The basics of determining the equivalence of an amount of money at a different point in time is essential in valuing securities, capital projects, and other transactions. Many financial decisions involve not one, but more than one cash flow that occurs at different points in time. Valuing these multiple cash flows is simply an extension of translating single values through time.

A. Valuing a series of cash flows

Consider the following: You plan to deposit $10,000 in one year, $20,000 in two years, and $30,000 in three years. If the interest earned on your deposits is 10 percent,

1. What will be the balance in the account at the end of the third year?
2. What is the value today of these deposits?

The balance in the account at the end of the third year is calculated as the sum of the future values, or $64,100:

<table>
<thead>
<tr>
<th>Today</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$10,000</td>
<td>$20,000</td>
<td>$30,000</td>
<td></td>
</tr>
<tr>
<td></td>
<td>12,100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>FV</td>
<td></td>
<td>22,000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What is the value of the deposits today? The value of these deposits today is calculated as the sum of the present values, or $48,160.28:
Why would we want to know the future value of a series? Suppose you are setting aside funds for your retirement. What you may want to know is how much you will have available at the time you retire. You’ll have to assume a specific return on your funds – that is, how much interest you can earn on your savings – but you can calculate how much you’ll have at some future point in time.

Why would we want to know the present value of a series? Suppose you are considering investing in a project that will produce cash flows in the future. If you know what you can earn on similar projects, what is this project worth to you today? How much would you be willing to pay for this investment? We can calculate the present value of the future cash flows to determine the value today of these future cash flows.

**Example 1: Value of a series of cash flows**

**Problem**
Suppose you deposit $100 today, $200 one year from today, and $300 two years from today, in an account that pays 4 percent interest, compounded annually.

1. What is the balance in the account at the end of two years?
2. What is the balance in the account at the end of three years?
3. What is the present value of these deposits?

**Solution**

1. What is the balance in the account at the end of two years?
   \[ FV = 100 \times (1.04)^2 + 200 \times (1.04) + 300 = $616.16 \]
2. What is the balance in the account at the end of three years?
   \[ FV = 100 \times (1.04)^3 + 200 \times (1.04)^2 + 300 \times (1.04) = $616.16 \times (1.04) \text{ or } $640.81 \]
3. What is the present value of these deposits?
   \[ PV = 100 + \frac{200}{(1.04)} + \frac{300}{(1.04)^2} = 100 + 192.31 + 277.37 = $569.68 \]

**B. A calculator short-cut**

In cases in which you require a present value of uneven cash flows, you can use a program in your financial calculator, the NPV program. The NPV program requires you to input all cash flow, beginning with the cash flow in the next period, in order, and specifying the interest rate. Using the earlier problem, you can calculate the present value of the uneven series of cash flows as:

---

1 NPV represents net present value - the present value of all future cash flows.

2 If there is no cash flow you must input a “0” to hold the time period’s place in the program – otherwise, the cash flow will receive an incorrect time value of money.
Note that there is no short-cut in most calculators for the future value of an uneven series of cash flows. To calculate the future value of an uneven series of cash flows, you need to calculate the future value of each of the individual cash flows and then sum these future values to arrive at the future value of the series.

C. Annuities

An annuity is a series of even cash flows. Because the cash flows are the same amount, the math is simpler. Suppose you have a series of three cash flows, each of $1,000. The first cash flow occurs one year from today, the second occurs two years from today, and the third occurs three years from today. The present value of this series is:

\[
PV = \frac{1,000}{(1+i)^1} + \frac{1,000}{(1+i)^2} + \frac{1,000}{(1+i)^3} = \sum_{t=1}^{3} \frac{1,000}{(1+i)^t}
\]

Using the notation CF to represent the periodic cash flow, we can represent this as

\[
PV = \sum_{t=1}^{N} \frac{CF}{(1+i)^t} = CF \sum_{t=1}^{N} \frac{1}{(1+i)^t}
\]

The term \( \sum_{t=1}^{N} \frac{1}{(1+i)^t} \) is the annuity discount factor.

There are different types of annuities in financial transactions, which differ in terms of the timing of the first cash flow:

- An ordinary annuity is an annuity in which the first cash flow is one period in the future.
- An annuity due is an annuity in which the first cash flow occurs today.
- A deferred annuity is an annuity in which the first cash flow occurs beyond one period from today.

