SINGULAR PERTURBATIONS IN A NON-LINEAR VISCOELASTICITY

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Abstract

A non-linear equation in viscoelasticity of the form

\[
\rho u_{tt}(t, x) = \phi(u_{\rho}(t, x))_x + \int_{-\infty}^{t} F(t-s)\phi(u_{\rho}(s, x))_x ds + \rho g(t, x) + f(x), \quad t \geq 0, \quad x \in [0, 1],
\]

(0.1)

\[
u^\rho(t, 0) = v^\rho(t, 1) = 0, \quad t \geq 0,
\]

(0.2)

\[
u^\rho(s, x) = \nu^\rho(s, x), \quad s \leq 0, \quad x \in [0, 1],
\]

(0.3)

(where \(\phi\) is non-linear) is studied when the density \(\rho\) of the material goes to zero. It will be shown that when \(\rho \downarrow 0\), solutions \(u^\rho\) of the dynamical system (0.1)-(0.3) approach the unique solution \(w\) (which is independent of \(t\)) of the steady state obtained from (0.1)-(0.3) with \(\rho = 0\). Moreover, the rate of convergence in \(\rho\) is obtained to be \(\|u^\rho - w\|_{L^2} \leq K\sqrt{\rho}\) and \(\|u^\rho_x - w_x\|_{L^2} \leq K\sqrt{\rho}\) for some constant \(K\) independent of \(\rho\).

1 INTRODUCTION.

Let us begin with the following quasi-static approximation studied in MacCamy [11],

\[
u_{tt}(t) = -A(0)g(\nu(t)) - \int_{0}^{t} A'(t-s)g(\nu(s))ds + F(t),
\]

(1.1)

and

\[0 = -A(0)g(\nu(t)) - \int_{0}^{t} A'(t-s)g(\nu(s))ds + F(t).
\]

(1.2)

Here \(A(t)\) is a bounded and linear operator and \(g\) is a non-linear and unbounded operator in a Hilbert space. It is shown in [11] that if \(F(t)\) approaches a constant vector \(F(\infty)\) as

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$t \to \infty$, then, under appropriate conditions, one has

$$g(u(t)) \to A(\infty)^{-1}F(\infty) \text{ weakly in } H, \quad t \to \infty,$$

(1.3)

$$g(w(t)) \to A(\infty)^{-1}F(\infty) \text{ in } H, \quad t \to \infty,$$

(1.4)

where $u$ and $w$ are solutions of (1.1) and (1.2) respectively. This result motivates the procedure of using the quasi-static approximation in viscoelasticity, which drops the “acceleration” term $u_{tt}$ when $t$ is large. That is, use $w$ to approximate $u$.

Now, let us look at the following non-linear equation in viscoelasticity,

$$\rho u_{tt}(t, x) = \phi(u_x(t, x)) + \int_{-\infty}^{t} F(t - s) \phi(u_x(s, x)) ds + \rho g(t, x) + f(x), \quad t \geq 0, \quad x \in [0, 1],$$

(1.5)

$$u^\rho(t, 0) = u^\rho(t, 1) = 0, \quad t \geq 0; \quad u^\rho(s, x) = v^\rho(s, x), \quad s \leq 0, \quad x \in [0, 1],$$

(1.6)

which can be found in e.g., Dafermos and Nohel [4] and MacCamy [13]. Here $u$ is the displacement, $\rho g$ is the body force, $f$ is the external force, and $\rho$ is the density of the material. Same as in MacCamy [13], we assume that $\phi$ on $\mathbb{R}$ is non-linear, $\phi(0) = 0$, and there is a constant $c_0 > 0$ such that $\phi' \geq c_0$ on $\mathbb{R}$.

For Eq.(1.5)-(1.6), we propose the singular perturbation problem in the following sense: show that when $\rho \downarrow 0$, the solutions of (1.5)-(1.6) approach the solutions of the equation obtained from (1.5)-(1.6) with $\rho = 0$. It will be shown that the solution of (1.5)-(1.6) with $\rho = 0$ exists uniquely and is independent of $t$, i.e., in static-state. Thus, this singular perturbation can also be regarded as a quasi-static approximation.

