is computed using simple interest is therefore:

\[
\text{FV}_{\text{simple}} = \text{PV} + \text{PV} \times i \times N = \text{PV} (1 + iN)
\]

If the present value is $1,000 and interest is simple interest at 5 percent per year, the future value after four periods is:

\[
\text{FV}_{\text{simple}} = $1,000 + $1,000 \times (0.05) \times (4) = $1,000 (1 + 0.2) = $1,200
\]

The interest paid on interest in compounding is the difference between the future values with compound and simple interest.

**B. Frequency of compounding**

If interest is compounded more frequently than once per year, you need to consider this in any valuation problem involving compounded interest. Consider the following scenario:

$1,000 is deposited in account at the beginning of the period and 12% interest paid on the account, with interest compounded quarterly. This means that at the end of the first quarter, the account has a balance of:

\[
\text{FV}_{1\text{st quarter}} = $1,000 \times (1 + 0.12/4) = $1,000 \times (1 + 0.03) = $1,030
\]

The following quarters’ balances are calculated in a like manner, with interest paid on the balance in the account:

- \[
\text{FV}_{2\text{nd quarter}} = $1,030.00 \times (1 + 0.03) = $1,060.90
\]
- \[
\text{FV}_{3\text{rd quarter}} = $1,060.90 \times (1 + 0.03) = $1,092.73
\]
- \[
\text{FV}_{4\text{th quarter}} = $1,092.73 \times (1 + 0.03) = $1,125.51
\]
Therefore, at the end of one year, there is a balance of $1,000 \times (1 + 0.03)^4 = $1,125.51.

When you face a situation in which interest is compounded more frequently than on an annual balance, you need to adjust both the number of periods and the interest rate accordingly.

If interest is compounded continuously (that is, instantaneously), the compound factor uses the exponential function, $e$, the inverse of the natural logarithm. The compound factor is:

$$e^{\text{annual interest rate} \times \text{number of years}} = e^{\text{APR} \times \text{years}}$$

If annual interest is 10 percent, continuously compounded, the compound factor for one year is:

$$e^{0.10} = 1.1052$$

For two years, the factor is:

$$e^{(0.10) \times 2} = e^{0.20} = 1.2214$$

For 10 years, the factor is:

$$e^{(0.10) \times 10} = e^{1} = 2.7183$$

Using a calculator, you need to use the math function $e^x$. For example, suppose you want to calculate the future value of $1,000 invested five years at 4 percent, with interest compounded continuously. The future value is:

$$FV_5 = $1,000 \times e^{(0.04 \times 5)} = $1,000 \times e^{0.2} = $1,000 \times 1.2214 = $1,221.40$$

Using Microsoft Excel®, you use the exponential worksheet function, EXP:

$$=1000*EXP(0.04*5)$$

where the value in parentheses is the exponent.

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2 The number of periods is therefore the number of compounding periods, not the number of years.
3. Discounting

Translating a value back in time -- referred to as **discounting** -- requires determining what a future amount or cash flow is worth today. Discounting is used in valuation because we often want to determine the value today of some future value or cash flow (e.g., what a bond is worth today if it promised interest and principal repayment in the future).

The equation for the present value is a rearrangement of the basic valuation equation that we saw earlier:

\[
PV = \frac{FV}{(1 + i)^n},
\]

where:
- \(PV\) is the present value (today's value),
- \(FV\) the future value (a value or cash flow sometime in the future),
- \(i\) the interest rate per period, and
- \(n\) the number of compounding periods.

As you can see, there are many ways to represent solving for the present value.

From the formula for the present value we know that

- as the number of discount periods, \(n\), becomes larger, the discount factor becomes smaller and the present value becomes less, and
- as the interest rate per period, \(i\), increases, the discount factor becomes smaller and the present value becomes less.

### Example 4: Continuous compounding

**Problem**

Suppose you invest $1,000 today in an account that pays 9 percent interest, compounded continuously. What will be the value in this account at the end of ten years?

**Solution:**

\[
FV = 1,000 \times e^{0.09 \times 10} \\
= 1,000 \times e^{0.9} \\
= 1,000 \times (2.4596) = 2,459.60
\]

### Example 5: Continuous versus discrete compounding

**Problem**

Suppose you invest $5,000 in an account that earns 10 percent interest. How much more would you have after 20 years if interest is compounded continuously instead of compounded semi-annually?

**Solution:**

\[
\begin{align*}
FV_{\text{continuously}} &= 5,000 \times e^{0.1 \times 20} \\
&= 5,000 \times (7.3891) = 36,945.28 \\
FV_{\text{semi-annually}} &= 5,000 \times (1 + 0.05)^{40} \\
&= 5,000 \times (7.0400) = 35,199.94 \\
\text{Difference} &= 36,945.28 - 35,199.94 \\
&= 1,745.34
\end{align*}
\]

### Example 6: Discounting

**Problem**

Suppose that you wish to have $20,000 saved by the end of five years. And suppose you deposit funds today in account that pays 4 percent interest, compounded annually. How much must you deposit today to meet your goal?

**Solution**

Given: \(FV = 20,000\); \(n = 5\); \(i = 4\%\)

\[
PV = \frac{20,000}{(1 + 0.04)^{5}} = \frac{20,000}{1.21665} \\
PV = 16,438.54
\]
period, i, becomes larger, the discount factor becomes smaller and the present value becomes less.