The example that we just completed is an example of an ordinary annuity. You can see the timing issue when comparing the time lines associated with each. Consider the following 3-cash flow annuities: the ordinary annuity, an annuity due, and a deferred annuity with a deferral of three periods.
CF represents the periodic cash flow amount. In the case of an annuity, this amount is the same each period. Because of the time value of money, the valuation of these annuities, whether we are referring to the present value or the future value, will be different.

i) Valuing an ordinary annuity

The ordinary annuity is the most common annuity that we’ll encounter, though deferred annuities and annuities due do occur with some frequency as well. The future value of an ordinary annuity is simply the sum of the future values of the individual cash flows. Consider a three-payment ordinary annuity that has payments of $1,000 each and a 5 percent interest rate.

The future value of this annuity is:

<table>
<thead>
<tr>
<th>Today</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000.0</td>
</tr>
<tr>
<td>PV</td>
<td>1,102.5</td>
<td>1,050.0</td>
<td></td>
</tr>
<tr>
<td>FV</td>
<td>3,152.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The present value of this annuity is:

<table>
<thead>
<tr>
<th>Today</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>CF</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
</tr>
<tr>
<td>PV</td>
<td>$952.381</td>
<td>907.029</td>
<td>863.838</td>
</tr>
<tr>
<td>PV =  $2,723.248</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We can represent the value of the annuity in more general terms. Let t indicate a time period, CF represent the individual cash flow, and let N indicate the number of cash flows. The future value is the sum of the future values of the cash flows:
The present value of an ordinary annuity can be represented as:

\[
PV = \sum_{t=1}^{N} \frac{CF}{(1+i)^t} = CF \sum_{t=1}^{N} \frac{1}{(1+i)^t} = CF \left(1 - \frac{(1+i)^N}{i}\right)
\]

We used the notation “CF” to indicate a cash flow. In the case of an annuity, this cash flow is the same each period. The term \(\sum_{t=0}^{N-1} (1+i)^t\) is referred to as the **future value annuity factor** and the term \(\sum_{t=1}^{N} \frac{1}{(1+i)^t}\) is referred to as the **present value annuity factor**.

In financial calculator applications, the cash flow associated with an annuity is referred to as a payment, or PMT.

Consider another example. Suppose you wish to calculate the present value of a four-payment ordinary annuity that has annual payments of $5,000 each. If the interest rate is 5 percent, the present value is $17,729.75. Using a calculator, we input the known values (i.e., N, I, PMT) and solve for PV.\(^3\)

Referring to the timeline above, you can see that the value you calculated occurs one period before the first cash flow (i.e., today).

Now suppose you wish to calculate the future value. You use the same inputs, but simply solve for the future value instead of the present value, resulting in a value of $21,550.63. Referring to the timeline above, you can see that the value you calculated occurs at the same time as the last cash flow, which in this example is at the end of the fourth year.

\(^3\) Be sure that your calculator is set for one payment per period and in the END mode.
Example 2: Valuing an annuity

Problem
Consider a four-payment annuity in which the payment is $2,500 and the interest rate is 6 percent.
1. What is the present value of this annuity?
2. What is the future value of this annuity?

Solution
1. PV = $8,662.76
2. FV = $10,936.54

Note:
FV of ordinary annuity = (1 + i)^N
PV of ordinary annuity
$10,936.54 = (1 + 0.06)^4
$8,662.76

ii) Valuing an annuity due

An annuity due is like an ordinary annuity, yet the first cash flow occurs immediately, instead of one period from today. This means that each cash flow is discounted one period less than each cash flow in a similar payment ordinary annuity.

\[ FV = CF(1 + i)^1 + CF(1 + i)^2 + \ldots + CF(1 + i)^{N+1} = CF \sum_{t=0}^{N} (1+i)^{t+1} \]

\[ PV = \sum_{t=1}^{N} \frac{CF}{(1+i)^{t-1}} = \sum_{t=1}^{N} \frac{1}{(1+i)^{t-1}} \]

Consider the example of a three-payment annuity due in which the payments are $1,000 each and the interest rate is 5 percent. The future value of this annuity due is $3,310.125:

<table>
<thead>
<tr>
<th>Today</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$1,157.625</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$1,102.500</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$1,050.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>FV = $3,310.125</td>
</tr>
</tbody>
</table>

The present value of this three-payment annuity due is:

<table>
<thead>
<tr>
<th>Today</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1,000,000</td>
<td>$1,000</td>
<td>$1,000</td>
<td>$952.381</td>
</tr>
<tr>
<td>$952.381</td>
<td>907.029</td>
<td></td>
<td></td>
</tr>
<tr>
<td>PV = $2,859.410</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Comparing the values of the ordinary annuity with those of the annuity due, you’ll see that the values differ by a factor of (1+i):

The time value of money: Part II, A reading prepared by Pamela Peterson Drake
This factor represents the difference in the timing of the cash flows: the cash flows of the annuity due occur one period prior to the cash flows for a similar-payment ordinary annuity.

