Math 485

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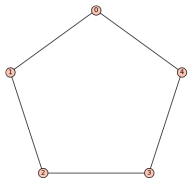
1 Strongly Regular Graphs

We say a graph $G(V, E)$ where $|V| = v$ is **strongly regular** if the following are true.

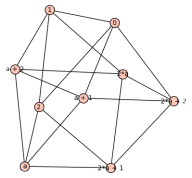
- G is a regular graph with valency k.
- Any two adjacent vertices have λ common neighbors.
- Any two non-adjacent vertices have μ common neighbors.

We can then say G has a parameter set (v, k, λ, μ).

Example: The cycle graph with 5 vertices is an SRG with parameters $(5, 2, 0, 1)$



The Paley(9) graph is an $srg(9, 4, 1, 2)$.



An important question regarding SRGs is whether it is possible to construct a graph for any given parameter set. Some parameter sets can easily be shown to be impossible using established bounds that will be discussed later, but other parameter sets are not easily dismissed using these methods. We will now discuss some of the tools used to determine the non-existence of certain parameter sets for SRGs.

We define the **adjacency matrix**, A, of G to be the $v \times v$ matrix, with vertices ordered such that the n-th row and n-th column correspond the n-th vertex of the graph. Then, if v_i is adjacent to v_j, $A(i,j) = 1$, or if v_i is not adjacent to v_j then $A(i,j) = 0$.

We also similarly define the **Laplacian matrix**, L, to be the $v \times v$ matrix where entries corresponding to adjacent vertices are -1 and the degree of each vertex for the diagonal entry. For n-regular graphs, $L = nI - A$.

Example: The adjacency and Laplacian matrices of the cycle graph with 5 vertices are as follows.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}$$

For a graph $G = srg(v, k, \lambda, \mu)$ the adjacency matrix of G satisfies the following equation where J is the all ones matrix.

$$A^2 = kI + \lambda A + \mu(J - A - I)$$

Since G is k-regular, A has an eigenvalue of k corresponding to the all ones eigenvector. We can then use the fact that all other eigenvectors must be orthogonal to the all ones vector to derive the following quadratic equation which has solutions of the other eigenvalues of A.

$$c^2 + (\mu - \lambda)c + (\mu - k) = 0$$

We call the two solutions to this equation r and s. Now, using the fact that the $tr(A) = 0$ and the sum of the multiplicities of the eigenvalues must equal the number of vertices, we are left with the following system of equations which can be used to solve for the multiplicities of r and s, which we will call f and g.

$$k + fr + gs = 0$$
$$f + g + 1 = v$$

Also, since the Laplacian matrix for SRGs can be expressed as $L = kI - A$, we know the spectrum of L is as follows where the multiplicities of each eigenvalue is the same as the corresponding one for A.

$$k_L = 0$$

$$r_L = k - r$$

$$s_L = k - s$$

This shows us that based solely off the parameters of an SRG, we are able to obtain full spectral information for both the adjacency and Laplacian matrices. In further sections we will use this spectral information to gain information about other graph invariants that we can use to show non-existence for certain parameter sets of SRGs.

Matrix Tree Theorem:

Given the Laplacian of a srg(v, k, λ, μ) and the spectrum of L $= r^f, s^g, 0$

The size of the critical group of the graph is $|K(\gamma)| = \frac{r^f s^g}{v}$

Which is equivalent to $(Z/pZ)^{e_1} \oplus (Z/p^2Z)^{e_2}(p^1)^{e_1}(p^2)^{e_2}(p^3)^{e_3}$
From this we look at v_p as the "exponent of p" where

$$v_p|k(\gamma)| = e_1 + 2e_2 + 3e_3 + ...$$

Giving

$$A^2 + 0A + 0I = \mu J$$

$$(A - r_A I)(A - s_A I) = \mu J$$

$$(L - rI)(L - sI) = \mu J$$

$$L^2 - (r + s)L + rsI = \mu J$$

$$rsI = L(L - (r + s)I)$$

2 Critical Groups

Let $G = (V, E)$ be a graph with vertices $V = \{v_1, v_2, \ldots, v_n\}$. Consider assigning to each v_i a number such that the vertices sum to 0. We will call these numbers the *chip values* of our vertex set. Define *chip firing* from a vertex v_i as a function $f_i : \mathbb{Z}^V \to \mathbb{Z}^V$ that increases the chip value of each neighbor of v_i by 1 and decreases the chip value of v_i by the degree of v_i.