When $\phi$ is linear, (1.5)-(1.6) is studied in Grimmer and Liu [6], where linearity is used to subtract the solution $w$ of (1.5)-(1.6) with $\rho = 0$ from the solutions $u^\rho$ of (1.5)-(1.6). Then an equation for $Q^\rho \equiv u^\rho - w$ is formulated and the method of energy estimate is employed to show that $(u^\rho - w) = Q^\rho \to 0$ as $\rho \to 0$.

When $\phi$ is non-linear but $f \neq 0$, it is shown in [6] that the solution $w$ of (1.5)-(1.6) with $\rho = 0$ is $w = 0$. Thus the equation for $Q^\rho \equiv u^\rho - w = u^\rho$ is the same as Eq.(1.5)-(1.6) (with $f = 0$). Therefore, it is indicated in [6] that the energy estimate method can be modified to show that $(u^\rho - w = u^\rho) = Q^\rho \to 0$ as $\rho \to 0$.

Now, in this paper, we look at the case where $\phi$ is non-linear and $f \neq 0$. It will be seen that this case is complicated than the previous cases. For example, the equation for
$Q^\rho \equiv u^\rho - w$ also involves $w$. However, after some trials and errors, we found an appropriate energy function for $Q^\rho$ so that the method of the energy estimate used in [6] can also be extended here to show that $(u^\rho - w =) Q^\rho \to 0$ as $\rho \to 0$. Moreover, the rate of convergence in $\rho$ is obtained to be $\|u^\rho - w\|_{L^2} \leq K\sqrt{\rho}$ and $\|u^\rho_x - w_x\|_{L^2} \leq K\sqrt{\rho}$ for some constant $K$ independent of $\rho$, as a by-product of our energy estimate in this paper. (The rate of convergence was not discovered in [6].)

Related studies of singular perturbations can be found in, for example, Chow and Lu [1], Fattorini [5], Hale and Raugel [8], Grimmer and Liu [6], and Liu [9, 10].

2 SINGULAR PERTURBATIONS.

Note that the existence and uniqueness of solutions of Eq.(1.5)-(1.6) (with $\rho > 0$) were obtained in [4, 7, 12, 13], and we are only interested in singular perturbations in this paper, so we will assume that Eq.(1.5)-(1.6) (with $\rho > 0$) has a unique solution $u^\rho$ for every $\rho > 0$. Also note that we first assume that the “history” $v^\rho$ satisfies Eq.(1.5) on $\mathbb{R}^-$. Then we will see that if $v^\rho$ is only specified on $\mathbb{R}^-$ (may not satisfy Eq.(1.5)), then with essentially the same proof, we can obtain the similar results.

Now we can state and prove our main results with the following hypothesis:

(H). $1 + \hat{F}(\lambda) \neq 0$ for $\text{Re}\lambda \geq 0$. $F$ and $F' \in L^1(\mathbb{R}^+)$. $F = 0$ on $\mathbb{R}^-$. $f \in C[0, 1]$. $\|v^\rho_f(s, \cdot)\|_{L^2}$ and $\|g(-s)\|_{L^2}$ are bounded for $s \leq 0$.

Here $\hat{F}$ is the Laplace transform of $F$, and $L^2 = L^2[0, T]$.

**Theorem 2.1.** Assume that the hypothesis (H) is satisfied. Then there is a unique $w$, which is independent of $t$, such that

\begin{align*}
0 &= \phi(w_x(x))_x + \int_{-\infty}^t F(t-s)\phi(w_x(x))_x ds + f(x), \; t \in \mathbb{R}, \; x \in [0, 1], \quad (2.1) \\
w(0) = w(1) &= 0. \quad (2.2)
\end{align*}

(This equation is obtained from (1.5)- (1.6) with $\rho = 0$.)

