Hypercube Graphs and the Representation Theory of Symmetric Groups

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An Honors College Project Presented to

the Faculty of the Undergraduate

College of Science and Mathematics

James Madison University

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Accepted by the faculty of the College of Science and Mathematics, James Madison University, in partial fulfillment of the requirements for the Honors College.

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PUBLIC PRESENTATION

This work is accepted for presentation, in part or in full, at Virginia State University on April 29, 2023.

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April 2023

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Acknowledgements

This work would not have been possible without the invaluable assistance and feedback from my advisor Dr. Joshua Ducey. I am also grateful to readers Dr. David Duncan and Dr. Brant Jones for providing their knowledge and experience. Additionally, financial support from the Madison Award for Academic Excellence, Lisa Persson Helms '87 Scholarship Endowment for Mathematics, and Droms Memorial Scholarship Endowment for Mathematics contributed to the success of this project.

Research from this project will be presented at the Mathematical Association of America MD-DC-VA Spring 2023 meeting at Virginia State University.

Abstract

Hypercube graphs are an interesting class of graph which exhibit many regularity properties that make them useful in a variety of applications. Two important algebraic invariants of hypercube graphs are the critical group and the Smith group. While the Smith group for hypercubes has been completely described, a description for the 2-Sylow subgroup of the critical group has not yet been discovered. In fact, there are currently no conjectures for what form this subgroup may take. We introduce a new approach to this problem using the representation theory of symmetric groups. This approach gives a new method of calculating the critical groups of hypercube graphs that may be useful in deriving a general formula.

Chapter 1

Introduction

In this work we will study a class of graphs known as hypercube graphs. These graphs are relatively easy to describe and have various regularity properties which have made them an interesting area of study in graph theory. Hypercubes are familiar and easy to describe due to the simplicity of the 2-cube (square) and 3-cube (cube), but they also quickly become less tangible and complex as we consider higher dimensional cubes. These higher dimensional hypercubes retain some of the nice properties of their lower dimensional analogues while also losing some of the physical interpretation which may aid the intuition of those who wish to study them. In this way hypercube graphs occupy a liminal space between the concrete and abstract, simultaneously familiar and foreign, regular yet also complex. Beyond their beauty as mathematical objects, higher dimensional hypercube graphs also find uses in a variety of different applications. For example, coding theory makes use of hypercube graph properties to make error detecting and error correcting codes. In quantum computing, hypercube graphs are rare examples of graphs that allow perfect state transfer between two vertices.

For graphs as important and well studied as hypercubes, it may be surprising that there are still unanswered questions pertaining to them. One such open problem is to find a formula for their critical groups. This question was posed along with two conjectures about the structure the critical group by Vic Reiner in 2001. In the following year Hua Bai proved Reiner's conjectures in [1] (although the paper was not published until 2003). Bai's results, although they describe most of the critical group's structure, did not provide a complete description. Crucially, the 2-Sylow subgoup of the critical group has not been described for powers of 2 greater than 2. Even more surprisingly, as of now, 20 years later, there is still not even a conjecture of what the full 2-Sylow subgroup might look like.

One reason for the difficulty of this problem is a lack of experimental data. As we will see, the critical group of a graph can be computed by an algorithm that performs row and column operations on a certain matrix. However, the size of that matrix is equal to the number of vertices of the graph it represents, and an n-dimensional hypercube graph has 2^n vertices. This means moving up one dimension doubles the number of rows and columns, and thus the size of the problem increases exponentially. For this reason very few critical groups of hypercubes can be solved by computer, and there are not enough data to find patterns or form conjectures.

The goal of this work is to provide a new lens through which to view the problem of the critical group of hypercube graphs, namely the representation theory of symmetric groups. Representations of symmetric groups are well-studied, and by viewing the vertices of hypercube graphs as subsets with the action of a symmetric group, the language and tools from representation theory become tools to formulate and prove conjectures about the seemingly random structure of hypercubes' critical groups.

A secondary goal of this work is to provide a resource for those interested in researching or teaching algebraic graph theory. Because of this, we will introduce the problem slowly and provide most of the necessary background in graph theory and algebraic graph theory. However, we will assume that the reader is familiar with concepts from group theory and linear algebra with modules. The structure of the text is as follows. Chapter 2 will introduce some basic graph theory concepts as well as two different perspectives for defining a hypercube graph. Chapter 3 dives deeper into algebraic invariants and introduces the critical group. Then, in chapter 4 we will begin developing the tools from representation theory that will guide the remainder of the text, the most important of these being the Specht series defined at the end of the chapter. Chapter 5 then applies the results of chapter 4 to a different problem from combinatorics related to critical groups. This demonstrates the method that we then adapt in chapter 6 for use in the hypercube problem.