Note that if

$$\vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is our chip values for each vertex, then $f_i(\vec{c}) = \vec{c} - L_i$ where L_i is the i'th column of the Laplacian matrix of G.

Define an equivalence relation $\sim$ on vertex value vectors c, c' by saying $c \sim c'$ if there exists a composition of chip firing functions from one chip value

arrangement to the other. The set of chip values summing to zero under this equivalence relation is a finite abelian group called the critical group.

The laplacian matrix is a map $L : \mathbb{Z}^V \to \mathbb{Z}^V$. Then

$$\mathbb{Z}^V / Im(L) = Coker(L) \cong K(G) \oplus \mathbb{Z}$$

Where $K(G)$ is the cirical group of G.

3 Smith Groups

In general, if M is a map from $\mathbb{Z}^V \to \mathbb{Z}^V$, we call $S(M) = \mathbb{Z}^V / Im(M)$ the **smith group** of M. In this way the critical group and the smith group are heavily related, as the smith group of the laplacian matrix of a graph G is $S(L) = K(G) \oplus \mathbb{Z}$ as before. Note that this group is almost finite abelian, just with an extra copy of $\mathbb{Z}$.

Definition 3.1. *Let $M : \mathbb{Z}^V \to \mathbb{Z}^V$. Then the **smith normal form** of M is the diagonal matrix derived from performing elementary integer row and column operations on M.*

Observation 3.2. *Let $S(M) = \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_k\mathbb{Z}$. Then the values $n_1, \ldots n_k$ are the diagonal entries of the smith normal form of M.*

Note that we can always construct the smith normal form of a matrix such that the diagonal entries are written in ascending order such that $s_i | s_{i+1}$.

Define

$$\mathbb{Z}_{(p)} := \{\frac{a}{b} \in \mathbb{Q} \mid p \not| \, b\}$$

Then the diagonal entries of the smith normal form of the laplacian matrix of a graph will look like

$$1, \ldots, p, \ldots, p^2, \ldots, p^3, \ldots, p^n, \ldots, 0$$

Definition 3.3. *let p be a fixed prime p Let e_i = the number of invariant factors exactly divisible by p^i.*

For $C : \mathbb{Z}^V \to \mathbb{Z}^V$ we have $M_i = \{\overrightarrow{x} \in \mathbb{Z}^V_{(p)} \mid p_i \text{ divides } L\overrightarrow{x}\}$

Note that

$$\mathbb{Z}^V_{(p)} = M_0 \supset M_1 \supset M_2 \supset \ldots$$

Then $\dim_p \overline{M_i} = e_i + e_{i+1} + \cdots + e_{i+n} + \dim \overline{\ker C}$.

Definition 3.4. *$N_i = \{p^{-i} C x \mid x \in M_i\}$. Then*

$$\dim_p \overline{N_i} = e_0 + e_1 + \cdots + e_i.$$

Using this we arrive at the following theorems

Theorem 3.5. *Let A be the adjacency matrix of $G = srg(v, k\lambda, \mu)$ and $M = A + bI$ be nonsingular. If*

$$p \| r_M,\ p \| s_M,\ p \not| v,\ p^\delta \| k_M,$$

then $S_p(M) \cong \mathbb{Z}/p^\delta\mathbb{Z} \oplus \mathbb{Z}^{v-1-2\widetilde{e_0}}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$, where

$$\widetilde{e_0} = \begin{cases} e_0 & \delta > 0 \\ e_0 - 1 & \delta = 0 \end{cases}.$$

Proof. Suppose that A is the adjacency matrix of an $srg(v, k, \lambda, \mu)$ and has spectrum, $spec(A) = [k]^1, [r]^f, [s]^g$. Then we know that $spec(M) = [k_M]^1, [r_M]^f, [s_M^g$, where $k_M = k + b, r_M = r + b, s_M = s + b$. Also suppose for a prime, p, that $p||r_M, p||s_M, p^\gamma||v$, and $p^\delta||k_M$. By Kirchhoff's Matrix Tree Theorem, we know $|S(A + bI)| = k_M r_M^f s_M^g$, so when we reduce modulo p, we obtain the following order for the p part of the Smith Group.