**Proof.** Similar to [6], we let $R$ be the function such that $R(s) = 0$, $s \leq 0$ and

\begin{equation}
R(t) = -F(t) - \int_0^t R(t-s)F(s)ds, \; t \geq 0, \quad (2.3)
\end{equation}
whose existence is studied in, e.g., [2, 3, 7]. Note that (2.3) can be written as
\[(\delta + R) * (\delta + F) = \delta,\] (2.4)
where
\[R * F(t) = \int_{-\infty}^{t} R(t - s)F(s)ds \quad \text{and} \quad \delta * H = H.\] (2.5)

Now, write (1.5) with \(\rho = 0\) as
\[-f(x) = (\delta + F) * \phi(u_x(t, x))_x.\] (2.6)
This implies
\[
\phi(u_x(t, x))_x = -(\delta + R) * f(x) = -\left[1 + \int_{0}^{\infty} R(s)ds\right]f(x) \\
= -\left[1 + \int_{0}^{\infty} F(s)ds\right]^{-1}f(x) \overset{\text{def}}{=} f_0(x). \quad \text{(2.7)}
\]
Thus we have
\[
\phi(u_x(t, x)) = \int_{0}^{x} f_0(r)dr + C, \quad \text{(2.8)} \\
u_x(t, x) = \phi^{-1}\left(\int_{0}^{x} f_0(r)dr + C\right). \quad \text{(2.9)}
\]

Therefore, the solution takes the following form
\[
w(x) \overset{\text{def}}{=} u(t, x) = \int_{0}^{x} \phi^{-1}\left(\int_{0}^{s} f_0(r)dr + C\right)ds + C_1. \quad \text{(2.10)}
\]

Taking into account of the boundary condition (1.6), we see that \(C_1 = 0\) and that we only need to verify that there is a unique constant \(C\) such that
\[
\int_{0}^{1} \phi^{-1}\left(\int_{0}^{s} f_0(r)dr + C\right)ds = 0. \quad \text{(2.11)}
\]
For this purpose, we first note that since \(\phi' \geq c_0 > 0\) on \(\Re\), one has \(\phi^{-1}(-\infty) = -\infty\) and \(\phi^{-1}(\infty) = \infty\). Thus there exists at least one \(C\) such that (2.11) is true.

Next, taking a derivative in \(C\) of the function
\[
G(C) \equiv \int_{0}^{1} \phi^{-1}\left(\int_{0}^{s} f_0(r)dr + C\right)ds, \quad \text{(2.12)}
\]
one gets
\[ \frac{1}{c_0} \geq G'(C) = \int_0^1 \frac{1}{\phi'(\phi^{-1} \left( \int_0^s f_0(r)dr + C' \right))} ds > 0. \] (2.13)

Therefore \( G(C) \) is strictly increasing in \( C \). Hence, there exists a unique \( C \) such that (2.11) is true. \( \square \)

**Theorem 2.2.** Assume that the hypothesis (H) is satisfied and that Eq.(1.5)-(1.6) has a unique solution \( u^\rho \) (on \( \mathbb{R} \)) for \( \rho > 0 \) (i.e., \( v^\rho \) satisfies Eq.(1.5)-(1.6) on \( \mathbb{R}^- \)). Let \( w \) be the unique solution of (1.5)-(1.6) with \( \rho = 0 \) (from Theorem 2.1). For \( T > 0 \) fixed and \( t \in [0, T], \; x \in [0, 1] \), define \( Q^\rho(t, x) \equiv u^\rho(t, x) - w(x) \) and

\[
E(t; \rho) \equiv \int_0^1 \left[ Q^\rho_t(t, x) \right]^2 dx + \frac{2}{\rho} \int_0^1 \int_0^{Q^\rho_x(t, x)} \left[ \phi(r + w(x)) - \phi(w(x)) \right] dr dx. \] (2.14)

If there exists a constant \( K_0 \) independent of \( \rho \) such that \( E(0, \rho) \leq K_0, \; \rho > 0 \), then as \( \rho \to 0 \), we have \( u^\rho(t, \cdot) \to w(\cdot) \) and \( u^\rho_x(t, \cdot) \to w_x(\cdot) \) in \( C([0, T], L^2[0, T]) \). Moreover, there exists a constant \( K \) independent of \( \rho \) such that

\[
\| u^\rho(t, \cdot) - w(\cdot) \|_{L^2} \leq K \sqrt{\rho}, \; \| u^\rho_x(t, \cdot) - w_x(\cdot) \|_{L^2} \leq K \sqrt{\rho}, \; t \in [0, T], \; \rho > 0. \] (2.15)

**Remark 2.1.** \( E(0, \rho) \) is bounded when, for example, \( v_\rho^\rho(0, x) \) is bounded and \( Q^\rho_x(0, x) = 0 \) (i.e., \( v_\rho^\rho(0, x) = w_x(x) \)), independently of \( \rho \).