When preparing this research, the author made significant use of SageMath. A library of methods related to Specht modules and symmetric group representation theory can be found at

https://github.com/sherwocj/representation_theory_sage.

Chapter 2

The Hypercube

2.1 Background

A hypercube, also known as an n-cube, is a generalization of the familiar three dimensional cube and two dimensional square, the 3-cube and 2-cube respectively. These higher dimensional cubes have many fascinating properties and are of great interest to mathematicians. These higher dimensional analogues of cubes are also the inspiration for another mathematical object, the hypercube graph (or n-cube graph), $Q(n)$.

Unlike hypercubes, which are geometric objects in some higher dimensional space, hypercube graphs are a way to represent the connections between a hypercube's vertices. While in a geometry class, you may have learned that a square is an equiangular and equilateral quadrilateral, our definition of an n-cube will make no mention of side length or angles. As long as the connections between vertices remain unchanged, altering the way the graph is drawn on paper does not change it. For an example, see Figure 2.1.

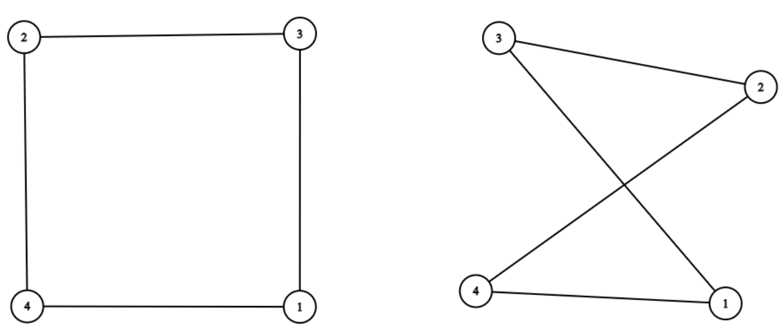


Figure 2.1: Two squares

It may seem odd at first to ignore so many details. However, this abstraction gives us one way to study higher-dimensional analogs of cubes that are difficult if not impossible to visualize. For example, a four dimensional hypercube cannot be constructed in the three spatial dimensions that we can perceive, much less on a two dimensional page, but we can easily draw a 4-cube graph. Furthermore, and perhaps more importantly, the vertices of a graph can represent almost anything, and so results derived about a hypercube graph can apply to any situation where things are connected in the same way as a hypercube, even if it does not make sense to impose a geometry. Because of this, hypercube graphs are useful in many areas of study including algebraic design theory, coding theory, and computer science [16] [29].

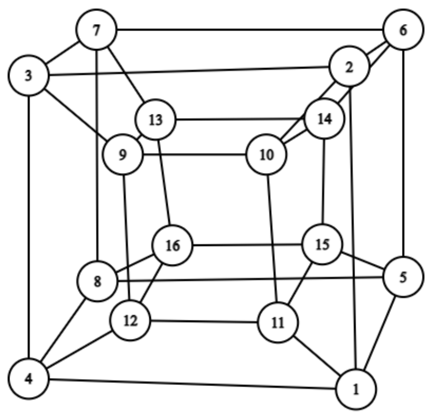


Figure 2.2: 4-cube

2.2 Some Graph Theory

For the rest of this chapter, we will introduce some basic definitions and concepts from graph theory and then explain how they relate to hypercube graphs.

A **graph** $\Gamma(V, E)$ consists of two sets, a vertex set V and an edge set E which is a subset of the the two element subsets of V. The vertex set describes the vertices of the graph, and the edge set describes how to connect the vertices. If v and w are elements of V, then they are connected by an edge when $\{v, w\}$ is in E. In this case we say v is **adjacent** to w, and we denote it by $v \sim w$. We call the set of vertices adjacent to a vertex v the **neighborhood** of v, denoted $N(v)$. A graph where no vertex is adjacent to itself and there is at most one edge between any two vertices is called a **simple graph**. Notice that the elements of E are sets, so the order in which the elements are written does not matter. This means that adjacency is symmetric, so if $v \sim w$, then also $w \sim v$. If we were to replace the sets with ordered pairs, we would obtain a different object called a **digraph** where adjacency is not necessarily symmetric.

