**DIRECTIONS:**

- **STAPLE** this page to the front of your homework (don’t forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will lose points if you do not circle your answers**.

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**Problem 1:** (2 points) Compute the volume of the solid bounded by the surface $z = \sin y$, the planes $x = 1$, $x = 0$, $y = 0$, and $y = \pi/2$, and the $xy$ plane.

$$
\int_0^1 \int_0^{\pi/2} \sin y dy dx = \int_0^1 - \cos y \bigg|_0^{\pi/2} dx = \int_0^1 dx = 1.
$$

**Problem 2:** (3 points) Let $D$ be the region bounded by the positive $x$ and $y$ axes and the line $12x + 4y = 12$. Compute

$$
\int \int_D (x^2 + y^2) \, dA.
$$

The solution is given by

$$
\int \int_D (x^2 + y^2) \, dA = \int_0^1 \int_0^{-3x+3} (x^2 + y^2) \, dy dx = \int_0^1 \left( x^2 y + \frac{y^3}{3} \right) \bigg|_0^{-3x+3} dx.
$$

With a little bit of algebraic manipulation, this becomes

$$
\int_0^1 \left[ 3x^2 - 3x^3 + 9(1 - x)^3 \right] \, dx = \frac{5}{2}.
$$

**Problem 3:** (2 points)
(a) Prove the Mean Value Theorem for Double Integrals. That is, suppose $f : D \to \mathbb{R}$ is continuous and $D$ is an elementary region. The for some point $(x_0, y_0)$ in $D$ we have
\[
\int \int_D f(x, y)dA = f(x_0, y_0)A(D),
\]
where $A(D)$ is the area of $D$.

**Proof:** Because $f$ is continuous on $D$, it attains its maximum, which we will call $M$, for some $(x_1, y_1) \in D$ and its minimum, which we will call $m$, for some $(x_2, y_2) \in D$. Hence we know for all $(x, y) \in D$,
\[
m \leq f(x, y) \leq M.
\]
Now the mean value inequality gives us
\[
m \cdot A(D) \leq \int \int_D f(x, y)dA \leq M \cdot A(D),
\]
where $A(D)$ is the area of the region $D$. Dividing through this inequality by the area $A(D)$, we see that
\[
m \leq \frac{1}{A(D)} \int \int_D f(x, y)dA \leq M.
\]
Now because this integral is bounded between the maximum and minimum of $f$, and because we know $f$ is continuous, we know that it must take on every value between $m$ and $M$. Hence, there must exist an $(x_0, y_0) \in D$ such that $f$ takes on just the right value such that
\[
\int \int_D f(x, y)dA = f(x_0, y_0)A(D).
\]

(b) Use the mean value theorem to show that if $D = [-1, 1] \times [-1, 2]$, then
\[
1 \leq \int \int_D \frac{1}{x^2 + y^2 + 1}dxdy \leq 6.
\]
Consider the function $f(x, y) = \frac{1}{x^2 + y^2 + 1}$. This function is the largest when the denominator is the smallest. (Note: this function is always positive). Hence $x^2 + y^2 + 1$ is smallest when $(x, y) = (0, 0)$. Hence the maximum of $f$ is
\[
M = f(0, 0) = 1.
\]
Similarly, $f$ is smallest when the denominator is largest which clearly occurs at $(1, 2)$ or $(-1, 2)$, in which case
\[
m = f(1, 2) = f(-1, 2) = \frac{1}{6}.
\]
Now since the area of the domain $D$, is $A(D) = 6$, the Mean Value Inequality yeilds
\[
1 \leq \int \int_D \frac{1}{x^2 + y^2 + 1}dxdy \leq 6.
\]

**Problem 4:** (3 points) Evaluate the following integrals.

(a) (1 point) $\int_0^4 \int_{y/2}^2 e^{x^2}dxdy$
Changing the order of integration we find
\[
\int_{0}^{2} \int_{y/2}^{x} e^{x^2} \, dy \, dx = \int_{0}^{2} e^{x^2} \, dy \, dx = \int_{0}^{2} 2xe^{x^2} \, dx.
\]
By substituting \( u = x^2 \Rightarrow du = 2x \, dx \) and \( x = 0 \rightarrow u = 0, x = 2 \rightarrow u = 4 \), we find
\[
\int_{0}^{2} 2xe^{x^2} \, dx = \int_{0}^{4} e^{u} \, du = e^4 - 1.
\]

(b) (1 point) \( \int_{0}^{1} \int_{0}^{x} \int_{0}^{y} (y + xz) \, dz \, dy \, dx \).

This problem can be easily integrated without changing the order of integration so we will do so.
\[
\int_{0}^{1} \int_{0}^{x} \int_{0}^{y} (y + xz) \, dz \, dy \, dx = \int_{0}^{1} \int_{0}^{x} \left( \frac{yz}{2} + \frac{x^2z^2}{2} \right) \, dy \, dx = \int_{0}^{1} \int_{0}^{x} \left( \frac{y^2}{2} + \frac{xy^2}{2} \right) \, dy \, dx.
\]
Integrating again we obtain
\[
\int_{0}^{1} \left( \frac{x^3}{3} + \frac{x^4}{6} \right) \, dx = \frac{1}{12} + \frac{1}{30}.
\]

(c) (1 point) \( \int \int_{W} zdxdydz \), where \( W \) is the region bounded by \( x + y + z = a \) (with \( a > 0 \)), \( x = 0 \), \( y = 0 \), and \( z = 0 \). This integral can be written as
\[
\int_{0}^{a} \int_{0}^{a-x} \int_{0}^{a-x-y} z \, dz \, dy \, dx = \frac{1}{2} \int_{0}^{a} \int_{0}^{a-x} [(x - a) + y] \, dy \, dx.
\]
By integrating (and accounting for the correct number of negative signs)
\[
-\frac{1}{6} \int_{0}^{a} (x - a)^3 \, dx = \frac{a^4}{24}.
\]