Using a financial calculator to value an annuity due requires changing the mode from END to BEG or BEGIN. Once in the BEG or BEGIN mode, you can input the values as you did with the ordinary annuity. 4

Consider a five-payment annuity due with an annual payment of $3,000 and an interest rate of 6 percent. The present value of this annuity due is $13,395.317.

<table>
<thead>
<tr>
<th></th>
<th>Value of ordinary annuity</th>
<th>Value of annuity due</th>
<th>value of annuity due</th>
<th>value of ordinary annuity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present value</td>
<td>$2,723.248</td>
<td>$2,859.410</td>
<td>1.05</td>
<td></td>
</tr>
<tr>
<td>Future value</td>
<td>$3,152.500</td>
<td>$3,310.125</td>
<td>1.05</td>
<td></td>
</tr>
</tbody>
</table>

Using a financial calculator to value an annuity due requires changing the mode from END to BEG or BEGIN. Once in the BEG or BEGIN mode, you can input the values as you did with the ordinary annuity. 4

Consider a five-payment annuity due with an annual payment of $3,000 and an interest rate of 6 percent. The present value of this annuity due is $13,395.317.

<table>
<thead>
<tr>
<th>TI 83/84 Using TVM Solver</th>
<th>HP10B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set BEGIN N = 5 I% = 6 FV = 0 PMT = 3000 PV = Solve</td>
<td>Set BEG N = 5 I% = 6 PV = 0 PMT = 3000 FV = Solve</td>
</tr>
</tbody>
</table>

The future value of this annuity due is $17,925.956.

4 Caution: A common mistake is to leave the calculator in the annuity due mode when calculating other, non-due problems.

5 Therefore, this is an annuity due pattern of cash flows. It would be lousy public relations for a lottery commission to say, “Congratulations, you’ll get your first check in one year.”, so most lotteries begin payments immediately.
iii) Valuing a deferred annuity

A deferred annuity is an annuity in which the first cash flow occurs beyond the end of the first period. The key to solving for the present value is to break down the analysis into manageable steps. For example, if you are solving for a present value of a deferred annuity, the two steps are:

1. Solve for present value of ordinary annuity
2. Discount this present value to today

Consider a deferred annuity that consists of three payments of $1,000 and an interest rate of 5 percent. If the annuity is deferred three periods, the first cash flow occurs at the end of the third period, the second cash flow occurs at the end of the fourth period, and so on. A simple way of solving this problem is to first calculate the present value at the end of the second period, treating this as an ordinary annuity, and then discounting this present value ($\text{PV}_2$) two periods to the today ($\text{PV}_0$), producing a present value of the deferred annuity of $2,470.066:

**Step 1: Calculate the present of a three-payment ordinary annuity**

<table>
<thead>
<tr>
<th>Today</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
</tr>
<tr>
<td>$952.381</td>
<td>907.029</td>
<td>863.838</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{PV}_2 = 2,723.248$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TI 83/84 Using TVM Solver**
- **N** = 3
- **I%** = 5
- **FV** = 0
- **PMT** = 1000
- **PV** = Solve

**HP10B**
- **3 N**
- **5 I/YR**
- **1000 PMT**
- **PV**

**Step 2: Calculate the present value of a lump sum amount**

<table>
<thead>
<tr>
<th>Today</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$1,000</td>
<td>$1,000</td>
<td>$1,000</td>
</tr>
<tr>
<td>$952.381</td>
<td>907.029</td>
<td>863.838</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{PV}_2 = 2,723.248$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$\text{PV}_0 = 2,470.066$
There is no built-in function for deferred annuities on your financial calculator, so when faced with a deferred annuity you need to break it down into steps and execute each step using your calculator's functions.

There are many types of deferred annuities that you may encounter in financial management. Solving these depends on what information is given. Like all time value of money problems, there is one unknown element that we want to solve for. In an deferred annuity, this unknown may be the amount of savings that lead up to withdrawals, the amount of withdrawals given a savings program, the number of savings deposits to make to satisfy planned withdrawals, and the number of withdrawals possible given a savings program. No matter the problem, it can usually be broken into two or three manageable pieces.

### Example 3: Present value of a deferred annuity

**Problem**

Suppose that you wish to have a balance in your savings account when you retire at 65 years of age such that you can make withdrawals of $10,000 each year for 20 years, starting with your 66th birthday. How much must you deposit on your 35th birthday in an account paying 5 percent interest, compounded annually, so that you can meet your goal?

