Smith groups of integer matrices in the Bose-Mesner algebra of the Johnson scheme.

Josh Ducey
James Madison University

University of Wisconsin - Milwaukee
AMS Session: The algebras and special functions around association schemes

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In this talk I will describe work done at a 2023 summer REU at James Madison University, with my colleague Brant Jones and our students Jacob Gathje, Lauren Engelthaler, Isabel Pfaff and Jenna Plute.

Outline

Any integer matrix M, say $m \times n$, determines a finitely generated abelian group called the *Smith group* of M.

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Using unimodular matrices, M can be brought into a *diagonal form*

$$EMF = D = diag(d_1, d_2, \cdots, d_r, 0, \cdots, 0)$$

$$S(M) \;\cong\; \mathbb{Z}/d_1\mathbb{Z} \,\oplus\, \mathbb{Z}/d_2\mathbb{Z} \,\oplus\, \cdots \,\oplus\, \mathbb{Z}/d_r\mathbb{Z} \,\oplus\, \mathbb{Z}^{n-r}\,.$$

Interesting example: Inclusion of subsets

Let $r \leq s \leq n - r$. Denote by $W_{r,s}$ the incidence matrix with rows indexed by the r-subsets and columns indexed by the s-subsets of an n-element set.

$$W_{r,s}(R,S) = \begin{cases} 1 \text{ if } R \subseteq S, \\ 0 \text{ otherwise.} \end{cases}$$

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where $0 \leq j \leq r$ and $\mu_j = \binom{n}{j} - \binom{n}{j-1}$. (Wilson, 1990)

Wilson:

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We study:

$$\left(L_r^T\right)^{-1} W_{r,s} L_s^T = ?$$

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We study:

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$P_r := L_r^T$

$$P_r^{-1} W_{r,s} P_s = ?$$

Bier's P Matrix

$n = 5$
$P_2 =$

Standard $\leq$ 2-Subsets

2-Subsets

	$\overline{\emptyset}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{24}$	$\overline{25}$	$\overline{34}$	$\overline{35}$	$\overline{45}$
$\overline{12}$	1	1	0	0	0	0	0	0	0	0
$\overline{13}$	1	0	1	0	0	0	0	0	0	0
$\overline{14}$	1	0	0	1	0	0	0	0	0	0
$\overline{15}$	1	0	0	0	1	0	0	0	0	0
$\overline{23}$	1	1	1	0	0	0	0	0	0	0
$\overline{24}$	1	1	0	1	0	1	0	0	0	0
$\overline{25}$	1	1	0	0	1	0	1	0	0	0
$\overline{34}$	1	0	1	1	0	0	0	1	0	0
$\overline{35}$	1	0	1	0	1	0	0	0	1	0
$\overline{45}$	1	0	0	1	1	0	0	0	0	1

$P_r^{-1} W_{r,s} P_s = U$, where

$$U = \left(\begin{array}{c|c|c|c|c}
f_{0,0}\widetilde{W_{0,0}} & f_{0,1}\widetilde{W_{0,1}} & f_{0,2}\widetilde{W_{0,2}} & \cdots & f_{0,s}\widetilde{W_{0,s}} \\ \hline
0 & f_{1,1}\widetilde{W_{1,1}} & f_{1,2}\widetilde{W_{1,2}} & \cdots & f_{1,s}\widetilde{W_{1,s}} \\ \hline
0 & 0 & \ddots & \ddots & \vdots \\ \hline
0 & 0 & 0 & \ddots & f_{r-1,s}\widetilde{W_{r-1,s}} \\ \hline
0 & 0 & 0 & 0 & f_{r,s}\widetilde{W_{r,s}}
\end{array}\right)$$

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The $\widetilde{W_{i,j}}$ are the inclusion matrices of *standard* i-subsets vs *standard* j-subsets. The $f_{i,j}$ are easily determined.

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The $\widetilde{W_{i,j}}$ are the inclusion matrices of *standard* i-subsets vs *standard* j-subsets. The $f_{i,j}$ are easily determined.
This same result holds even when $W_{r,s}$ is the incidence matrix describing intersection in a set of size ℓ.

Outline

1. Diagonal forms of integer matrices
2. Association matrices of Johnson Scheme
3. Some Examples

Let $A = A_{n,k,\ell}$ be the distance $k - \ell$ association matrix of the Johnson scheme $J(n, k)$.

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$$P_k^{-1} A P_k = U,$$

where

$$U = \left(\begin{array}{c|c|c|c|c}
\lambda_{0,\ell} I & f_{0,1,\ell}\widetilde{W_{0,1}} & f_{0,2,\ell}\widetilde{W_{0,2}} & \cdots & f_{0,k,\ell}\widetilde{W_{0,k}} \\ \hline
0 & \lambda_{1,\ell} I & f_{1,2,\ell}\widetilde{W_{1,2}} & \cdots & f_{1,k,\ell}\widetilde{W_{1,k}} \\ \hline
0 & 0 & \ddots & \ddots & \vdots \\ \hline
0 & 0 & 0 & \ddots & f_{k-1,k,\ell}\widetilde{W_{k-1,k}} \\ \hline
0 & 0 & 0 & 0 & \lambda_{k,\ell} I
\end{array}\right)$$

We discovered a diagonal form for the $\widetilde{W_{i,j}}$, analogous to Wilson's form, and all of these standard inclusion matrices can be diagonalized at once.

