One sample problem

Some notations for population parameters:
Population mean $E(y) = \mu$, where y is a random variable.
variance: $Var(y) = V(y) = \sigma^2 = E(y-\mu)^2$
standard deviation: $\sigma = \sqrt{\sigma^2}$.
Normal distribution: $y \sim N(\mu, \sigma^2)$. ($\sim$ stands for distributed as)
Standard normal distribution: $z \sim N(0, 1)$.
Fact: If $y \sim N(\mu, \sigma^2)$, then $z = \frac{y-\mu}{\sigma} \sim N(0, 1)$.

Recall
sample mean: $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$. sample variance: $s^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}$,
sample standard deviation: $s = \sqrt{s^2}$.

sampling distribution of the sample mean

$E(\bar{y}) = \mu, \sigma_{\bar{y}} = \frac{\sigma}{\sqrt{n}}$.
If the population distribution is normal with mean μ and standard deviation σ, then $\bar{y} \sim N(\mu, \frac{\sigma^2}{n})$. If the population distribution is not normal but the sample size is relatively large ($n \geq 30$), then approximately $\bar{y} \sim N(\mu, \frac{\sigma^2}{n})$

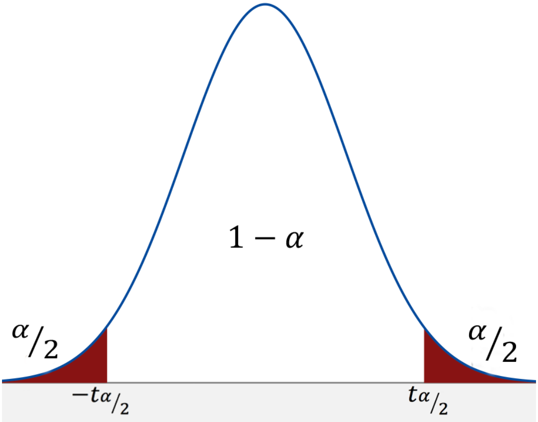
standard error of the mean: $s_{\bar{y}} = \frac{s}{\sqrt{n}}$ which is an estimate of $\sigma_{\bar{y}}$. i.e., use s to replace unknown σ so $s_{\bar{y}}$ can be computed from data.

confidence interval for μ

Suppose a random sample $y_1, y_2, \cdots, y_n \sim N(\mu, \sigma^2)$, then $\frac{\bar{y}-\mu}{s/\sqrt{n}}$ has a t distribution with degrees of freedom $\nu = n-1$.

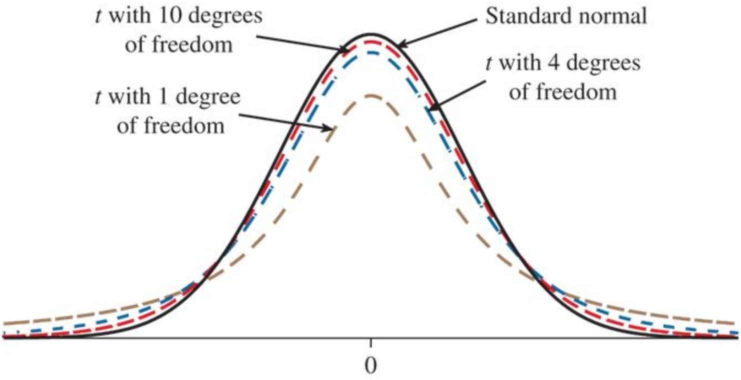
$(1-\alpha)100\%$ confidence interval for μ :
$\bar{y} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}$, where the critical value $t_{\alpha/2}$ depends on the confidence level and the degrees of freedom. The area between $-t_{\alpha/2}$ and $t_{\alpha/2}$ equals $1-\alpha$.
Confidence interval often has the form
point estimate$\pm$ margin of error
and margin of error$=$ multiplier$\times$ standard error of point estimate
In this example, point estimate for μ is $\bar{y}$, the standard error of $\bar{y}$ is $\frac{s}{\sqrt{n}}$ and the multiplier is the t critical value.

t critical value

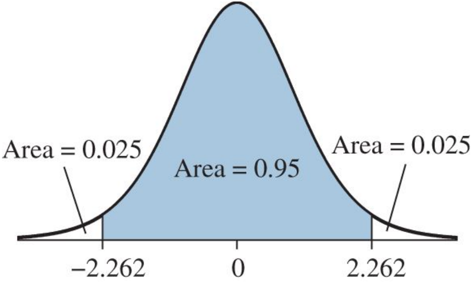


Note the area between $-t_{\alpha/2}$ and $t_{\alpha/2}$ is $1-\alpha$, the area to the left of $t_{\alpha/2}$ is $1-\frac{\alpha}{2}$.