**Solution**

**Step 1: Solve for the present value of the withdrawals.**

Given information: CF = $10,000; i = 5%; N = 20

\[
PV = \frac{10,000}{(1 + 0.05)^{20}} = 124,622
\]

This is the balance required at the time of your 65th birthday (that is, at the end of 30 periods). Why 65th and not 66th? Because we used an ordinary annuity approach to solving this, which means that the PV of the series occurs one period prior to the first cash flow -- in other words, by using the ordinary annuity "short-cut" the PV is on your 65th birthday, not on your 66th birthday.

Note: from 35 to 36 is one period, 35 to 37 is two periods, ... , 35 to 65 is 30 periods).

**Step 2: Solve for the present value of your 65th birthday balance**

Given: FV = $124,622; N = 30; i = 5%

\[
PV = \frac{124,622}{(1 + 0.05)^{30}} = 28,834.72
\]

Check it out:

\[
PV = 28,834.72 \times (1 + 0.05)^{30} = 28,834.72 \times (4.32194) = 124,622
\]
Example 4: Solving for the amount of the deferred annuity

Problem
Suppose you deposit $1,000 in an account at the end of each year, starting next year, for thirty years (thirty deposits). Your goal is to live off of the savings for twenty years, starting the year after your last deposit.

If you can earn 6 percent on your deposits and your withdrawals in retirement are an even, annual amount, what is the amount you can withdraw once you retire?

Solution
Step 1: Determine the future value of the deposits

Given:
- PMT = $1,000
- N = 30
- I = 6%

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & \ldots & 30 & 31 & 32 & \ldots & 50 \\
D & D & D & D & \ldots & D & W & W & W & \ldots & W \\
\end{array}
\]

\[FV_{30} \text{ of deposits: } $79,058.129\]

Step 2: Determine the withdrawal

Given:
- PV of withdrawals = $79,058.129
- N = 20
- I = 6%

\[
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & \ldots & 30 & 31 & 32 & \ldots & 50 \\
D & D & D & D & \ldots & D & W & W & W & \ldots & W \\
\end{array}
\]

\[PV_{30} \text{ of withdrawals} \]

Solve for PMT

\[PMT = $6,892.65\]

D. Loan amortization

Earlier, you learned how to value an annuity. If an amount is loaned and then repaid in installments, we say that the loan is amortized. Therefore, loan amortization is the process of calculating the loan payments that amortize the loaned amount. We can determine the amount of the loan payments once we know the frequency of payments, the interest rate, and the number of payments.

Consider a loan of $100,000. If the loan is repaid in twenty-four installments and if the interest rate is 5 percent per year, we calculate the amount of the payments by applying the relationship:

\[PV = CF \times \text{(present value annuity factor for } N=24 \text{ and } i=0.05/12=0.0042)\]

We are given the following:

\[PV = $100,000\]
\[I = 5%/12 = 0.4167\%\]
N = 24

And we want to solve for the payment. The payment, PMT, is $4,378.19.

Therefore, the monthly payments are $4,378.19 each. In other words, if payments of $4,378.19 are made each month for twenty-four months, the $100,000 loan will be repaid and the lender earns a return that is equivalent to a 5 percent APR on this loan.

Using a financial calculator,

<table>
<thead>
<tr>
<th>TI 83/84 Using TVM Solver</th>
<th>HP10B</th>
</tr>
</thead>
<tbody>
<tr>
<td>N = 24</td>
<td>2 N</td>
</tr>
<tr>
<td>I% = .05/12</td>
<td>.05/12 I/YR</td>
</tr>
<tr>
<td>PV = -100000</td>
<td>-100000 PV</td>
</tr>
<tr>
<td>PMT = Solve</td>
<td>PMT</td>
</tr>
</tbody>
</table>

We can also use a spreadsheet to perform this calculation. In Microsoft's Excel, we can solve for the monthly payment using the PMT function:

=PM(T(rate, number of payments, amount of loan, future value, timing indicator)

where the timing indicator is 1 for beginning of the period flows (i.e., annuity due) and 0 for end of period flows (i.e., ordinary annuity).

=PM(.05/12,24,-100000,0,0)

We can calculate the amount of interest and principal repayment associated with each loan payment using a loan amortization chart, as shown in Exhibit 1.
The principle amount of the loan declines as payments are made. The proportion of each loan payment devoted to the repayment of the principle increases throughout the loan period from $3,970.47 for the first payment to $4,368.94 for the last payment. The decline in the loan's principle is shown graphically in Exhibit 2.

