DIRECTIONS:

• **STAPLE** this page to the front of your homework (don’t forget your name!).
• Show all work, clearly and in order **You will lose points if you work is not in order.**
• When required, **do not forget the units**!
• Circle your final answers. **You will lose points if you do not circle your answers.**

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**Problem 1:** (2 points) Verify Green’s theorem for \( F = -x^2y \mathbf{i} + xy^2 \mathbf{j} \), where \( D \) is the disk \( x^2 + y^2 \leq 4 \).

First we note that \( D \) is simple and \( P = -x^2y \), \( Q = xy^2 \) are both \( C^1 \) on \( D \). Hence, Green’s theorem will apply. Green’s Theorem states that

\[
\int_{\partial D} P \, dx + Q \, dy = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy.
\]

A clockwise parametrization of \( \partial D \) is \( c(t) = (2 \cos t, 2 \sin t) \) with \( t \in [0, 2\pi] \). Then

\[
\int_{\partial D} P \, dx + Q \, dy = 32 \int_0^{2\pi} \cos^2 t \sin^2 t \, dt = 32 \int_0^{2\pi} \sin^2 t \, (1 - \sin^2 t) \, dt = 32 \left[ \int_0^{2\pi} \sin^2 t \, dt - \int_0^{2\pi} (\sin^2 t)^2 \, dt \right].
\]

Using the formula \( \sin^2 t = \frac{1}{2} (1 - \cos 2t) \) the integral becomes

\[
32 \left[ \frac{1}{2} \int_0^{2\pi} (1 - \cos 2t) \, dt - \frac{1}{4} \int_0^{2\pi} (1 - 2 \cos 2t + \cos^2 2t) \, dt \right] = 32 \left[ \pi - \frac{\pi}{2} + \frac{1}{8} \int_0^{2\pi} (1 - \cos 4t) \, dt \right].
\]

Using the formula \( \cos^2 2t = \frac{1}{2} (1 + \cos 4t) \), the integral becomes

\[
32 \left[ \frac{\pi}{2} + \frac{1}{8} \int_0^{2\pi} (1 + \cos 4t) \, dt \right] = 32 \left[ \frac{\pi}{2} - \frac{\pi}{4} \right] = 8\pi.
\]
Now looking at the right hand side of greens theorem we find
\[
\int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy = \int \int_D \left( x^2 + y^2 \right) \, dA = \int_0^{2\pi} \int_0^2 r^3 \, dr \, d\theta = 2\pi \int_0^2 r^2 \, dr = 8\pi.
\]
Q.E.D.

**Problem 2:** (1 point) Let \( P(x, y) = -\frac{y}{x^2 + y^2} \) and \( Q = \frac{x}{x^2 + y^2} \). Assuming \( D \) is the unit disk, investigate why Green’s theorem fails for this \( P \) and \( Q \).

Green’s theorem is not applicable because it requires \( P \) and \( Q \) to be differentiable on the domain \( D = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \} \). However, we can see that these functions are not even bounded, so they certainly cannot be differentiable at the origin \((0, 0) \in D \). To show they are not bounded, consider the limit as \((x, y) \to (0, 0) \) along the y-axis for \( P \) (which is \(-\infty\)).

**Problem 3:** (2 points) Use Green’s theorem to find the area between the ellipse \( x^2/9 + y^2/4 = 1 \) and the circle \( x^2 + y^2 = 25 \).

We parametrize the outer boundary, the circle, in a positive, or counterclockwise, motion, so that the normal is outward to the circle and the boundary to inner boundary, the ellipse, in a negative, or clockwise direction. That is
\[
\partial D = \begin{cases} 
  c_1(t) = (5 \cos t, 5 \sin t) & \text{for } t \in [0, 2\pi], \\
  c_2(t) = (3 \cos t, -2 \sin t) & \text{for } t \in [0, 2\pi]. 
\end{cases}
\]

Then the area of \( D \) is given by
\[
A = \frac{1}{2} \left( \int_{c_1} x \, dy - y \, dx + \int_{c_2} x \, dy - y \, dx \right) = \frac{1}{2} \left[ 25 \int_0^{2\pi} dt - 6 \int_0^{2\pi} dt \right] = 19\pi.
\]

**Problem 4:** (2 points) Verify Stokes’s theorem for the surface defined by \( x^2 + y^2 + 5z = 1 \) where \( z \geq 0 \), oriented by an upward normal for the vector field
\[
F = (xz, yz, x^2 + y^2).
\]

We first note that \( \partial S = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \} \) and \( F \) is continuous and differentiable on the \( S, \partial S \), and the domain \( D = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 \} \). Since we may write
\[
z = f(x, y) = \frac{1}{5} \left( 1 - x^2 - y^2 \right),
\]
the surface is a graph and the upward facing normal is given by
\[
N = \left( \frac{2}{5}x, \frac{2}{5}y, 1 \right).
\]

Stokes theorem states that
\[
\int \int_D \nabla \times F \cdot dS = \oint_{\partial S} F \cdot ds.
\]

Considering the left hand side, calculating \( \nabla \times F = (y, -x, 0) \), we see that
\[
\int \int_D \nabla \times F \cdot dS = \int \int_D (\nabla \times F) \cdot N \, dA = \int \int_D 0 \, dA = 0.
\]
Looking at the right hand side, we can parametrize the boundary of $S$ with $c(t) = (\cos t, \sin t, 0)$ for $t \in [0, 2\pi]$. Hence $c'(t) = (-\sin t, \cos t, 0)$. So $F(c(t)) = (0, 0, 1)$. Hence

$$\oint_{\partial S} F \cdot ds = \int_c F(c(t)) \cdot c'(t) dt = 0.$$  
Q.E.D.

**Problem 5:** (3 points) Let $S$ be the surface defined by $y = 10 - x^2 - z^2$ with $y \geq 1$, oriented with a rightward pointing normal. Let $F = (2xyz + 5z, e^x \cos (yz), x^2 y)$. Determine

$$\int \int_S \nabla \times F \cdot dS.$$  
(Hint: You will need to use an indirect approach.)

First we note that the surface, $S$, is a graph such that $y = f(z, x) = 10 - x^2 - z^2$ with $y \geq 1$ such that the boundary $\partial S = \{(z, x) \in \mathbb{R}^2 | x^2 + z^2 = 9\}$. This means that we have a rightward pointing normal of $N = (2z, 2x, 1)$. Now, we may define a new surface $S'$ such that $y = 1$ and $x^2 + z^2 = 9$ (i.e. the disc at $y = 1$ of radius 3) such that $\partial S' = \partial S$. Then the normal to $S'$ is simply $n = j$. Hence by Stokes’s theorem

$$\int \int_S \nabla \times F \cdot dS = \int \int_{S'} \nabla \times F \cdot dS = \int \int_{S'} (\nabla \times F) \cdot ndS.$$  
Calculating $\nabla \times F = (x^2 + ye^x \sin xz, 5, e^x \cos yz - 2xz)$, the integral becomes

$$\int \int_{S'} 5dxdy = 45\pi.$$