**Proof of Theorem 2.2.** We first verify that

\[
\int_0^t [\phi(r + s) - \phi(s)] dr \geq \frac{c_0}{2} t^2, \; t, s \in \mathbb{R}. \] (2.16)

For this purpose let us use the Mean Value Theorem and get

\[
\int_0^t [\phi(r + s) - \phi(s)] dr = \int_0^t \phi'(\xi) r dr. \] (2.17)

If \( t > 0 \), then \( r \geq 0 \) and

\[
\int_0^t \phi'(\xi) r dr \geq c_0 \int_0^t r dr = \frac{c_0}{2} t^2. \] (2.18)

If \( t < 0 \), then \( r \leq 0 \) and

\[
\int_0^t \phi'(\xi) r dr = \int_t^0 \phi'(\xi)(-r) dr \geq c_0 \int_t^0 (-r) dr = \frac{c_0}{2} t^2. \] (2.19)
Next, we show that for the \( E(t; \rho) \) defined by (2.14) with \( E(0; \rho) \leq K_0 \), there exists a constant \( K_1 \) independent of \( \rho \) such that \( E(t; \rho) \leq K_1, \rho > 0, t \in [0, T] \).

For this end we first note that from (2.16), one has

\[
\int_0^1 \int_0^{Q^\rho_{tt}(t,x)} \left[ \phi(r + w_x(x)) - \phi(w_x(x)) \right] dr dx \geq \frac{c_0}{2} \int_0^1 \left[ Q^\rho_{tt}(t,x) \right]^2 dx \geq 0. \tag{2.20}
\]

Then, observe that since we assumed that \( u^\rho \) satisfies Eq.(1.5) on \( \mathcal{R} \), the equation for \( Q^\rho(t, x) \equiv u^\rho(t, x) - w(x) \) is

\[
\rho Q^\rho_{tt}(t, x) = \left[ \phi(Q^\rho_x(t, x) + w(x)) - \phi(w_x(x)) \right]_x
+ \int_{-\infty}^t F(t - s) \left[ \phi(Q^\rho_x(s, x) + w(x)) - \phi(w_x(x)) \right]_x ds
+ \rho g(t, x) \tag{2.21}
\]

for \( t \in \mathcal{R} \). Using (2.5), this can be written as

\[
\rho \left( Q^\rho_{tt}(t, x) - g(t, x) \right) = (\delta + F) \ast \left[ \phi(Q^\rho_x(t, x) + w(x)) - \phi(w_x(x)) \right]_x, \quad t \in \mathcal{R}. \tag{2.22}
\]

Now, note that from [6, 14] one has \( R(\infty) = 0 \). Hence,

\[
\left[ \phi(Q^\rho_x(t, x) + w(x)) - \phi(w_x(x)) \right]_x = \rho(\delta + R) \ast \left( Q^\rho_{tt}(t, x) - g(t, x) \right)
\]

\[
= \rho \left( Q^\rho_{tt}(t, x) - g(t, x) + \int_{-\infty}^t R(t - s) \left[ Q^\rho_{tt}(s, x) - g(s, x) \right] ds \right)
\]

\[
= \rho \left( Q^\rho_{tt}(t, x) - g(t, x) + R(0)Q^\rho_{tt}(t, x) + \int_{-\infty}^t R(t - s)Q^\rho_{tt}(s, x) ds \right.
\]

\[
- \int_{-\infty}^t R(t - s)g(s, x) ds \right). \tag{2.23}
\]

Next, take a derivative of \( E(t; \rho) \) in \( t \) and use the boundary condition (1.6) to get

\[
\frac{d}{dt} E(t; \rho) = 2 \int_0^1 Q^\rho_{tt}(t, x)Q^\rho_{tt}(t, x) dx + \frac{2}{\rho} \int_0^1 \left[ \phi(Q^\rho_x(t, x) + w(x)) - \phi(w_x(x)) \right] Q^\rho_{tt}(t, x) dx
\]

\[
= 2 \int_0^1 Q^\rho_{tt}(t, x)Q^\rho_{tt}(t, x) dx - \frac{2}{\rho} \int_0^1 \left[ \phi(Q^\rho_x(t, x) + w(x)) - \phi(w_x(x)) \right]_x Q^\rho_{tt}(t, x) dx. \tag{2.24}
\]