You’ll notice that the decline in the remaining principle is not a linear, but is curvilinear due to the compounding of interest.

You can download the Excel spreadsheet that created this table and the corresponding graph here.

---

**Exhibit 1: Loan amortization on a $100,000 loan for twenty-four months and an interest rate of 5 percent per year**

<table>
<thead>
<tr>
<th>Payment</th>
<th>Loan payment</th>
<th>Interest on the loan (= 0.05/12 \times \text{(remaining principle)})</th>
<th>Principal paid off = loan repayment - interest</th>
<th>Remaining principle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Today</td>
<td>$0.00</td>
<td>$0.00</td>
<td>$0.00</td>
<td>$100,000.00</td>
</tr>
<tr>
<td>1</td>
<td>$4,387.14</td>
<td>$416.67</td>
<td>$3,970.47</td>
<td>$96,029.53</td>
</tr>
<tr>
<td>2</td>
<td>$4,387.14</td>
<td>$400.12</td>
<td>$3,987.02</td>
<td>$92,042.51</td>
</tr>
<tr>
<td>3</td>
<td>$4,387.14</td>
<td>$383.51</td>
<td>$4,003.63</td>
<td>$88,038.88</td>
</tr>
<tr>
<td>4</td>
<td>$4,387.14</td>
<td>$366.83</td>
<td>$4,020.31</td>
<td>$84,018.57</td>
</tr>
<tr>
<td>5</td>
<td>$4,387.14</td>
<td>$350.08</td>
<td>$4,037.06</td>
<td>$79,981.51</td>
</tr>
<tr>
<td>6</td>
<td>$4,387.14</td>
<td>$333.26</td>
<td>$4,053.88</td>
<td>$75,927.63</td>
</tr>
<tr>
<td>7</td>
<td>$4,387.14</td>
<td>$316.37</td>
<td>$4,070.77</td>
<td>$71,856.85</td>
</tr>
<tr>
<td>8</td>
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<td>$4,087.74</td>
<td>$67,769.12</td>
</tr>
<tr>
<td>9</td>
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<td>$282.37</td>
<td>$4,104.77</td>
<td>$63,664.35</td>
</tr>
<tr>
<td>10</td>
<td>$4,387.14</td>
<td>$265.27</td>
<td>$4,121.87</td>
<td>$59,542.48</td>
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<tr>
<td>11</td>
<td>$4,387.14</td>
<td>$248.09</td>
<td>$4,139.05</td>
<td>$55,403.44</td>
</tr>
<tr>
<td>12</td>
<td>$4,387.14</td>
<td>$230.85</td>
<td>$4,156.29</td>
<td>$51,247.14</td>
</tr>
<tr>
<td>13</td>
<td>$4,387.14</td>
<td>$213.53</td>
<td>$4,173.61</td>
<td>$47,073.54</td>
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<tr>
<td>14</td>
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<td>$4,191.00</td>
<td>$42,882.54</td>
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<tr>
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<td>$4,208.46</td>
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<tr>
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<td>$161.14</td>
<td>$4,226.00</td>
<td>$34,448.08</td>
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<tr>
<td>23</td>
<td>$4,387.14</td>
<td>$36.33</td>
<td>$4,350.81</td>
<td>$4,368.94</td>
</tr>
<tr>
<td>24</td>
<td>$4,387.14</td>
<td>$18.20</td>
<td>$4,368.94</td>
<td>$0.00</td>
</tr>
</tbody>
</table>
2. Determining the unknown interest rate

Let’s say that you have $1,000 to invest today and in five years you would like the investment to be worth $2,000. What interest rate would satisfy your investment objective? We know the present value (PV = $1,000), the future value (FV = $2,000) and the number of compounding periods (n=5). Using the basic valuation equation:

\[ FV = PV \times (1 + i)^n, \]

and substituting the known values of FV, PV, and n,

\[ $2,000 = $1,000 \times (1 + i)^5 \]

Rearranging, we see that the ratio of the future value to the present value is equal to the compound factor for five periods at some unknown rate:

\[ \frac{2,000}{1,000} = (1 + i)^5 \]

where 2.000 is the compound factor.

We therefore have one equation with one unknown, i. We can determine the unknown interest rate either mathematically or by using the table of compound factors. Using the table of factors, we see that for five compounding periods, the interest rate that produces a compound factor closest to 2.000 is 15 percent per year.

