CALCULUS CONCEPTS

1) LIMITS

\[ |x| = \sqrt{x^2} \geq 0, \quad |x| \text{ is the distance } x \text{ is from 0}, \quad |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases} \]

\[ |A - B| \text{ is the distance } A \text{ is from } B \]

\[ \lim_{x \to a} f(x) = L \text{ means if } 0 < |x - a| \text{ is small enough then } |f(x) - L| \text{ is nearly 0} \]

\[ \lim_{x \to a} f(x) = L \text{ means if } x \neq a, \text{ but the distance } x \text{ is from } a \text{ is small enough then the distance } f(x) \text{ is from } L \text{ is nearly 0.} \]

SOME LIMIT RULES

Suppose \( c \) is a number and \( \lim_{x \to a} f(x) = L \) and \( \lim_{x \to a} g(x) = M \)

1. \( \lim_{x \to a} c = c \)
2. \( \lim_{x \to a} x = a \)
3. \( \lim_{x \to a} f(x) \pm g(x) = L \pm M \)
4. \( \lim_{x \to a} cf(x) = cL \)
5. \( \lim_{x \to a} fg(x) = LM \)
6. \( \text{If } M \neq 0 \text{ then } \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \)
7. \( \text{If } M = 0 \text{ and } L = 0 \text{ then } \lim_{x \to a} \frac{f(x)}{g(x)} \text{ DOES NOT EXIST(DNE)} \)
8. \( \lim_{h \to 0} \frac{\sin h}{h} = 1 \) \( \lim_{x \to 0^+} \frac{1}{x} = \infty \) (DNE)
9. \( \lim_{x \to 0^-} \frac{1}{x} = -\infty \) (DNE)
10. \( \lim_{x \to -0} \frac{1}{x} = 0 \)
11. \( \lim_{x \to +0} \frac{1}{x} = 0 \)

2) CONTINUITY

The function \( f \) is continuous at \( x = a \) if and only if \( \lim_{x \to a} f(x) = f(a) \)

The function \( f \) is continuous if it is continuous at each number in its domain.
SOME CONTINUITY RULES

Suppose $c$ is a number, the function $f$ is continuous at $x = a$, the function $g$ is continuous at $x = a$ and the function $p$ is continuous at $x = f(a)$

(1) $cf$ is continuous at $x = a$
(2) $f \pm g$ is continuous at $x = a$
(3) $fg$ is continuous at $x = a$
(4) If $g(a) \neq 0$ then $\frac{f}{g}$ is continuous at $x = a$
(5) $p \circ f$ is continuous at $x = a$
(6) polynomials are continuous

If $f$ is continuous on $[a,b]$ then $f$ attains a maximum and a minimum value on $[a,b]$.

Intermediate Value Theorem: If $f$ is continuous on $[a,b]$ and $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$ then there is a $c$ in $(a,b)$ so that $f(c) = y$.

3) DERIVATIVES

Let the function $f$ be defined on $[a,b]$ then the line through $(a, f(a))$ and $(b, f(b))$ is called the secant line and its slope is given by $m = \frac{f(b) - f(a)}{b - a}$

$\Delta f(x) = f(x + h) - f(x)$

The difference quotient of the function $f$ is

$$\frac{\Delta f}{\Delta x}(x) = \frac{\Delta f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{h}.$$ 

If $\lim_{h \to 0} \frac{\Delta f}{\Delta x}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists then we say the function $f$ is differentiable at $x$ and write $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The function $f$ is called differentiable if it is differentiable at each $x$ in its domain.

SOME DIFFERENTIATION RULES
Let $c$ be a number, $f, g$ be differentiable functions

1. \[ \frac{d}{dx} c = 0 \]
2. \[ \frac{d}{dx} (f \pm g(x)) = f'(x) \pm g'(x) \]
3. \[ \frac{d}{dx} cf(x) = cf'(x) \]
4. \[ \frac{d}{dx} fg(x) = f'(x)g(x) + f(x)g'(x) \]
5. \[ \frac{d}{dx} \frac{f}{g}(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \]
6. Chain Rule: \[ \frac{d}{dx} f \circ g(x) = \frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \]

7. \[ \frac{d}{dx} r^x = r^x \ln r \]
8. \[ \frac{d}{dx} \sin x = \cos x \]
9. \[ \frac{d}{dx} \cos x = -\sin x \]
10. Differentiable functions are continuous, but a continuous function may not be differentiable.

Mean Value Theorem: If the function $f$ is differentiable on $(a, b)$ and continuous on $[a, b]$ then there is a number $c \in (a, b)$ so that $f'(c) = \frac{f(b) - f(a)}{b - a}$. (The slope of the secant line is equal to at least the slope of one tangent line. The secant line is parallel to a tangent line.)

If $f'(a) > 0$ then $f \uparrow$ on an interval containing $a$. If $f'(a) < 0$ then $f \downarrow$ on an interval containing $a$.

4) INTEGRALS

Let the function $f$ be defined on $[a, b]$ then $\sum_{j=1}^{n} f(x_j^*) \Delta x_j$ is called the Riemann Sum of $f$ on $[a, b]$ with partition $P = \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$ and where $\Delta x_j = x_j - x_{j-1}$. 

If \( \lim_{||P|| \to 0} \sum_{j=1}^{n} f(x_j^*) \Delta x_j \) where \( ||P|| = \max\{\Delta x_j, j = 1, \ldots, n\} \) exists then we say \( f \) is integrable on \([a, b]\) and we write \( \int_{a}^{b} f(x)dx = \lim_{||P|| \to 0} \sum_{j=1}^{n} f(x_j^*) \Delta x_j \).

SOME INTEGRAL RULES

If \( f \) is continuous on \([a, b]\) then \( f \) is integrable on \([a, b]\).

If \( F'(x) = f(x) \) on \([a, b]\) then \( \int_{a}^{b} f(x)dx = F(b) - F(a) \).

If \( F(x) = \int_{a}^{x} f(t)dt \) on \([a, b]\) then \( F'(x) = f(x) \).

If \( f \) is continuous on \([a, b]\) then \( \int_{a}^{b} f(x)dx = F(b) - F(a) = f(c)(b - a) \) for some \( c \in (a, b) \).

\[ \int f(x)dx = F(x) + c \] if and only if \( F'(x) = f(x) \).

\[ y'(x) = f(x, y(x)) \]; \( y(x_0) = x_0 \) if and only if \( y(x) = y_0 + \int_{x_0}^{x} f(s, y(s)) \, ds \) for \( f \) continuous on a region containing \((x_0, y_0)\).

Let \( c \) be a number, \( f, g \) be integrable functions

(1) \( \int_{a}^{b} c \, dx = cx_{a}^{b} = c(b - a) \)

(2) \( \int_{a}^{b} f \pm g(x) \, dx = \int_{a}^{b} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx \)

(3) \( \int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx \)

(4) Chain Rule: \( \int_{a}^{b} f(g(x))g'(x) \, dx = f(g(x))_{a}^{b} = \int_{g(a)}^{g(b)} f(u) \, du \)

(5) Integration By Parts: \( \int_{a}^{b} f(x)g'(x) \, dx = f(x)g(x)_{a}^{b} - \int_{a}^{b} g(x) f'(x) \, dx \)