Then, replace (2.23) into it to obtain

\[
\frac{d}{dt} E(t; \rho) = 2 \int_0^1 Q^\rho_{tt}(t, x)Q^\rho_{tt}(t, x) dx - 2 \int_0^1 \left( Q^\rho_{tt}(t, x) - g(t, x) \right)
\]

\[
- \int_{-\infty}^t R(t - s)g(s, x) ds \right). \tag{2.25}
\]
\begin{align*}
+ R(0)Q^p_t(t, x) + \int_{-\infty}^t R'(t-s)Q^p_t(s, x)ds - \int_{-\infty}^t R(t-s)g(s, x)ds \right) Q^p_t(t, x) dx \\
= 2 \int_0^1 \left( g(t, x) - R(0)Q^p_t(t, x) - \int_{-\infty}^t R'(t-s)Q^p_t(s, x)ds \\
+ \int_{-\infty}^t R(t-s)g(s, x)ds \right) Q^p_t(t, x) dx \\
\leq \|g(t, \cdot)\|_{L^2}^2 + \left( 2 + 2|R(0)| \right) \|Q^p_t(t, \cdot)\|_{L^2}^2 \\
+ \int_{-\infty}^t |R'(t-s)| \left[ \|Q^p_t(s, \cdot)\|_{L^2}^2 + \|Q^p_t(t, \cdot)\|_{L^2}^2 \right] ds \\
+ \int_0^1 \left[ \int_{-\infty}^t |R(t-s)g(s, x)|ds \right]^2 dx \\
\leq \left( 2 + 2|R(0)| \right) + \int_0^\infty |R'(s)|ds \|Q^p(t, \cdot)\|_{L^2}^2 \\
+ \int_0^t |R'(t-s)|\|Q^p_t(s, \cdot)\|_{L^2}^2 ds \\
+ \|g(t, \cdot)\|_{L^2}^2 + \int_{-\infty}^0 |R'(t-s)|\|Q^p(s, \cdot)\|_{L^2}^2 ds + \int_0^1 \left[ \int_{-\infty}^t |R(t-s)g(s, x)|ds \right]^2 dx.
\end{align*}

Now, note that \( \|Q^p_t(t, \cdot)\|_{L^2} \leq E(t; \rho) \) by (2.20). Then from above one gets

\[
\frac{d}{dt} E(t; \rho) \leq \left( 2 + 2|R(0)| \right) + \int_0^\infty |R'(s)|ds \] E(t; \rho) \\
+ \int_0^t |R'(t-s)|E(s; \rho) ds \\
+ \|g(t, \cdot)\|_{L^2}^2 + \int_{-\infty}^0 |R'(t-s)|\|Q^p(s, \cdot)\|_{L^2}^2 ds + \int_0^1 \left[ \int_{-\infty}^t |R(t-s)g(s, x)|ds \right]^2 dx \\
\leq HE(t; \rho) + \int_0^t |R'(t-s)|E(s; \rho) ds + P,
\]

(2.24)

where \( H \) and \( P \) are constants defined in a obvious way.

Similar to [6], we can use the standard arguments in differential inequality to obtain a constant \( K_1 \) independent of \( \rho \) such that \( E(t; \rho) \leq K_1, \ t \in [0, T], \ \rho > 0 \). Therefore, (2.20) implies

\[
\frac{c_0}{\rho} \int_0^1 \left[ Q^p_x(t, x) \right]^2 dx \leq E(t; \rho) \leq K_1, \ t \in [0, T], \ \rho > 0.
\]

(2.25)

Now, note that the boundary condition in (1.6) implies

\[
\|Q^p(t, \cdot)\|_{L^2} \leq \|Q^p(t, \cdot)\|_{L^2}.
\]

(2.26)
Thus we can let \( K \equiv \sqrt{K_1/c_0} \) and obtain
\[
\|Q^\rho(t, \cdot)\|_{L^2} \leq \|Q_x^\rho(t, \cdot)\|_{L^2} \leq K\sqrt{\rho}, \quad t \in [0, T], \ \rho > 0. \tag{2.27}
\]
This proves the Theorem. \( \Box \)

**Remark 2.2.** Here, the proof of \( Q^\rho(t, x) \to 0 \) as \( \rho \to 0 \) is different from [6], and is short and direct, and can also provide the rate of convergence in \( \rho \).