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If $k \leq \frac{n+1}{3}$, there exists a sequence $\{E_r\}$ of unimodular matrices satisfying

$$E_i \widetilde{W_{i,j}} E_j^{-1} = \widetilde{D_{i,j}} = diag \left(\binom{j-s}{i-s}^{\mu_s - \mu_{s-1}} \right),$$

where $0 \leq s \leq i$ and $\mu_s = \binom{n}{s} - \binom{n}{s-1}$.

$k = 3$

$$U \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
\lambda_0 & f_{0,1} & 0 & \cdots & 0 & f_{0,2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & f_{0,3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & \lambda_1 & 0 & \cdots & 0 & 2f_{1,2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 3f_{1,3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & & \vdots & 0 & f_{1,2} & 0 & \cdots & \cdots & \cdots & 0 & 0 & f_{1,3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 & f_{1,2} & 0 & \cdots & 0 & 0 & \cdots & 0 & f_{1,3} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
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0 & 0 & \cdots & \cdots & 0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 3f_{2,3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
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\end{array}\right]$$

$k = 3$

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U \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
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$k = 3$

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\end{array}\right]
$$

Main Theorem

Let $A = A_{n,k,\ell}$ where $k \leq \frac{n+1}{3}$.
Define, for $s = 0, 1, \cdots, k$, the $(k+1-s) \times (k+1-s)$ matrix $\mathcal{M}_s(A)$

$$\mathcal{M}_s(A)(i,j) = \binom{j-s}{i-s} f_{i,j,\ell},$$

where $f_{i,j,\ell} = \sum_{v=0}^{i} (-1)^{i+v} \binom{i}{v} \binom{k-v}{\ell-v} \binom{n-k-j+v}{k-\ell-j+v}$.
(We index the matrix entries $s \leq i, j \leq k$.)
Then

$$S(A) \cong \bigoplus_{s=0}^{k} S(\mathcal{M}_s)^{\mu_s - \mu_{s-1}}.$$

Outline

Johnson graph adjacency, $k = 3$

$$\mathcal{M}_0 = \begin{bmatrix} 3(n-3) & 3 & 0 & 0 \\ 0 & 2n-9 & 4 & 0 \\ 0 & 0 & n-7 & 3 \\ 0 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} (n-3)(2n-9) & 2n & 4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & n-7 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\mathcal{M}_1 = \begin{bmatrix} 2n-9 & 2 & 0 \\ 0 & n-7 & 2 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3(2n-9)(n-7) \end{bmatrix}$$

$$\mathcal{M}_2 = \begin{bmatrix} n-7 & 1 \\ 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 3(n-7) \end{bmatrix}$$

$$\mathcal{M}_3 = \begin{bmatrix} -3 \end{bmatrix}.$$

Johnson graph adjacency, $k = 3$

Theorem

The adjacency matrix of the $k = 3$ Johnson graph $n \geq 7$ has a diagonal form with:

entry	with multiplicity	if ...
$3(n-7)$	$\binom{n}{2} - 2n + 1$	always
$3(2n-9)(n-7)$	$n - 2$	always
3	$\binom{n}{3} - 2\binom{n}{2} + n + 1$	always
$3(n-3)(n-7)(2n-9)/X, X$	1	always
1	$\binom{n}{2} - 2$	always

where

$$X = \gcd(3(n-3)(2n-9), (n-7)(n-3)(2n-9), 12, 2n(n-7), 3(n-7)).$$

Kneser graph adjacency, arbitrary k

We compute

$$\mathcal{M}_s(i,j) = (-1)^i \binom{j-s}{i-s} \binom{n-k-j}{k-j}.$$

Kneser graph adjacency, arbitrary k

We compute

$$\mathcal{M}_s(i,j) = (-1)^i \binom{j-s}{i-s} \binom{n-k-j}{k-j}.$$

Each of the diagonal entries $M_s(j,j)$ divides each entry $M_s(i,j)$ above it $(i < j)$. Thus every $\mathcal{M}_s$, and therefore $\mathcal{A}_{n,k,0}$, is unimodularly equivalent to the diagonal matrix of its eigenvalues.

Theorem

The adjacency matrix of the Kneser graph $\Gamma(n, k, \ell = 0)$ *for* $n \geq 3k - 1$ *has a diagonal form with:*

entries	*with multiplicity*	*or in terms of eigenvalues ...*
$\binom{n-k-j}{k-j}$	$\binom{n}{j} - \binom{n}{j-1}$	$[e_j]^{\mu_j}$ *for* $0 \leq j \leq k$

Integer matrices in BM-algebra of $J(n, k)$

Let $n = 12, k = 3$ and consider the association matrices $A_1 = \mathcal{A}_{12,3,1}$ and $A_2 = \mathcal{A}_{12,3,2}$. Let $B = A_1 + 3A_2$.