The **hypercube (n-cube) graph** $Q(n)$ is the graph with vertex set $V = \{0,1\}^n$, the set of n-tuples with 0 and 1 as elements, where two vertices are adjacent if they differ in exactly one position.

For example, the square graph (also called the 4-cycle, C_4, or $Q(2)$) would have the following vertex set and edge set:

$$V = \{(0,0),(0,1),(1,0),(1,1)\}$$

$$E = \{\{(0,0),(0,1)\},\{(0,0),(1,0)\},\{(0,1),(1,1)\},\{(1,0),(1,1)\}\}.$$

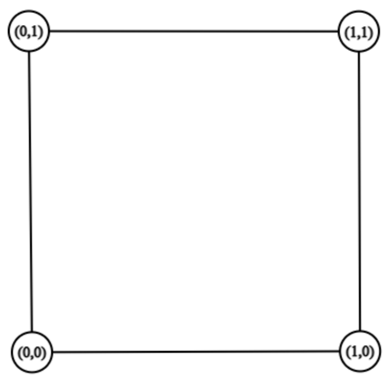


Figure 2.3: Square graph with tuple labeling

For an n-cube graph, each vertex is adjacent to n other vertices. The number of vertices adjacent to a given vertex v is called the **degree** of v. Graphs like hypercubes where each vertex has the same degree are called **regular**. If each vertex has a degree of n, we say the graph is **n-regular**. From the picture we can see that the square is a 2-regular graph. In general $Q(n)$ is n-regular. This is easy to see, since given an n-tuple there are n different places where we can change a 0 to 1 or vice versa to find a vertex that differs in exactly one position.

Hypercube graphs also satisfy another type of regularity. To define what this is, we first introduce the concept of distance between vertices. The **distance** between vertices v and w, $d(v,w)$, is the length of the shortest path between them, where the **length** of a path is defined to be the cardinality of its edge set. One way to figure out the distance between two vertices is to count the number of edges it takes to get from one vertex to the other. Adjacent vertices have a distance of 1, and for $Q(2)$, we can see that $d((0,0),(1,1)) = 2$. We also define the **diameter** of a graph to be the maximum distance between any two vertices on a graph. In general, the distance between any two vertices on a hypercube graph is the number of positions in which the n-tuples differ. This type of distance is known as the **Hamming distance**. The diameter of $Q(n)$ is n.

We say a graph with diameter D is **distance regular** if for any two vertices v and w such that $d(v,w) = i$ there are constants a_i, b_i, and c_i so that among the vertices in the neighborhood of w there are a_i distance i from v, b_i distance $i+1$ from v, and c_i distance $i-1$ from v. We can see that distance regularity implies regularity, by taking $i = 0$ and noticing that b_0 is the degree of a vertex and must therefore be the same for all vertices. Also since by assumption the distance between v and w is i, vertices adjacent to w must have a distance of

$i-1$, i, or $i+1$ to v. This means that if a distance regular graph is k-regular, then for any i, $a_i + b_i + c_i = k$. We can therefore write a_i in terms of b_i and c_i and express all the intersection numbers in an **intersection array** as follows:

$$\{b_0, b_1, ..., b_{D-1}; c_1, c_2, ..., c_D\}.$$

Theorem 2.2.1. *An n-cube is distance regular with the following intersection array.*

$$\{n, n-1, ..., 1; 1, 2, ..., n\}.$$

Proof. Suppose v and w are vertices of $Q(n)$ such that $d(v,w) = i$. Any vertex in the neighborhood of w must differ from w in exactly one position. This difference can be in either a position where v and w agree or in one where they differ. Since w and v differ in i positions, there are i neighbors of w that agree with v in one more position than w, and $n-i$ neighbors of w that agree in one fewer position than w. This accounts for all neighbors of w, and since v and w were chosen arbitrarily, it follows that $Q(n)$ is distance regular with the given intersection array. □

Hypercubes are also examples of bipartite graphs. A **bipartite graph** is a graph where the vertex set can be split into two disjoint subsets such that within in each subset there are no two adjacent vertices and each vertex is contained in one of the subsets.