We can determine the interest rate more precisely, however, by solving for i mathematically:

\[ 2 = (1 + i)^5. \]

Taking the fifth root of both sides, and representing this operation is several equivalent ways,

\[ 1 + i = 2^{1/5} = 20^{0.20} \]

You’ll need a calculator to figure out the fifth root:
\[2^{0.20} = 1.1487 = (1 + i)\]
\[i = 1.1487 - 1 = 0.1487 \text{ or } 14.87\%.

Therefore, if you invested $1,000 in an investment that pays 14.87 percent compounded interest per year, for five years, you would have $2,000 at the end of the fifth year. We can formalize an equation for finding the interest rate when we know PV, FV, and n from the valuation equation and notation: \(FV = PV (1 + i)^n\).

Using Algebra,

\[i = \left(\frac{FV}{PV}\right)^{1/n} - 1\]

As an example, suppose that the value of an investment today is $100 and the expected value of the investment in five years is expected to be $150. What is the annual rate of appreciation in value of this investment over the five year period?

We can use the math or financial programs in a calculator to solve for i, which is 8.447 percent.

Or, we can use a spreadsheet. Using the Excel function RATE,

\[=\text{RATE}(\text{number of periods, periodic payment, PV, FV, 0, -10})\]

\[=\text{RATE}(5, 0, -100, 150, 0, 10)\]

**Example 5: Calculating the interest rate**

**Problem**

Suppose you borrow $1,000, with terms that you will repay in a lump-sum of $1,750 at the end of three years. What is the effective interest rate on this loan?

**Solution**

\[PV = $1,000\]
\[FV = $1,750\]
\[n = 3\]

\[i = \left(\frac{$1,750}{$1,000}\right)^{1/3} - 1 = 20.51\%\]

**A. Application: Determining growth rates**

There are many applications in which we need to determine the rate of change in values over a period of time. If values are increasing over time, we refer to the rate of change as the growth rate. To make comparisons easier, we usually specify the growth rate as a rate per year. We can use the information about the starting value (the PV), the ending value (the FV), and the number of periods to determine the rate of growth of values over this time period. To make comparisons among investments, we typically need to determine the average annual growth rate.

For example, consider an investment that has a value of $100 in year 0, a value of $150 at the end of the first year and a value of $200 at the end of two periods. What is this investment's annual growth rate?

\[PV = $100\]
\[FV = $200\]
\[n = 2\]

\[i = \left(\frac{$200}{$100}\right)^{1/2} - 1 = 20.5 - 1 = 1.4142 - 1 = 41.42\%\]
Checking our work,

\[ \$100 (1 + 0.4142)(1 + 0.4142) = \$200 \]

Therefore, \$100 grows to \$200 at the rate of 41.42% per year.

This rate is the geometric average of the annual growth rates. To see this, consider the two annual growth rates for this example. The growth rate in the first year is \((\$150 - 100)/100\), or 50 percent. The growth rate for the second year is \((\$200 - 150)/150 = 33.3333\) percent. The geometric average is calculated as:

\[
\text{geometric average rate} = \left( (1 + i_1)(1 + i_2) \ldots (1 + i_n) \right)^{1/n} - 1
\]

which for this example is:

\[
\text{geometric average rate over the two years} = \left( (1 + 0.50)(1 + 0.3333) \right)^{1/2} - 1 = 41.42\%
\]

This is different from the arithmetic average rate of \((0.50 + 0.3333)/2 = 41.67\) percent. The arithmetic average is not appropriate because it does not consider the effects of compounding.

---

**Example 6: Determining growth rates**

Consider the growth rate of dividends for Bell Atlantic. Bell Atlantic paid dividends of \$2.36 per share in 1990 and \$2.76 in 1994. We have dividends for two different points in time: 1990 and 1994. With 1990 dividends as the present value, 1994 dividends as the future value, and \(n=4\):

\[
\left( \frac{\$2.76}{\$2.36} \right)^{1/4} - 1 = 3.99\%
\]

Therefore, Bell Atlantic’s dividends grew at a rate of almost 4% per year over this four year period.

Looking at an earlier period, in 1986 Bell Atlantic paid \$2.04 in dividends. The rate of growth over the period 1986-1990 is:

\[
\text{growth rate} = \left( \frac{\$2.36}{\$2.04} \right)^{1/4} - 1 = 3.71\%
\]

The growth rate from 1986 through 1994 falls between 3.71% and 3.99%:

\[
\text{growth rate} = \left( \frac{\$2.76}{\$2.04} \right)^{1/8} - 1 = 3.85\%
\]

---

**Example 7: Beanie Babies**

**Problem**

Susie Sweetie bought Iggy the Iguana two years ago for \$5. Today, Iggy is worth \$7. What is the average annual return that Susie has earned on her Iggy investment?