$$|S_p(M)| = p^\delta p^f p^g = p^{\delta+f+g}$$

We then conclude that $S_p(A)$ has the following form.

$$S_p(M) = (\mathbb{Z}/p\mathbb{Z})^{e_1} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{e_2} \oplus (\mathbb{Z}/p^\delta\mathbb{Z})$$

If we let $M : \mathbb{Z}_p^v \to \mathbb{Z}_p^v$, then, since $p \not| v$ we can decompose the domain into $\mathbb{Z}_{(p)}\mathbf{1} \oplus Y$, where Y is the set of vectors that sum to 0. M respects this decomposition so will send vectors in Y to other vectors in Y. It follows from the equation $(M - r_M I)(M - s_M I) = \mu J$, that once we restrict the domain of M to Y, $(M - r_M I)(M - s_M I) = 0$. This means that with this restriction rs kills the group, so the maximum invariant factor is p^2. When we consider the whole domain we gain one more possible factor from our eigenvalue, p^δ, as a result of the all ones vector and its eigenvalue,k. This gives us the form of our group from above.

We know that the number of invariant factors is the same of the number of entries on the diagonal of $A + bI$, and this formula gives us a new way to calculate the order of the group, giving us the following system of equations

$$e_0 + e_1 + e_2 + 1 = f + g + 1$$
$$e_1 + 2e_2 + 1 = \delta + f + g$$

Solving this system of equations gives us the following result.

$$e_1 = f + g + 1 - \delta - 2e_0$$

$$e_2 = \delta + e_0 - 1$$

The result of the theorem follows directly.

□

Theorem 3.6. *Let A be the adjacency matrix of $G = srg(v, k\lambda, \mu)$ and $M = A + bI$ be nonsingular. If*

$$p^a \| r_M,\ p \not| s_M,\ p \not| v,\ p^\delta \| k_M,$$

then $S_p(M) \cong \mathbb{Z}/p^\delta\mathbb{Z} \oplus (\mathbb{Z}/p^a\mathbb{Z})^f$

Proof. We take a similar approach to this theorem as Theorem 3.5. The spectrum of M is the same as before but this time the order of the p-part of our group is not affected by the eigenvalue that p does not divide. Therefore,

$$S_p(M) = p^\delta p^f = p^{\delta+f}$$

.

When we restrict M to Y we again see that rs kills the critical group giving us a maximum invariant factor of e_a. As a consequence,

$$e_1 + 2e_2 + 3e_3 + ... + ae_a = ag$$

This implies that $g/geqe_a$.

However we also know that $e_0 + e_1 + e_2 + ... + e_a = f + g + 1$, which implies $g \leq e_a$. Therefore $e_a = g$. For the same reason as before we also have an extra invariant factor of p^δ, and the result follows. □

Theorem 3.7. *Let A be the adjacency matrix of $G = srg(v, k\lambda, \mu)$ and $M = A + bI$ be nonsingular. If*

$$p^a \| s,\ p \not| r,\ p^\gamma \| v\ (\gamma \geq 1),\ p \| k\ (\delta = 1),$$

then if all elementary divisors p^i have $i \leq a$, $S_p(A) \cong \mathbb{Z}/p\mathbb{Z} \oplus (\mathbb{Z}/p^a\mathbb{Z})^g$.

Otherwise, there exists exactly one elementary divisor p^b with $b > a$, and $S_p(A) \cong (\mathbb{Z}/p^a\mathbb{Z})^{g-1} \oplus \mathbb{Z}/p^{a+1}\mathbb{Z}$.

Proof. □

Theorem 3.8. *Let A be the adjacency matrix of $G = srg(v, k\lambda, \mu)$ and $M = A + bI$ be nonsingular. If*

$$p \not| s,\ p \not| r,\ p^\delta \| k,$$

Then $S_p(A) \cong \mathbb{Z}/p^\delta\mathbb{Z}$.