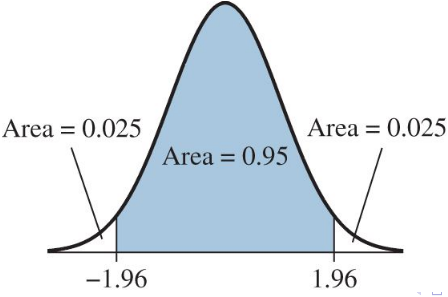
t distribution



t and z critical value



t critical value with d.f. $= 9$



z critical value

example

Seven subjects who identified themselves as Buddhist reported hours per week watching TV of:
2,1,1,3,2,3,2.
1). Estimate the sample mean $\bar{y}$ and sample standard deviation s.
2). Construct a 90% confidence interval for μ, the mean TV watching time per week for all Buddhists.
$\bar{y} = 2, s = 0.816$
$n = 7, df = n - 1 = 6, t_{0.05} = 1.943$
A 90% confidence interval for μ is:
$\bar{y} \pm t_{0.05} \frac{s}{\sqrt{n}} = 2 \pm 1.943 * \frac{0.816}{\sqrt{7}} = [1.40, 2.60]$ hours.

Use R to get t critical value

Download R from
https://www.r-project.org/

```
R version 3.6.1 (2019-07-05) -- "Action of the Toes"
Copyright (C) 2019 The R Foundation for Statistical Computing
Platform: x86_64-apple-darwin15.6.0 (64-bit)

R is free software and comes with ABSOLUTELY NO WARRANTY.
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  Natural language support but running in an English locale

R is a collaborative project with many contributors.
Type 'contributors()' for more information and
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Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.

[R.app GUI 1.70 (7684) x86_64-apple-darwin15.6.0]

[Workspace restored from /Users/chen3lx/.RData]
[History restored from /Users/chen3lx/.Rapp.history]

>
```

```
  > qt(0.95,6)
[1] 1.94318
```

Note this is the t critical value for 90% CI. qt (p, d.f.) computes the quantile of a t distribution, here p is cumulative probability, that is, the area to the left of t under the t density curve.

If the confidence level is 95%, we should use
qt(0.975,6) to get the t critical value.

exercise

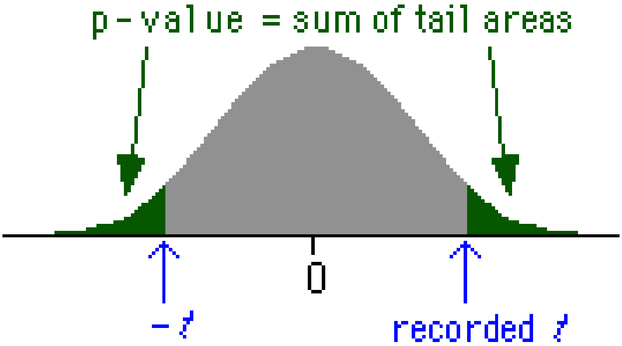
To discover the nature of the earth's atmosphere long ago, we can examine the gas in bubbles inside ancient amber. Measurements on specimens of amber from the late Cretaceous era (75 to 95 million years ago) give these percent of nitrogen: 63.4, 65.0, 64.4, 63.3, 54.8, 64.5, 60.8, 49.1, 51.0. (summary of data: $n = 9, \bar{y} = 59.59, s = 6.26$). Assume these observations are a random sample from the late Cretaceous atmosphere. Get a 95% CI for μ, the population mean.
Also perform a t test to examine if the mean nitrogen content in ancient air is equal to today's content, 78 percent.

A 95% CI is
$59.59 \pm 2.306 * \frac{6.26}{\sqrt{9}} = (54.78, 64.40)$ percent.
R code: qt(0.975, 8).

t test

$H_0 : \mu = 78$
$H_a : \mu \neq 78$
$t = \frac{59.59 - 78}{6.26/\sqrt{9}} = -8.82$
p-value $=$ 2 P$(t < -8.82) < 0.0001$
(R code: $>$ 2* pt(-8.82,8)).
pt (tvalue, d.f.) computes the cumulative probability of tvalue, i.e., the area to the left of tvalue under the t curve. so pt(-8.82,8) gives the probability $P(t < -8.82)$.
Reject H_0.

p-value for two tailed test



t test

General form of one sample t test:
1. $H_0 : \mu \leq \mu_0, Ha : \mu > \mu_0$. (right sided test)
2. $H_0 : \mu \geq \mu_0, H_a : \mu < \mu_0$. (left sided test)
3. $H_0 : \mu = \mu_0, Ha : \mu \neq \mu_0$. (two sided test)

the t statistic
$t = \frac{\bar{y}-\mu_0}{s/\sqrt{n}}$.
Let t^* be the observed test statistic.
P-value$=$ $P(t > t^*)$ for right sided test or
$P(t < t^*)$ for left sided test or
$2P(t > |t^*|)$ for two sided test.

example

A standard method of treating a disease has resulted in a mean survival time of 60 months. A new treatment given to a sample of 15 patients produced the following survival time:

61 55 68 62 65 54 70 63 56 51 72 63 76 53 71

Does the new treatment result in higher average survival time? Use $\alpha = 0.05$.

Based on data, $\bar{y} = 62.7, s = 7.7$.

$H_0 : \mu \leq 60, H_a : \mu > 60$.