**Solution**

\[
\begin{align*}
\text{PV} &= \$5 \\
\text{FV} &= \$7 \\
\text{n} &= 2 \\
\text{i} &= \left( \frac{7}{5} \right)^{1/2} - 1 \\
\text{i} &= 18.32\%
\end{align*}
\]

---

**B. Application: Determining the effective rate on loans**

We can calculate the effective annual rate for an installment loan in much the same manner that we can calculate the EAR for a savings account. Consider a loan of \$10,000 that is paid back in twenty-four monthly installments of \$470.74 each, with the first installment due at the end of the first month. Calculating the EAR corresponding to this loan requires us to first calculate the monthly rate and then translate this rate into the EAR.

We are given the following information:

\[
\begin{align*}
\text{PV} &= \$10,000 \\
\text{CF} &= \$470.74 \\
\text{N} &= 24
\end{align*}
\]

Using the ordinary annuity relation,

\[
\text{PV} = \text{CF} \times \text{(present value annuity factor, N=24, i = ?)}
\]

and substituting the known values,
we find that the present value annuity factor is 21.2431. Using a financial calculator, we find that the monthly rate, i, is 1 percent. Therefore, the effective annual rate on this loan is:

\[
EAR = (1 + 0.01)^{12} - 1 = 12.68\%
\]

The key to solving for the effective annual rate is to find the rate per compounding period. If the compounding period is less than one year, we can then use the relation above, inserting the rate per compounding period and the number of compounding periods in a year to determine the effective annual rate.

3. Determining the number of compounding periods

Let's say that you place $1,000 in a savings account that pays 10 percent compounded interest per year. How long would it take for that savings account balance to reach $5,000? In this case, we know the present value (PV=$1,000), the future value (FV=$5,000), and the interest rate (i=10% per year). What we need to determine is the number of periods.

Let's start with the basic valuation equation and insert the known values of PV, FV, and i:

\[
FV = PV (1 + i)^n
\]

\[
5,000 = 1,000 (1 + 0.10)^n
\]

Rearranging,

\[
(1 + 0.10)^n = 5.000
\]

Therefore, the compound factor is 5.000.

Like the determination of the unknown interest rate, we can determine the number of periods either mathematically or by using the table of compound factors. If we use the table of compound factors, we look down the column for the 10 percent interest rate to find the factor closest to 5.000 Then we look across the row containing this factor to find n. From the table of factors, we see that the n that corresponds to a factor of 5.000 for a 10 percent interest rate is between 16 and 17, though closer to 17. Therefore, we approximate the number of periods as 17.

Solving the equation mathematically is a bit more complex. We know that:

\[
5 = (1 + 0.10)^n
\]

We must somehow rearrange this equation so that the unknown value, n, is on one side of the equation and all the known values are on the other. To do this, we must use logarithms and a bit of algebra. Taking the natural log of both sides:
\[ \ln 5 = n \ln(1 + 0.10) \]

or

\[ \ln 5 = n \ln 1.10, \]

where "\( \ln \)" indicates the natural log. Substitute the values of the natural logs of 5 and 1.10,

\[ 1.6094 = n (0.0953) . \]

Rearranging and solving for \( n \),

\[ n = 16.8877 \] which means 17 whole compound periods.

Since the last interest payment is at the end of the last year, the number of periods years is 17 -- it would take 17 years for your $1,000 investment to grow to $5,000 if interest is compounded at 10 percent per year.

As you can see, given the present and future values, calculating the number of periods when we know the interest rate is a bit more complex than calculating the interest rate when we know the number of periods. Nevertheless, we can develop an equation for determining the number of periods, beginning with the valuation formula:

\[ FV = PV (1 + i)^n, \]

Using algebra and principles of logarithms,

\[ n = \frac{\ln FV - \ln PV}{\ln (1+i)} \]

Suppose that the present value of an investment is $100 and you wish to determine how long it will take for the investment to double in value if the investment earns 6 percent per year, compounded annually:

\[ n = \frac{\ln 200 - \ln 100}{\ln (1+0.06)} = 11.8885 \Rightarrow 12 \text{ years} \]

You'll notice that we round off to the next whole period. To see why, consider this last example. After 11.8885 years, we have doubled our money if interest were paid 88.85 percent the way through the twelfth year. But, we stated earlier that interest is paid at the end of each period -- not part of the way through. At the end of the eleventh year, our investment is worth $189.93, and at the end of the twelfth year, our investment is worth $201.22. So, our investment's value doubles by the twelfth period -- with a little extra, $1.22.