Proof. This follows immediately from Theorem 3.6, with $a = 0$. □

Theorem 3.9. *Let A be the adjacency matrix of $G = srg(v, k\lambda, \mu)$ and $M = A + bI$ be nonsingular. If*

$$p^a \| s,\ p \not| r,\ p \not| k,$$

Then $S_p(A) \cong (\mathbb{Z}/p^a\mathbb{Z})^g$.

Proof. This follows from Theorem 3.6 with $\delta = 0$ □

Theorem 3.10. *Let A be the adjacency matrix of $G = srg(v, k\lambda, \mu)$ and $M = A + bI$ be nonsingular. If*

$$p^2 \| s,\ p \| r,\ p \not| v,\ p^\delta \| k,$$

Then $S_p(A) \cong \mathbb{Z}/p^\delta\mathbb{Z} \oplus (\mathbb{Z}/p\mathbb{Z})^{f-\widetilde{e_0}} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{g-\widetilde{e_0}} \oplus (\mathbb{Z}/p^3\mathbb{Z})^{\widetilde{e_0}}$.

Proof. As before restrict the domain of M to Y. We know rs kills the group, which means that p^3 is our maximum elementary divisor. Therefore,

$$e_0 + e_1 + e_2 + e_3 = f + g + 1$$

$$e_1 + 2e_2 + 3e_3 = f + 2g$$

We also know $f \leq e_0 + e_1$ and $g \leq e_2 + e_3 + 1$, which implies $f + g \leq f + g + 1$. This gives us two cases $f = e_0 + e_1$ and $g + 1 = e_2 + e_3 + 1$ or $f + 1 = e_0 + e_1$ and $g = e_2 + e_3 + 1$.

However, e_i for $M|_Y$ is has the same value as e_{3-i}. This implies $e_3 = \widetilde{e_0}$. Knowing this determines the rest of the invariant factors. Combining this with the invariant factor of p^δ from the all ones vector and the result follows.

□

4 Existence of SRGs

A SRG with parameters v, k, λ, μ, where $\mu \neq 0$ has the following property: (v-k-1)μ = (k-λ-1)k.

Suppose there exists a SRG Γ with parameters v, k, λ, μ. Fix a vertex x then we define subconstituent $\Gamma_1(x)$ to be the vertices adjacent to x. Next, we define subconstituent $\Gamma_2(x)$ to be the vertices remaining. We know there are k vertices in $\Gamma_1(x)$ meaning there must be v-k-1 vertices in $\Gamma_2(x)$. By definition any two nonadjacent vertices share μ neighbors. So we pick any one vertex in $\Gamma_2(x)$ and know that there are μ edges from that vertex to a vertex in $\Gamma_1(x)$. Therefore, there are μ(v-k-1) edges connecting $\Gamma_2(x)$ to $\Gamma_1(x)$.

Similarly, we can work from $\Gamma_1(x)$ towards $\Gamma_2(x)$. There are k vertices in $\Gamma_1(x)$. Since every adjacent vertices has λ common neighbors, any given vertex in $\Gamma_1(x)$ must connect to x and also must connect to λ vertices that are also connected to x. The vertices that are connected to x are in $\Gamma_1(x)$. Therefore, a vertex in $\Gamma_1(x)$ has k - λ - 1 edges going to $\Gamma_2(x)$. Thus we can say that there are k(k-λ-1) edges connecting $\Gamma_1(x)$ to $\Gamma_2(x)$. The number of edges from $\Gamma_2(x)$ to $\Gamma_1(x)$ and from $\Gamma_1(x)$ to $\Gamma_2(x)$ is the same therefore (v-k-1)μ = (k-λ-1)k.

There are additional methods to limit to number of possible parameter sets including the Krein conditions and absolute bound.

From the paper "Generalized Krein Parameters of a SRG" by Vieira and Mano,

the Krein conditions are as follows. Suppose there exists a SRG Γ with parameters v, k, λ, μ, then the eigenvalues of Γ are k, θ, τ where:

$$\theta = (\lambda - \mu + \sqrt{(\lambda-\mu)^2 + 4(k-\mu)})/2$$
$$\tau = (\lambda - \mu - \sqrt{(\lambda-\mu)^2 + 4(k-\mu)})/2$$

Then the Krein conditions are:

$$(\theta+1)(k+\theta+2\theta\tau) \leq (k+\theta)(\tau+1)^2$$
$$(\tau+1)(k+\tau+2\theta\tau) \leq (k+\tau)(\theta+1)^2$$

From the paper "Strongly Regular Graphs" by Brouwer and Maldeghem, the absolute bound is as follows.