In the following, we will verify that if \( \nu^\rho \) is only specified on \( \mathbb{R}^- \) and may not satisfy Eq.(1.5), then we can still get the similar results. Because now, (2.21) becomes
\[
\rho Q_{tt}^\rho(t, x) = \left[ \phi(Q_x^\rho(t, x) + w_x(x)) - \phi(w_x(x)) \right]_x + \int_0^t F(t - s) \left[ \phi(Q_x^\rho(s, x) + w_x(x)) - \phi(w_x(x)) \right]_x ds + \rho g(t, x), \quad t \geq 0.
\]
And hence, (2.22) becomes
\[
\rho \left( Q_{tt}^\rho(t, x) - g(t, x) \right) = (\delta + F)^* \left[ \phi(Q_x^\rho(t, x) + w_x(x)) - \phi(w_x(x)) \right]_x + \int_0^t F(t - s) \left[ \phi(Q_x^\rho(s, x) + w_x(x)) - \phi(w_x(x)) \right]_x ds \tag{2.29}
\]
where the integration in \( \hat{\star} \) is from 0 to \( t \). Therefore (2.23) becomes
\[
\left[ \phi(Q_x^\rho(t, x) + w_x(x)) - \phi(w_x(x)) \right]_x = (\delta + R)^* \left\{ \rho \left( Q_{tt}^\rho(t, x) - g(t, x) \right) \right. \\
- \int_{-\infty}^0 F(t - s) \left[ \phi(Q_x^\rho(s, x) + w_x(x)) - \phi(w_x(x)) \right]_x ds \bigg\}
\]
\[
= \rho \left( Q_{tt}^\rho(t, x) - g(t, x) + \int_0^t R(t - s) \left[ Q_{tt}^\rho(s, x) - g(s, x) \right] ds \right) \\
- (\delta + R)^* \int_0^t F(t - s) \left[ \phi(Q_x^\rho(s, x) + w_x(x)) - \phi(w_x(x)) \right]_x ds \\
= \rho \left( Q_{tt}^\rho(t, x) - g(t, x) + R(0) Q_{tt}^\rho(t, x) - R(t) Q_{tt}^\rho(0, x) \right) \\
+ \int_0^t R'(t - s) Q_{tt}^\rho(s, x) ds - \int_0^t R(t - s) g(s, x) ds \\
- (\delta + R)^* \int_{-\infty}^0 F(t - s) \left[ \phi(Q_x^\rho(s, x) + w_x(x)) - \phi(w_x(x)) \right]_x ds. \tag{2.30}
\]
Thus, (2.24) will be changed to
\[
\frac{d}{dt} E(t; \rho) = 2 \int_0^1 Q_\rho^\varepsilon(t, x) Q_\rho^\varepsilon(t, x) dx - \frac{2}{\rho} \int_0^1 \left[ \phi(Q_\rho^\varepsilon(t, x) + w_x(x)) - \phi(w_x(x)) \right] Q_\rho^\varepsilon(t, x) dx
\]
\[
= 2 \int_0^1 Q_\rho^\varepsilon(t, x) Q_\rho^\varepsilon(t, x) dx - 2 \int_0^1 \left( Q_\rho^\varepsilon(t, x) - g(t, x) + R(0) Q_\rho^\varepsilon(t, x) \right)
\]
\[
- R(t) Q_\rho^\varepsilon(0, x) + \int_0^t R'(t - s) Q_\rho^\varepsilon(s, x) ds - \int_0^t R(t - s) g(s, x) ds \right) Q_\rho^\varepsilon(t, x) dx
\]
\[
+ \frac{2}{\rho} \int_0^1 \left\{ \frac{1}{\rho} \left( (\delta + R) s \right) \int_{-\infty}^0 F(t - s) \left[ \phi(Q_\rho^\varepsilon(s, x) + w_x(s)) \right] - \phi(w_x(s)) \right\} ds \right\} Q_\rho^\varepsilon(t, x) dx
\]
\[
\leq \|g(t, \cdot) + R(t) Q_\rho^\varepsilon(0, \cdot)\|_{L^2}^2 + \left( 3 + 2 |R(0)| \right) \|Q_\rho^\varepsilon(t, \cdot)\|_{L^2}^2
\]
\[
+ \int_0^t |R'(t - s)| \left[ \|Q_\rho^\varepsilon(s, \cdot)\|_{L^2}^2 + \|Q_\rho^\varepsilon(t, \cdot)\|_{L^2}^2 \right] ds
\]
\[
+ \int_0^t \left[ \int_0^1 |R(t - s) g(s, x)| |ds| \right] dx
\]
\[
+ \int_0^t \left\{ \frac{1}{\rho} \left( (\delta + R) s \right) \int_{-\infty}^0 F(t - s) \left[ \phi(Q_\rho^\varepsilon(s, x) + w_x(s)) \right] - \phi(w_x(s)) \right\} ds \right\} dx
\]
\[
\leq \left( 3 + 2 |R(0)| \right) \int_0^\infty |R'(s)| ds \|Q_\rho^\varepsilon(t, \cdot)\|_{L^2}^2
\]
\[
+ \int_0^t |R'(t - s)| \|Q_\rho^\varepsilon(s, \cdot)\|_{L^2}^2 ds
\]
\[
+ \|g(t, \cdot) + R(t) Q_\rho^\varepsilon(0, \cdot)\|_{L^2}^2 + \int_0^t \left[ \int_0^1 |R(t - s) g(s, x)| |ds| \right] dx
\]
\[
+ \int_0^t \left\{ (\delta + R) s \int_{-\infty}^0 F(t - s) \frac{1}{\rho} \left[ \phi(v_\rho^\varepsilon(s, x)) \right] - \phi(w_x(s)) \right\} ds \right\} dx
\]
\[
\leq HE(t; \rho) + \int_0^t |R'(t - s)| E(s; \rho) ds + \hat{P}.
\] (2.31)
Now, it is clear that we have the following result, which is similar to Theorem 2.2:

**Theorem 2.3.** Assume that the hypothesis (H) is satisfied and that Eq.(1.5)-(1.6) has a unique solution \( u^\rho \) (on \( \mathbb{R}^+ \)) for \( \rho > 0 \) (i.e., \( v^\rho \) is only specified on \( \mathbb{R}^- \) and may not satisfy Eq.(1.5)-(1.6) on \( \mathbb{R}^- \)). Let \( w \) be the unique solution of (1.5)-(1.6) with \( \rho = 0 \) (from Theorem 2.1). Assume further that for some constant \( C \) independent of \( \rho \),

\[
\frac{1}{\rho} \left| \phi(v^\rho_x(s, x)) - \phi(w_x(x)) \right| \leq C, \quad s \leq 0, \ x \in [0,1], \ \rho > 0. \tag{2.32}
\]

If there exists a constant \( K_0 \) independent of \( \rho \) such that \( E(0, \rho) \leq K_0, \ \rho > 0 \), then as \( \rho \to 0 \), we have \( u^\rho(t, \cdot) \to w(\cdot) \) and \( u^\rho_x(t, \cdot) \to w_x(\cdot) \) in \( C([0,T], L^2[0,T]) \). Moreover, there exists a constant \( K \) independent of \( \rho \) such that

\[
\| u^\rho(t, \cdot) - w(\cdot) \|_{L^2} \leq K \sqrt{\rho}, \quad \| u^\rho_x(t, \cdot) - w_x(\cdot) \|_{L^2} \leq K \sqrt{\rho}, \quad t \in [0,T], \ \rho > 0. \tag{2.33}
\]

**Remark 2.3.** (2.32) is satisfied if, for example, \( v^\rho_x(s, x) = w_x(x), \ s \leq 0, \ x \in [0,1], \ \rho > 0. \)

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**References**