Theorem 2.2.2. *Hypercube graphs are bipartite.*

Proof. Let v be a vertex of $Q(n)$. Define $S(v)$ to be the sum of the entries in its n-tuple. Suppose $w \sim v$. Then w differs from v in exactly one position with a 0 instead of a 1 or vice versa. In either case if $S(v)$ is even, then $S(w)$ is odd, and if $S(v)$ is odd, $S(w)$ is even. Therefore the sum of entries for any two adjacent vertices must have opposite parity, and we can partition the vertex set of $Q(n)$ into odd vertices and even vertices, where there are no adjacent vertices within each subset. This is the definition for a bipartite graph. □

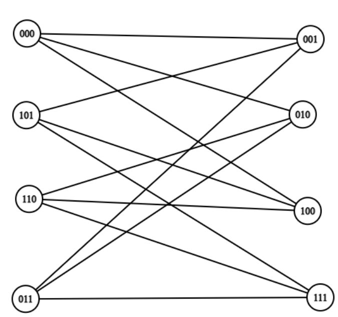


Figure 2.4: Example bipartite 3-cube

Another way to think about bipartite graphs is to imagine coloring the vertices using two colors so that no two adjacent vertices have the same color. We have shown that this is possible for hypercube graphs, but this is not the case for all graphs. One example of such a case is K_3, the complete graph on three vertices. This graph has three vertices that are all adjacent to each other. This graph cannot be colored with 2 colors since no matter which two vertices are colored first, there are no remaining options for the third. It is then interesting to ask how many colors is the fewest necessary to color a graph. This is called the graph's **chromatic number**. We have already shown that the chromatic number for a hypercube is 2. K_3 has a chromatic number of 3, and in general K_n has a chromatic number of n [17].

Often it is useful to represent the information of a graph using a matrix. Associating graphs with matrices allows us to translate questions about graphs into linear algebra problems. This has multiple advantages. Many matrix properties correspond to features of graphs, so we can use theorems from linear algebra to discuss graphs. Also, representing graphs as matrices, allows for the use of computers to aid in studying these objects.

One way to encode a graph's information is to use an **adjacency matrix**. To form an adjacency matrix for a graph Γ, first select an ordering for the vertices. Then, if the graph has k vertices, the adjacency matrix A will be the $k \times k$ matrix where $A(i, j) = 1$ if the i-th and j-th vertices in the chosen ordering are adjacent $A(i, j) = 0$ otherwise. See Figure 2.5 for an example adjacency matrix.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Figure 2.5: Adjacency Matrix for $Q(3)$

Since adjacency is a symmetric relation, adjacency matrices for graphs are symmetric. Also, for simple graphs, the trace of an adjacency matrix is 0, since vertices are not adjacent to themselves causing all diagonal entries to be 0. There is also more than one way to form an adjacency matrix for a given graph. Choosing a different ordering of the vertices corresponds to swapping rows and columns of A, making a new matrix that describes the same graph. We can also work the other way, and use an adjacency matrix to draw a corresponding graph, but as we have seen, multiple matrices can lead to the same graph. This brings us to the important question of how to tell if two graphs are equivalent.

This question of whether two graphs are "the same" is not just important when looking at adjacency matrices. It is possible that the same graph can arise in multiple different contexts, so it is necessary to be able to translate between different points of view so that information learned about a graph in one context can be extended for use in others. To illustrate this point, let us examine a new family of graphs.

Define the **subset-chain graph for an n-set** to be a graph with the vertex set $P([n])$, the power set of a set with n elements, where two subsets V and W are adjacent if one of the following hold. Either $|V| + 1 = |W|$ and $V \subset W$, or $|V| = |W| + 1$ and $W \subset V$. This may seem like an odd definition for a graph, but we can draw it to see if we gain any insight. This picture can be found in Figure 2.6.

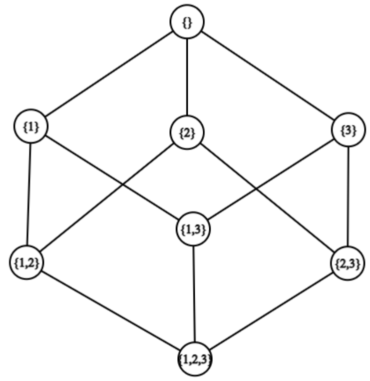


Figure 2.6: 3-set subset chain graph

From the picture we can see that this graph is equivalent to the 3-cube! We have just changed the labels from n-tuples to subsets. Even though the vertex sets and adjacency were defined differently, the subset relationship that we described has the same shape as n-tuples with a Hamming distance of 1. We can easily verify this fact for the 3-cube by selecting an appropriate ordering for vertices and comparing the adjacency matrix to that from Figure 2.5, but we wish to show this is true for all n-cubes. To accomplish this we introduce the concept of a graph isomorphism.