Suppose there exists a SRG Γ with parameters v, k, λ, μ, then the eigenvalues of Γ are k, θ, τ. The multiplicities of the eigenvalues f and g are defined as:

$$\text{f} = 1/2(n-1-\tfrac{(\theta+\tau)(v-1)+2k}{\theta-\tau})$$
$$\text{g} = 1/2(n-1+\tfrac{(\theta+\tau)(v-1)+2k}{\theta-\tau})$$

Then the absolute bounds are:

$$v \leq 1/2f(f+3)$$
$$v \leq 1/2g(g+3)$$

When the krein or absolute bound inequalities are not true, we know that the parameter set is not possible.

Example:
A SRG Γ with parameters (64, 21, 0, 10) can be shown to not exist using absolute bounds. The multiplicities f and g of the eigenvalue of Γ are f=56, g=7. Then $v \leq 1/2f(f+3)$ is $64 \leq \frac{56}{2}(56+3)$ or $64 \leq 28(59)$. This inequality is true. However the second absolute bound, $v \leq 1/2g(g+3)$, becomes $64 \leq \frac{7}{2}(7+3)$. When this expression is evaluated we get $64 \leq 35$. This inequality is not true therefore (64, 21, 0, 10) is not a possible parameter set for a strongly regular graph.
A similar strategy using Krein conditions allows us to reach the same conclusion. For the SRG(64, 21, 0, 10), $\theta = 1$ and $\tau = -11$. Then the second Krein condition is $(1+1)(21+11+2*1*11) \leq (21+1)(11+1)^2$. This expression is equivalent to $2(32+22) \leq (22)(12)^2$ or $108 \leq 22(144)$ thus this Krein condition does not hold and once again we have shown that (64, 21, 0, 10) is not a possible parameter set for a SRG.

In addition to using bounds to restrict possible parameter sets, it is also possible to use information on the groups of the SRG to find a contradiction. Again, suppose there exist a SRG Γ with parameters (64, 21, 0, 10). Then the spectrum of the adjacency matrix $A_\Gamma = 1^{56}, -11^7, 21^1$. Using the quadratic equation from section 1, $A^2 = kI + \lambda A + \mu(J - A - I)$ we get (A+11I)(A-I)=10J. That expression modulus 11 is $\bar{A}(\bar{A} - I) = 10J$. This expression tell us that A maps the columns of A - I to 10J, and differences of columns of $A - I$ are in the kernal of A. From theorem 3.6, we have that the 11 part of smith group of A is $S(A) \cong (\frac{\mathbb{Z}}{11\mathbb{Z}})^7$ When we evaluate modulus 11, the 7 instances of 11 in the Smith Normal Form all become 0. We know then that the dimension of the kernel of the SNF for the 11 part is 7. Now suppose we choose two adjacent vertices in this SRG Γ x and y. We define the set of all points connected to x in this graph to be subconstituent $X = \Gamma(x)$ and the set of all points connected to y in this graph to be subconstituent $Y = \Gamma(y)$. We also define Z to be the set of vertices that are not x, y, in X, or in Y. From the parameter set we see that the subgraph Z is composed of 11 unconnected copies of P_2. We can then arrange the columns of $A - I$ such that x is next to y and adjacent vertices in Z are next to each other.

$$A - I = \left[\begin{array}{c|c|cccccc|c} -1 & 1 & 0 & 0 & 0 & 0 & ... & 1 & 0 \\ \hline 1 & -1 & 0 & 0 & 0 & 0 & ... & 0 & 1 \\ \hline 0 & 0 & -1 & 1 & 0 & 0 & ... & ? & ? \\ 0 & 0 & 1 & -1 & 0 & 0 & ... & ? & ? \\ 0 & 0 & 0 & 0 & -1 & 1 & ... & ? & ? \\ 0 & 0 & 0 & 0 & 1 & -1 & ... & ? & ? \\ . & . & . & . & . & . & ... & . & . \\ . & . & . & . & . & . & ... & . & . \\ . & . & . & . & . & . & ... & . & . \\ \hline 1_{20} & 0_{20} & ? & ? & ? & ? & ? & ? & \\ \hline 0_{20} & 1_{20} & ? & ? & ? & ? & ? & ? & \end{array}\right]$$

If we take the difference of the first twelve pairs of columns we see that there are 12 linearly independent vectors in the kernel of A. This is a contradiction to the fact that the group implies a kernel of dimension 7. Therefore a strongly regular graph with this parameter set does not exist.

