The Smith group and the critical group of the Grassmann graph of lines in a finite projective space.

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This is joint work with Peter Sin.

Outline

1. The skew-lines and Grassmann graphs
2. Integer invariants
3. Results
   - p-part
   - p'-part

- Γ, graph with vertices the 2-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$ ($q = p^t$), adjacent when far apart

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- Strongly regular with parameters

$$\begin{aligned} v' &= \left[{n \atop 2} \right]_q \\ k' &= q(q+1)\left[{n-2 \atop 1} \right]_q \\ \lambda' &= \left[{n-1 \atop 1} \right]_q + q^2 - 2 \\ \mu' &= (q+1)^2. \end{aligned}$$

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- $|K(\Gamma)|$ counts number of spanning trees

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$$\dim_{\mathbb{F}_p} \overline{M_i} = \dim_{\mathbb{F}_p} \overline{\ker(A)} + e_i + e_{i+1} + \cdots$$
and
$$\dim_{\mathbb{F}_p} \overline{N_i} = e_0 + e_1 + \cdots + e_i.$$

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- A has spectrum

$$q^4 \begin{bmatrix} n-2 \\ 2 \end{bmatrix}_q,\ -q^2 \begin{bmatrix} n-3 \\ 1 \end{bmatrix}_q,\ q$$

with respective multiplicities $1, \begin{bmatrix} n \\ 1 \end{bmatrix}_q - 1, \begin{bmatrix} n \\ 2 \end{bmatrix}_q - \begin{bmatrix} n \\ 1 \end{bmatrix}_q$.

Skew lines, A

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$$\begin{bmatrix} n \\ 1 \end{bmatrix}_q - 1 \leq e_{2t} + \cdots + e_{3t} \qquad (1)$$
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$$\begin{bmatrix} n \\ 2 \end{bmatrix}_q - \begin{bmatrix} n \\ 1 \end{bmatrix}_q \leq e_0 + \cdots + e_t. \tag{2}$$

- In general, if $A_{r,s}$ denotes the zero-intersection incidence matrix between r-spaces and s-spaces, then we have that
$$-A_{r,s} \equiv A_{r,1}A_{1,s} \pmod{p^t}$$

Skew lines, A

Theorem (Brouwer-D-Sin 2011)

Let e_i denote the multiplicity of p^i as an elementary divisor of $A_{2,1}A_{1,2}$.

1. $e_{4t} = 1$.
2. *For $i \neq 4t$,*

$$e_i = \sum_{\vec{s} \in \Gamma(i)} d(\vec{s}),$$

where

$$\Gamma(i) = \bigcup_{\substack{\alpha+\beta=i \\ 0 \leq \alpha \leq t \\ 0 \leq \beta \leq t}} {}_{\beta}\mathcal{H} \cap \mathcal{H}_{\alpha}.$$

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- L': no p-part

Notation: The vector space has dimension n over a field of $q = p^t$ elements.

(n, p, t)	matrix	(elem. div. : multiplicity)
(4, 2, 1)	A	$(2:14)$, $(2^2:8)$, $(2^3:6)$, $(2^4:1)$
	L	$(2:14)$, $(2^2:8)$, $(2^3:6)$ $(5:13)$ $(7:19)$
	A'	$(2:1)$ $(3:8)$, $(3^2:14)$
	L'	$(3:8)$, $(3^2:13)$ $(5:13)$ $(7:19)$

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(n, p, t)	matrix	(elem. div. : multiplicity)
(4, 2, 2)	A	$(2:16), (2^2:220), (2^4:32), (2^5:16), (2^6:36), (2^8:1)$
	L	$(2:16), (2^2:220), (2^4:32), (2^5:16), (2^6:36)$ $(3:1), (3^2:271)$ $(7:271)$ $(17:83)$
	A'	$(2^2:1)$ $(3:84)$ $(5:190), (5^2:84)$
	L'	$(3:271)$ $(5:190), (5^2:83)$ $(7:271)$; $(17:83)$

p' part

- ℓ, a prime different than p
- Structure of $\mathbb{F}_\ell \mathrm{GL}(n, q)$-permutation module on 2-spaces is able to be understood. (few composition factors)
- We rely heavily on work of G. James
- $\mathbb{F}_\ell^{V(\Gamma)}$ has descending filtration with subquotients as Specht modules
- Straightforward arithmetic conditions determine the composition factors and multiplicities of the Specht modules