A **graph isomorphism** is a one-to-one correspondence between vertex sets that preserves adjacency. This means given two graphs, $\Gamma_1(V_1, E_1)$ and $\Gamma_2(V_2, E_2)$, the function $f : V_1 \to V_2$ is a graph isomorphism if f is a bijection and for any v, w in V_1, $v \sim w$ if and only if $f(v) \sim f(w)$ [17]. This definition formalizes the intuitive idea that two graphs are equivalent if you can relabel the vertices of one to obtain the other. We can also use this definition to show that the n-cube is equivalent to an n-set subset chain graph.

Theorem 2.2.3. *The n-cube is isomorphic to the subset chain graph for an n-set.*

Proof. Let V_1, V_2 denote the vertex sets for the n-cube and subset chain graph for an n set respectively. Define $f : V_2 \to V_1$ to be the function that maps X, a subset of $[n]$, to $f(X)$, the n-tuple where for $1 \leq j \leq n$, $f(X)$ has a 1 in the jth position if j is an element of X and 0 in the jth position otherwise. This is clearly a bijection, so now we must show that adjacency is preserved. Suppose $\{X, Y\}$ is an edge in E_2. Without loss of generality, suppose $X \subset Y$ and $|X| + 1 = |Y|$. Since $X \subset Y$, the tuple $f(Y)$ will have a 1 in every position where $f(X)$ has a 1. Also, since there is one element in Y that is not in X, $f(Y)$ will contain a 1 in exactly one position where $f(X)$ contains a 0. This means that $f(X)$ and $f(Y)$ have a Hamming distance of 1. Therefore $\{f(X), f(Y)\}$ is in the edge set E_1. A similar argument will show that if $\{f(X), f(Y)\}$ is in E_1, then $\{X, Y\}$ is an edge in E_2. Therefore f is a graph isomorphism, and the n-cube is isomorphic to the subset chain graph for an n-set. □

Since we have shown that these two graphs are equivalent, moving forward we will refer to both graphs as hypercubes or n-cubes regardless of whether we are using the tuple or subset point of view.

An isomorphism from a graph to itself is called a **graph automorphism**. The set of automorphisms for a graph Γ is denoted $Aut(\Gamma)$ and forms a group under function composition. The identity of $Aut(\Gamma)$ is the identity function that sends each vertex to itself. Graphs with a small automorphism group have few symmetries, so it is generally possible to distinguish vertices from each other. Graphs with large automorphism groups exhibit more symmetry and have more vertices that if unlabeled would be indistinguishable from each other.

Lemma 2.2.4. *For any vertex v in $Q(n)$, an automorphism of $Q(n)$ is uniquely determined by where it maps v and its neighbors.*

Proof. Suppose f is a graph automorphism, and $f(v) = (a_1, a_2, ..., a_n)$. Then the neighbors of v must map to tuples of the form $(a_1, a_2, ..., b_j, ..., a_n)$ where $1 \leq j \leq n$ and b_j is the opposite element as a_j. Suppose that a mapping of the neighbors of v to these elements has been chosen. We now examine where f maps a vertex w distance 2 away from v. The vertex $f(w)$ must also be distance 2 away from $f(v)$, so it will take the form $(a_1, ..., a_{j-1}, b_j, a_{j+1}..., a_{k-1}, b_k, a_{k+1}, ..., a_n)$ for $1 \leq k \leq n$ where $j < k$. However, since f preserves adjacency, $f(w)$ must also be adjacent to the images of vertices that are adjacent to both v and w. By assumption these two vertices have already been mapped to specific tuples, say $(a_1, a_2, ..., b_l, ..., a_n)$ and $(a_1, a_2, ..., b_m, ..., a_n)$. This means that the only option is for w to map to the tuple that differs from $f(v)$ in positions l and m. Therefore the images of all vertices distance two from v are determined, and by a similar argument, the rest of the vertices in $Q(n)$ are also determined. □

Theorem 2.2.5. $|Aut(Q(n))| = n!\ 2^n$

Proof. By the previous lemma, the cardinality of the automorphism group of $Q(n)$ is equal to the number of possible ways to map a vertex and its neighbors to the n-cube. A vertex can map to any vertex on the hypercube, of which there are 2^n. This property is called **vertex transitivity**. For a proof that $Q(n)$ is vertex transitive see lemma 3.1.1 in [17]. There are then $n!$ ways to arrange the neighbors of v. Therefore, there are $n!\ 2^n$ total automorphisms. □