5 Connection with Automorphisms

We say $\phi : V \to V$ is a **graph automorphism** if for any vertices $v, w \in V$, if v is adjacent to w, then $\phi(v)$ is adjacent to $\phi(w)$. Graph automorphisms can be thought of as the subset of permutations of the vertices of a graph which preserve adjacency. The set of automorphisms of a graph under the binary operation of

composition forms a group, $Aut(G)$. For some graphs for which existence is unknown, information about constraints on its automorphism group is known. In this section we will discuss the connection between the critical group of a graph and its automorphisms.

For any automorphism, P, of a graph. There is an induced automorphism of the critical group of the graph. If our graph has an integer labeling, then P permutes the order of entries in the labeling. taking one element of the critical group and mapping it to another element. For some graphs there are automorphisms that have the property that for any element, g, in the critical group, $Pg = g$ i.e. all elements of the critical group are mapped to themselves. In this case we call the automorphism **group preserving**. If an automorphism is not group preserving, then we call it **group permuting**.

For any graph the identity automorphism is group preserving. Also the composition of any two group preserving automorphisms is also group preserving. As such we can see that the set of group preserving automorphisms,P_0 is itself a subgroup of $Aut(G)$. In fact we know this must be a normal subgroup since $P_0 = ker(\psi)$ where $\psi : Aut(G) \to Aut(K(G))$ is the induced homomorphism between the automorphism group of the graph and the automorphism group of the graph's critical group.

Example: For the 5-cycle, $G = srg(5, 2, 0, 1)$, we know that $Aut(G) \cong D_5$ and $K(G) \cong \mathbb{Z}/5\mathbb{Z}$. In this case the group preserving automorphisms are those that correspond to rotations of the pentagon, so $P_0 = ker(\psi) \cong \mathbb{Z}/5\mathbb{Z}$, which is in fact a normal subgroup of D_5. Also by the first isomorphism theorem, we know that $Aut(G)/ker(\psi) \cong Im(\psi) \cong \mathbb{Z}/2\mathbb{Z} \subset Aut(K(G))$.

In general, however, a graph having many group preserving automorphisms seems to be rare. The next example will show how a similar analysis can also give us information when there are very few group preserving automorphisms.

Example: For the Petersen Graph, $G = srg(10, 3, 0, 1)$, we know that $Aut(G) \cong S_5$ and $K(G) \cong \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/10\mathbb{Z})^3$. In this case the only group preserving automorphism is the identity automorphism, so $ker(\psi)$ is isomorphic to the trivial group. By the first isomorphism theorem, this means that $Aut(G) \cong Im(Aut(G)) \subset Aut(K(G))$.

6 Missing More Graph

Moore Graph:

Let G be a regular graph with degree k and diameter d whose number of vertices is bounded by

$$1 + d\sum_{i=0}^{-1}(d-1)^i$$

This bound can also be viewed in the form $v \leq 1 + k + k(k-1) + ...k(k-1)$ When this relation is equivalent, the graph is referred to as a Moore Graph. Their are four Moore graphs known to possibly exist but only three have been discovered; the 5 Cycle, the Peterson graph, and the Hoffman-Singleton graph.

Hoffman-Singleton Theorem:

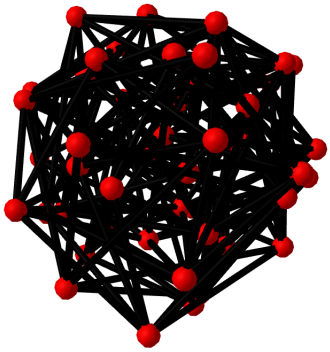
Let G be a Moore graph with a girth of 5 and diameter 2,then it is a regular graph with degree

$$k \in \{2, 3, 7, 57\}$$

The elements in this set in order correspond to degrees of the 5 cycle graph, the Peterson graph, and the Hoffman Singleton graph. A Moore graph with diameter 2 and degree 57 has yet to be discovered and is known as the missing Moore graph.