Therefore, the present value is influenced by both the interest rate (i.e., the discount rate) and the number of discount periods.

A. Calculator and spreadsheet solutions

If you want to take a value to power (e.g., \((1 + 0.05)^8\)), you generally key in the base (e.g., 1.05) and then use the key marked as ^ or y^x: \((1 + 0.05)^8 = 1.4775\). To invert a value (e.g., \(1/1.4775\)), use the key marked 1/x: \(1/1.4775 = 0.6768\).

Calculate the present value of $6,000 to be received in eight years if the interest rate is 5 percent per year:

\[
PV = \frac{6,000}{(1.05)^8} = 4,061.04
\]

The present value is $4,061.04. You will notice that the calculator displays the present value as a negative number. This has to do with the way the program is written for the calculator; it is written such that \(0 = FV - PV(1+i)^n\).

Using TVM Solver

HP10B

\[
\begin{align*}
N &= 8 \\
I &= 5 \\
FV &= 6000 \\
PMT &= 0 \\
PV &= \text{Solve}
\end{align*}
\]

The present value can also be calculated using Microsoft Excel.

The present value function is:

\[
=PV(rate, nper, pmt, fv, type)
\]

where

- \text{rate} is the interest rate, stated in decimal form;
- \text{nper} is the number of periods;
- \text{pmt} is 0 in this case because we are assuming no other cash flows;
- \text{fv} is the future value; and
- \text{type} reflects the timing (0 or end of period, the usual assumption, and 1 for the beginning of the period).

Suppose we want to calculate the present value of $500,000 to be received in ten years, with an interest rate of 7 percent. The entry in the Excel spreadsheet is:

\[
=PV(.07, 10, 0, 500000, 0)
\]

and because the default assumption is that an amount is received at the end of the period, we can also write this entry as:

\[
=PV(.07, 10, 0, 500000)
\]

The answer is $254.174.65. An Excel spreadsheet demonstrating the use of the PV function for this problem (using several different interest rates), along with a chart of the present values, is available here.

B. Frequency of compounding

Suppose that interest is compounded more frequently than annually. We must therefore adjust both i and n to reflect this more frequent compounding. Consider an example: calculate the present value of $10,000 to be received at the end of five years if the annual interest rate is 6 percent, compounded semi-annually. If the annual rate is 6 percent, the semi-annual rate is 6 percent / 2 = 3
percent. The number of semi-annual periods is 5 years x 2 times per years = 10. Therefore, the
present value of this $10,000 is calculated as:

\[ PV = \$10,000 \left[ \frac{1}{1 + 0.03} \right]^{10} = \$10,000 \times (0.7441) = \$7,440.94 \]

If interest is compounded continuously, the present value is calculated as:

\[ PV = \frac{1}{e^{0.06 \times 5}} = \frac{1}{e^{0.30}} \]

\[ PV = \$10,000 \times (1 / 1.3499) = \$10,000 \times (0.7408) = \$7,408.18 \]

We can see the difference the frequency of compounding makes in Exhibit 2, in which this last example has been expanded to include quarterly and continuous compounding. For a given annual percentage rate, the greater the frequency of compounding, the greater is the effective interest rate and the smaller is the present value. But as you can also see in the Exhibit, the difference may be quite small.
### Example 7: Present value with quarterly compounding

**Problem**
How much would you have to deposit today in an account that pays 4 percent annual interest, compounded quarterly, if you wish to have a balance of $100,000 at the end of ten years?

**Solution**
Given information:  
- \( FV = $100,000 \)
- \( i = 4\% / 1 = 1\% \)
- \( n = 10 \times 4 = 40 \) quarters

\[
PV = $100,000 \times \left( \frac{1}{1 + 0.01} \right)^{40}
\]

\[
PV = $100,000 \times (0.6717) = $67,165.31
\]

### Example 8: Present value with continuous compounding

**Problem**
How much would you have to deposit today in an account that pays 4 percent annual interest, compounded continuously, if you wish to have a balance of $100,000 at the end of ten years?

**Solution**
Given information:  
- \( FV = $100,000 \)
- \( i = 4\% \)
- \( n = 10 \) years

\[
PV = $100,000 \times \left( \frac{1}{e^{0.04 \times 10}} \right)
\]

\[
PV = $100,000 \times (0.67032) = $67,032
\]

### Example 9: Solving for a present value

**Problem**
Which of the following requires the least amount of a deposit today?

a. A balance of $10,000 four years from today that has grown from a sum deposited in an account that pays 8 percent interest, compounded quarterly.

b. A balance of $10,000 five years from today that has grown from a sum deposited in an account that pays 7 percent interest, compounded annually.

c. A balance of $10,000 ten years from today that has grown from a sum deposited in an account that pays 4 percent interest, compounded continuously.

d. A balance of $10,000 eight years from today that has grown from a sum deposited in an account that pays 4 percent interest, compounded semi-annually.

**Solution**

\[
PV_a = \frac{10,000}{(1 + 0.02)^{16}} = $7,284.46
\]

\[
PV_b = \frac{10,000}{(1 + 0.07)^{5}} = $7,129.86
\]

\[
PV_c = \frac{10,000}{e^{0.04 \times 10}} = $6,703.20
\]

\[
PV_d = \frac{10,000}{(1 + 0.02)^{16}} = $7,284.46
\]

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