The tables of factors can be used to approximate the number of periods. The approach is similar to the way we approximated the interest rate. The compounding factor in this example is 2.0000. and the discounting factor is 0.5000 (that is, \( FV/PV = 2.0000 \) and \( PV/FV = 0.5000 \)). Using the table of compound factors, following down the column corresponding to the interest rate of 6 percent, the compound factor closest to 2.0000 is for 12 periods. Likewise, using the table of discount factors, following down the column corresponding to the interest rate of 6 percent, the discount factor closest to 0.5000 is for 12 periods. We can use the table of annuity factors in a like manner to solve for the number of payments in the case of an annuity.

We could also perform this calculation using a spreadsheet. Using Excel's NPER function

\[ =\text{NPER}(i,pmt,pv,fv,type) \]

\[ =\text{NPER}(.06,0,-100,200,0) \]
**Example 9: Determining the number of periods**

**Problem**
How long does it take to double your money if the interest rate is 5% per year, compounded annually?

**Solution**

\[ PV = \$1 \]
\[ FV = \$2 \]
\[ i = 5\% \]
\[ n = \frac{\ln 2 - \ln 1}{\ln 1.05} = \frac{0.6931 - 0}{0.0488} = 14.2029 \implies 15\text{ years} \]

**Problem**
How long does it take to triple your money if the interest rate is 5% per year, compounded annually?

**Solution**

\[ PV = \$1 \]
\[ FV = \$3 \]
\[ i = 5\% \]
\[ n = \frac{\ln 3 - \ln 1}{1.05} = \frac{1.0986 - 0}{0.0488} = 22.5123 \text{ years} \implies 23\text{ years} \]

**Problem**
How long does it take to double your money if the interest rate is 12% per year, compounded quarterly?

**Solution**

\[ PV = \$1 \]
\[ FV = \$2 \]
\[ i = \frac{12\%}{4} = 3\% \]
\[ n = \frac{0.6931 - 0}{0.0296} = 23.4155 \text{ quarters} \implies 24 \text{ quarters} = 6\text{ years} \]

4. **Valuing a perpetual stream of cash flows**

Some cash flows are expected to continue forever. For example, a corporation may promise to pay dividends on preferred stock forever. Or, a company may issue a bond that pays interest every six months, forever. How do you value these cash flow streams? When we calculated the present value of an annuity, we took the amount of one cash flow and multiplied it by the sum of the discount factors that corresponded to the interest rate and number of payments. But what if the number of payments extends forever -- into infinity?

A series of cash flows that occur at regular intervals, forever, is a **perpetuity**. Valuing a perpetual cash flow stream is just like valuing an ordinary annuity. It looks like this:

\[
PV = \sum_{t=1}^{\infty} \frac{CF}{(1+i)^t} = CF \sum_{t=1}^{\infty} \frac{1}{(1+i)^t}
\]

As the number of discounting periods approaches infinity, the summation approaches \(1/i\). To see why, consider the present value annuity factor for an interest rate of 10 percent, as the number of payments goes from 1 to 200:

<table>
<thead>
<tr>
<th>Number of payments in the annuity</th>
<th>Present value annuity discount factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0909</td>
</tr>
<tr>
<td>10</td>
<td>6.1446</td>
</tr>
<tr>
<td>50</td>
<td>9.9148</td>
</tr>
<tr>
<td>100</td>
<td>9.9993</td>
</tr>
<tr>
<td>1000</td>
<td>10.000</td>
</tr>
</tbody>
</table>

As the number of payments increases, the factor approaches 10, or 1/0.10. Therefore, the present value of a perpetual annuity is very close to \(1/i\).
Suppose you are considering an investment that promises to pay $100 each period forever, and the interest rate you can earn on alternative investments of similar risk is 5 percent per period. What are you willing to pay today for this investment?

\[ PV = \frac{100}{0.05} = 2000. \]

Therefore, you would be willing to pay $2,000 today for this investment to receive, in return, the promise of $100 each period forever.

Let's look at the value of a perpetuity from a different angle. Suppose that you are given the opportunity to purchase an investment for $5,000 that promises to pay $50 at the end of every period forever. What is the periodic interest per period -- the return -- associated with this investment?

We know that the present value is PV = $5,000 and the periodic, perpetual payment is CF = $50. Inserting these values into the formula for the present value of a perpetuity:

\[ PV = \frac{5000}{i}. \]

Solving for i,

\[ i = \frac{50}{5000} = 0.01 \text{ or } 1\% \text{ per period.} \]

Therefore, an investment of $5,000 that generates $50 per period provides 1 percent compounded interest per period.