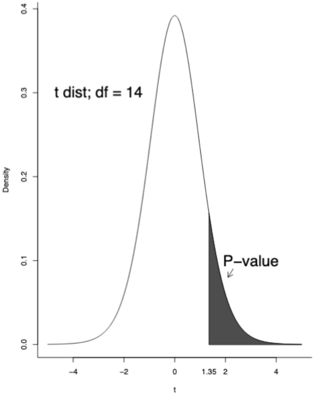
$t = \frac{\bar{y}-\mu_0}{s/\sqrt{n}} = \frac{62.7-60}{7.7/\sqrt{15}} = 1.35$.

p-value$=P(t > 1.35) = 0.101$.

R code: 1-pt(1.35,14)

Fail to reject H_0.

p-value for right sided test



Type I and Type II error

Type I error: Reject H_0 when it is true.
Type II error: Fail to reject H_0 when it is false.
α : prob. of making Type I error
β : prob. of making Type II error.
$1-\beta$: **power** of the test. Prob. of accepting H_a when it is true.

Example: An observation y comes from a normal distribution with μ and $\sigma = 1$. Test $H_0 : \mu = 0$ vs $H_a : \mu = 2$.
Test statistic: y.
Rejection region : $y > 1.645$, i.e., reject H_0 if $y > 1.645$.
$\alpha = P(\text{Type I error}) = P(H_0 \text{rejected when it is true}) = P(y > 1.645 \text{when} \mu = 0) = P(z > 1.645) = 0.05$.

$\beta = P(\text{type II error }) = P(H_0 \text{is not rejected when} \mu = 2) = P(y \leq 1.645 \text{when} \mu = 2) = P(z < -0.355) = 0.3613$.
R code for normal distribution pnorm(z) computes the cumulative probability up to z. qnorm(p) computes the quantile and the cumulative probability up to the quantile is p.
so pnorm(-0.355) gives $P(z < -0.355)$

Compute α and β for Rejection region: $y > 1.96$.

Power of the test

p. 33.
example: $y_1, \cdots, y_{25} \sim N(\mu, 1)$.
Test $H_0 : \mu = 0$ vs $H_a : \mu = 1$ with significance level $\alpha = 0.05$.
Test statistic is $z = \frac{\bar{y}-0}{\sigma/\sqrt{n}}$
Reject H_0 if $z > 1.645$ or
$\bar{y} > 1.645\frac{\sigma}{\sqrt{n}}$.
power$= 1 - \beta = P(\bar{y} > 1.645\frac{\sigma}{\sqrt{n}}; \mu = 1)$
$= P(z > \frac{(1.645\sigma/\sqrt{n})-1}{\sigma/\sqrt{n}}) = P(z > -3.355) = 0.9996$.
Q: what sample size is needed if we want $1 - \beta \geq 0.90$?

Solution

power$= 1 - \beta = P(\bar{y} > 1.645\frac{\sigma}{\sqrt{n}}; \mu = 1)$
$= P(z > \frac{(1.645\sigma/\sqrt{n})-1}{\sigma/\sqrt{n}})$
Note $P(z > -1.28) = 0.90$, so
let $\frac{(1.645\sigma/\sqrt{n})-1}{\sigma/\sqrt{n}} = -1.28$,
we have $\frac{(1.645/\sqrt{n})-1}{1/\sqrt{n}} = -1.28$, and
$n \approx 9$.

Power and sample size

H_a is true usually means there exists an effect and a test has high power means the test has a high probability of detecting an effect if it exists. Intuitively it is easier to detect an effect if its size is big. The standardized effect size can be expressed as $E = \frac{|\mu_a - \mu_0|}{\sigma}$, where μ_0, μ_a are mull and alternative values of μ. Table 2.3 shows the power of a one-sided one sample t test for given E and n at fixed $\alpha = 0.05$.

```
survival = c(61, 55, 68, 62, 65, 54, 70, 63, 56, 51, 72,
             63, 76, 53, 71) #write data in a vector

mean(survival) # compute the sample mean
var(survival) # compute the sample variance
sd(survival) # compute the sample standard deviation

t.test(survival)

#t.test performs t test: default for 0 null value and
two sided test, produces 95% CI.

t.test(survival, mu=60, conf.level=0.90)
#null value 60, 90% CI
t.test(survival, mu=60, alternative="less")
#left sided test
t.test(survival, mu=60, alternative="greater")
#right sided test
```

```
> survival = c(61, 55, 68, 62, 65, 54, 70, 63, 56, 51, 72,
+              63, 76, 53, 71)
> t.test(survival,mu=60,alternative="greater")

One Sample t-test

data:  survival
t = 1.3387, df = 14, p-value = 0.101
alternative hypothesis: true mean is greater than 60
95 percent confidence interval:
 59.15805      Inf
sample estimates:
mean of x
 62.66667
```

```
> t.test(survival,mu=60)

One Sample t-test

data:  survival
t = 1.3387, df = 14, p-value = 0.202
alternative hypothesis: true mean is not equal to 60
95 percent confidence interval:
 58.39415 66.93918
sample estimates:
mean of x
 62.66667
```