Integer matrices in BM-algebra of $J(n, k)$

Let $n = 12, k = 3$ and consider the association matrices $A_1 = \mathcal{A}_{12,3,1}$ and $A_2 = \mathcal{A}_{12,3,2}$. Let $B = A_1 + 3A_2$. We compute the $\mathcal{M}_s$ matrices for B:

$$\mathcal{M}_0(B) = \mathcal{M}_0(A_1) + 3\mathcal{M}_0(A_2) = \begin{bmatrix} 189 & 33 & 3 & 0 \\ 0 & 57 & 22 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

$$\mathcal{M}_1(B) = \mathcal{M}_1(A_1) + 3\mathcal{M}_1(A_2) = \begin{bmatrix} 57 & 11 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -6 \end{bmatrix}$$

$$\mathcal{M}_2(B) = \mathcal{M}_2(A_1) + 3\mathcal{M}_2(A_2) = \begin{bmatrix} 2 & 1 \\ 0 & -6 \end{bmatrix}$$

$$\mathcal{M}_3(B) = \mathcal{M}_3(A_1) + 3\mathcal{M}_3(A_2) = \begin{bmatrix} -6 \end{bmatrix}$$

$$
\begin{aligned}
S(B) &\cong S(\mathcal{M}_0(B))^1 \oplus S(\mathcal{M}_1(B))^{10} \oplus S(\mathcal{M}_2(B))^{43} \oplus S(\mathcal{M}_3(B))^{100} \\
&\cong \left(\mathbb{Z}/3\,\mathbb{Z}^2 \oplus \mathbb{Z}/14364\,\mathbb{Z}\right) \oplus (\mathbb{Z}/2\,\mathbb{Z} \oplus \mathbb{Z}/342\,\mathbb{Z})^{10} \\
&\quad \oplus (\mathbb{Z}/12\,\mathbb{Z})^{43} \oplus (\mathbb{Z}/6\,\mathbb{Z})^{100}.
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&\quad \oplus (\mathbb{Z}/12\mathbb{Z})^{43} \oplus (\mathbb{Z}/6\mathbb{Z})^{100}.
\end{aligned}
$$

So we can compute Smith groups of Laplacians $L = dI - A$, etc.

Kneser graph Laplacian, $k = 3$

$$\mathcal{M}_0 = \begin{bmatrix} 0 & \frac{1}{2}(n-4)(n-5) & n-5 & 1 \\ 0 & -\frac{1}{6}(n-4)(n-5)n & -2(n-5) & -3 \\ 0 & 0 & -\frac{1}{6}(n-1)(n-5)(n-6) & 3 \\ 0 & 0 & 0 & -\frac{1}{6}(n^2-10n+27)(n-2) \end{bmatrix}$$

$$\mathcal{M}_1 = \begin{bmatrix} -\frac{1}{6}(n-4)(n-5)n & -(n-5) & -1 \\ 0 & -\frac{1}{6}(n-1)(n-5)(n-6) & 2 \\ 0 & 0 & -\frac{1}{6}(n^2-10n+27)(n-2) \end{bmatrix}$$

$$\mathcal{M}_2 = \begin{bmatrix} -\frac{1}{6}(n-1)(n-5)(n-6) & 1 \\ 0 & -\frac{1}{6}(n^2-10n+27)(n-2) \end{bmatrix}$$

$$\mathcal{M}_3 = \begin{bmatrix} -\frac{1}{6}(n^2-10n+27)(n-2) \end{bmatrix}$$

Kneser graph Laplacian, $k = 3$

Theorem

The classical Laplacian of the $k = 3$ Kneser graph $\Gamma(n, 3, \ell = 0)$ for $n \geq 7$ has a diagonal form with:

entries	*with multiplicity*
$\frac{1}{6}(n^2 - 10n + 27)(n - 2)$	$\binom{n}{3} - 2\binom{n}{2} + n$
$\frac{1}{36}(n^2 - 10n + 27)(n - 1)(n - 2)(n - 5)(n - 6)$	$\binom{n}{2} - 2n + 1$
1	$\binom{n}{2} - n$
$\frac{\frac{1}{216}(n^2-10n+27)(n-1)(n-2)(n-4)(n-5)^2(n-6)n}{X}, X$	$n - 2$
$\frac{\frac{1}{36}(n^2-10n+27)(n-4)(n-5)^2(n-6)}{Y}, Y$	1
0	1

where X and Y are gcds of some minors, given on the next slide.

$$X = \gcd\left(\frac{(n-4)(n-5)n}{3}, \frac{(n^2-7n+18)(n-5)}{6}, \frac{(n^2-10n+27)(n-2)(n-4)(n-5)n}{36}\right)$$

$$Y = \gcd\left(n-5,\ \frac{3(n-4)(n-5)}{2},\ \frac{(n-1)(n-3)(n-5)}{3}\right)$$

Thank you for your attention!