Hoffman-Singleton Graph:

Let G be a 7-regular undirected graph with 50 vertices and 175 edges, this graph contains 1,260 5 cycle graphs and is the unique $srg(50, 7, 0, 1)$ that is the highest-order Moore graph known to exist.



Suppose we have a projective geometry PG(3,2) which is the subspace of a 4 dimensional vector space over $\mathbb{F}_2 = \{0, 1\}$ In this space there are 15 points and 35 lines where each point is isomorphic to the Fano plane. For example

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

With their being 3 subsets of the degree $[7] = \{1, 2, ...7\}$, their are $\binom{7}{3} = 35$ total lines. Discarding points that are adjacent such as $(1, 2, 3) \sim (7, 6, 5)$ makes up a set of triplets

$$\{(1, 2, 3), (1, 4, 5), (1, 6, 7), (2, 4, 6), (2, 5, 7), (3, 5, 6), (3, 4, 7)\}$$

Where each element links down to 400 possible locations resulting in a collection of 2,850 total lines. With each set of triplets being located in exactly 3 Fano planes, the incidence between the 35 triplets and 15 Fano planes creates the projective geometry PG(3,2) with a collection of 15 "points" and 35

"lines". The Hoffman Singleton Graph is then constructed by placing a vertex at the 15 Fano planes and 35 triplets and by connecting each Fano plane to its corresponding 7 triplets.

7 n-Cube

We will now begin a discussion of the hypercube graph. First, a few preliminaries.

Definition 7.1. *Let G be a graph. We call G **distance regular** if the following hold*

- *G is regular*
- *The number of vertices distance i from $v \in v(G)$ and j from $w \in V(G)$ depends only on $k = d(v, w)$.*

*If G is distance regular with diameter d, then the **intersection array** of G is*

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\},$$

where $\forall\ u, v \in V(G)$, b_j is the number neighbors of u distance $j+1$ from v, and c_j is the number of neighbors of u at distance $j-1$ from v.

Fix $d \geq 1$. Define

$$Q_d := \{(a_1, a_2, \ldots, a_d) \mid a_i = 0 \text{ or } 1\}.$$

Say $(a_1, a_2, d \ldots, a_d) \sim (b_1, b_2, \ldots, b_d)$ if these vectors differ in exactly 1 position. This is a distance regular graph called the **d-cube graph**.

The critical group of the hypercube is known for all p-parts except for $p = 2$. This remains as an open problem.
Consider the case of Q_3. The veritces of Q_3 are elements of $\{0, 1\}^3$. We will consider a function $\rho : \mathbb{Z}[x_1, x_2, x_3] \to \mathbb{Z}^{Q_3}$. This is a function that sends polynomials to the vertices of our cube graph. We want to turn this into a monomial basis, as since $x = 0$ or 1, $x^2 = x$. To do this, condier $S = (x_{2-x_1, x_2^2-x_2, x_3^2-x_3})$ and let

$$\bar{p} : \mathbb{Z}[x_1, x_2, x_3] \to \mathbb{Z}^{Q_3}.$$

This gives us the monomial basis. Let $X_I = \prod_{i \in I} x_i$ for $I \subset \{1, 2, , 3\}$. Let

$$\alpha(X_I) = \sum_{i \in I} (X_{I \setminus \{i\} - X_I}) + \sum_{i \in I} x_i = (n - 2|I|)X_I + \sum_{J \subset I,\ |J| = |I|-1} X_J$$

Let $w_{i,i+1}$ be the $i, i+1$ subset inclusion matrix. Then

$$W_{0,1} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$W_{1,2} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$W_{2,3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

With respect to the standard basis, α gives us the adjacency matrix. With respect to the monomial basis, α gives us the matrix

$$\widetilde{A} = \left[\begin{array}{c|ccc|cccc} 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline & & & & & & & \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline & & & & & & & \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ \hline & & & & & & & \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

Note that $W_{0,1}, W_{1,2}$, and $W_{2,3}$ Appear as submatrices of this matrix.