An example: Grassmann, L'

Parameters:

$$k' = q(q+1)\begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q, \quad \mu' = (q+1)^2, \quad \lambda' = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + q^2 - 2,$$

$$r = q(q+1)\begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q - q^2\begin{bmatrix} n-3 \\ 1 \end{bmatrix}_q + 1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q,$$

$$s = q(q+1)\begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q + (q+1) = (q+1)\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q.$$

$$|K(\Gamma')| = \frac{\begin{bmatrix} n \\ 1 \end{bmatrix}_q^{\,f} \left((q+1)\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q\right)^g}{\begin{bmatrix} n \\ 2 \end{bmatrix}_q}$$

$$= \begin{bmatrix} n \\ 1 \end{bmatrix}_q^{\,f-1} (q+1)^{g+1} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q^{\,g-1}.$$

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- $|K(\Gamma')| = \left[{n \atop 1} \right]_q^{f-1} (q+1)^{g+1} \left[{n-1 \atop 1} \right]_q^{g-1}.$

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- $|K(\Gamma')| = {\left[{n \atop 1} \right]_q}^{f-1}(q+1)^{g+1}{\left[{n-1 \atop 1} \right]_q}^{g-1}$.
- Case: $\ell \nmid q+1$, $\ell \mid \left[{n \atop 1} \right]_q$, and $\ell \nmid \left[{n-1 \atop 1} \right]_q$

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- Case: $\ell \nmid q+1$, $\ell \mid \left[{n \atop 1}\right]_q$, and $\ell \nmid \left[{n-1 \atop 1}\right]_q$
- Let $a = v_\ell(\left[{n \atop 1}\right]_q)$. Then $v_\ell(|K(\Gamma')|) = a(f-1)$.

An example: Grassmann, L'

We have

$$f \leq \dim_F \overline{M}_a = 1 + \sum_{i \geq a} e_i,$$

Therefore,

$$a(f-1) = v_\ell(|K(\Gamma')|) = \sum_{i \geq 0} ie_i \geq \sum_{i \geq a} ie_i \geq a \sum_{i \geq a} e_i \geq a(f-1)$$

Since we must have equality throughout, it follows that $e_i = 0$ unless $i = 0$ or a, and that $e_a = f - 1$. Then as $\sum_{i \geq 0} e_i = g + f$, we have $e_0 = g + 1$.
Can abstract this argument:

Lemma

Let M, and $\phi \in \mathrm{End}_R(M)$. Let d be the π-adic valuation of the product of the nonzero elementary divisors of ϕ, counted with multiplicities. Suppose that we have an increasing sequence of indices $0 < a_1 < a_2 < \cdots < a_h$ and a corresponding sequence of lower bounds $b_1 > b_2 > \cdots > b_h$ satisfying the following conditions.

1. $\dim_F \overline{M}_{a_j} \geq b_j$ *for* $j = 1, \ldots, h$.
2. $\sum_{j=1}^{h}(b_j - b_{j+1})a_j = d$, *where we set* $b_{h+1} = \dim_F \overline{\mathrm{Ker}(\phi)}$.

Then the following hold.

1. $e_{a_j}(\phi) = b_j - b_{j+1}$ *for* $j = 1, \ldots, h$.
2. $e_0(\phi) = \dim_F \overline{M} - b_1$.
3. $e_i(\phi) = 0$ *for* $i \notin \{0, a_1, \ldots, a_h\}$.

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- Invariants of the Kneser graph on 2-subsets of an n-element set can be handled with similar theory. Much fewer cases.
- The critical group has order:

$$\frac{\left[\frac{n(n-3)}{2}\right]^f \left[\frac{(n-4)(n-1)}{2}\right]^g}{\frac{n(n-1)}{2}} = \frac{n^{f-1}(n-1)^{g-1}(n-3)^f(n-4)^g}{2^{f+g-1}}$$

where $f = n-1$ and $g = \frac{n(n-3)}{2}$

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- Result:

$$\begin{aligned} e_1 &= g + 1 - f, \\ e_a &= f - 1, \\ e_0 &= f, \\ e_i &= 0, \text{ otherwise.} \end{aligned}$$

Thank you for your attention!