8 Markov Decision Process

MDP:

A Markov decision process or MDP is a framework for decision making under uncertainty, it is a 5 tuple

$$MDP(S, A, \{P_{sa}\}, \gamma, R)$$

S – States that represent the surrounding environment of an agent

A - Actions an agent can take in the various states.

$\{P_{sa}\}$ - State transition probabilities, the likelihood of taking a given action given the current state

γ – Discount factor, a scalar that denotes how much the future is valued

R – Rewards, where the model is aiming to reach

The goal of a MDP is to fit data to a function. There is data and a function(model) is desired that will find parameters that the model can use to find a policy that can be used to fit to new data.

$$\pi : a \mapsto s$$

A policy π produces a path, when in a new state it tells the agent what action should be performed. It is a mapping from states to actions with a goal of maximizing total payoffs or rewards. This payoff is determined from the amount of rewards gathered form an agent and the goal is to find an optimal policy π^* that maximizes the sum of current and futures rewards.

For a policy π,

$$V^{\pi} : s \mapsto R$$

V^{π} is a function mapping the states to rewards where $V^{\pi}(s)$ is the value function for the policy π. It is the expected utility from an agent following a policy from a state s. If various states are selected and a policy is run in those states, there will be a collection of expected payoffs that will all differ, $V^{\pi}(s)$ is the average of those payoffs.

$$V^{\pi}(s) = R(s_0) + \gamma \sum_{s\prime} P_{s\pi(s\prime)} V^{\pi}(s\prime)$$

Where $R(s_0)$ is the immediate rewards ,γ is the discount factor and $P_{s\pi(s)} V^{\pi}(s\prime)$ are the future rewards. Here $s\prime$ is drawn from $P_{s\pi(s)}$ which implies future states are drawn from the current states optimal policies. Because in a given state an agent will take an action $a = \pi(s)$, $s\prime$ will be drawn from P_{sa} where $s = \pi(s)$ resulting in

$$V^{\pi}(s) = R(s) + \gamma \sum_{s\prime} P_{s\pi(s)} V^{\pi}(s\prime)$$

Which gives a linear system of equations with the number of states equations and number of states unknowns that can be solved to obtain the value function which can be used to find a policy π. Although a policy has been given from the current state s, we seek to find the optimal policy. There are a a very large amount of possible policies, potentially infinite, but we seek the best. The next step is to find the optimal value function for a state, $V^*(s)$.

Optimal Value Function:

$$V^*(s) = max\, V^{\pi}(s)$$

Equivalent to Bellman's equation:

$$V^* = R(s) + max_a\, \gamma \sum_{s\prime} P_{sa}(s\prime) V^*(s\prime)$$

Where $R(s)$ is the immediate rewards .and $P_{sa}(s\prime)V^*(s\prime)$ is if the agent takes action a, what is the expected future reward.

These equation look at all possible policies and their value function returns and picks the largest one. From this the optimal policy

$$\pi^* = argmax_a \sum_{s\prime} P_{sa}(s\prime) V^*(s\prime)$$

Tells the agent in the state s what is the best action, what is the action that maximizes total payoff. This is the action taken from Bellman's equation. Gamma isn't necessary for this equation as constants do not affect the value of a argmax so it can be eliminated.

Value Iteration:

This optimal action can be obtained by creating a vector that is the size of the states, setting $V(s) = 0$ for all states s which is the estimated value of every state, repeatedly update the expected value for every state using Bellman's equation and then use synchronous gradient descent to update θ(vector parameters) simultaneously. For this synchronous update the state transition probabilities $\{P_{sa}\}(s\prime)$ are set to 0 for all states, the right hand side of Bellman's equations is then computed for all states, simultaneously overwrite all values, while simultaneously updating all state values on the left hand side. If $V^*(s)$ can be computed, which is not always known due to possibly infinite number of states for a given MDP then V^* can then be used to find π^* which gives a way to decide for every state s the optimal action a which allows the computation of the argmax which can then be used to compute the optimal action for every state. Giving

$$V^*(s) = V^{\pi *}(s) \geq V^{\pi}(s)$$

For every policy π and state s.