Chapter 3

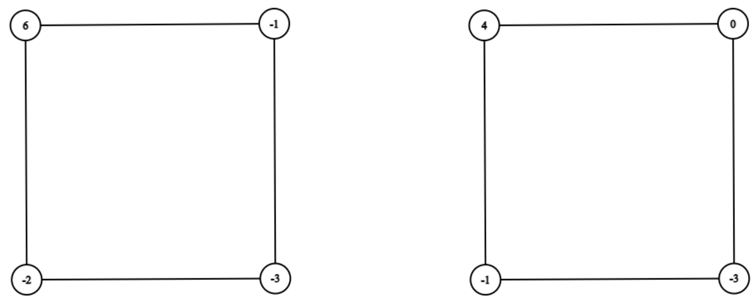
Algebraic Invariants

3.1 Chip-Firing

Now that we have introduced some concepts from graph theory and have defined the hypercube graph in a couple of different ways, we now focus our attention to algebraic graph theory. This subject specializes in looking at the algebraic properties of graphs, often looking at eigenvalues and canonical forms of matrices associated with graphs such as adjacency matrices. One important algebraic invariant of a graph is known as the **critical group**, which we will introduce here and will play a major role in the remainder of our study.

Imagine placing poker chips on each of the vertices of a graph. We can then associate an integer with each vertex representing the number of chips on it. Negative chips are allowed. By choosing an ordering for the vertex set we can represent all chip information with a vector that we will refer to as the **state** of the graph. We also introduce a new operation called **chip-firing**: when a vertex is chip-fired it sends a chip to each of its neighbors, and therefore loses a number of chips equal to its degree. We can also do the inverse operation where a vertex can take a chip from each of its neighbors, increasing its chip number by its degree. (This is equivalent to chip-firing each vertex except the one in question.) We can keep track of the state of a graph through chip-firing using a **state vector**, a vector where each entry corresponds to a vertex of the graph and the value of the entry is the number of chips on that vertex. Refer to Figures 3.1 and 3.2 for an example of chip-firing on a graph and its state vector.

Now that we have introduced this chip firing game, we can ask a few questions. Firstly, from any given starting position, is it possible to obtain any other? We can immediately see that the answer is no, since the number of chips on the graph remains constant after each chip firing. We therefore might adjust our question to ask: For any integer k, is it always possible to go between any two states whose sum is k through chip firing? This question is slightly more difficult to answer, but by experimenting a little bit, one can see that for the square graph the following states, which all sum to 0, are all inaccessible to each



(a) Before (b) After

Figure 3.1: Chip firing on the upper left vertex

$$\begin{pmatrix} 6 \\ -1 \\ -3 \\ -2 \end{pmatrix} \to \begin{pmatrix} 4 \\ 0 \\ -3 \\ -1 \end{pmatrix}$$

Figure 3.2: State vector before and after chip firing

other using chip firing operations:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

In fact if we define an equivalence relation on states of the square graph where two states are in the same equivalence class if you can chip fire one into the other, then these four states represent the only possible classes when the sum of chips on all vertices is 0.

To aid in our study of chip firing we introduce a new matrix that will keep track of how much the chip count on each vertex changes. Define the **Laplacian matrix** $L(\Gamma)$ of a graph Γ to be the matrix where if $i \neq j$, then $L(i,j) = 1$ when the i-th and j-th vertices are adjacent and $L(i,j) = 0$ otherwise, and the ith diagonal entry is the negative of the degree of the i-th vertex. Note that the first part of this definition is identical to that for the adjacency matrix. The only difference is on the diagonal, where we have included a term to account for the loss in chips when a vertex is chip fired. Also, the reader will note that in common references such as [5] the Laplacian is defined to be $-L$. However, this change does not fundamentally affect the algebraic properties of the matrix that are important to us and will aid in the clarity of our presentation. We can write the relationship between the adjacency matrix and Laplacian matrix for a given graph as follows where D is the diagonal matrix with the degrees of the vertices:

$$L = A - D.$$

For n-regular graphs, it follows that $L = A - nI$. Therefore for $Q(n)$,

$$L(Q(n)) = A(Q(n)) - nI$$

$$\begin{pmatrix} -3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -3 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -3 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -3 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -3 \end{pmatrix}$$

Figure 3.3: Laplacian Matrix for $Q(3)$

We can now keep track of the state of the graph throughout chip firing using the following formula, where S_1 is our new state, S_0 is the original state, and v is a vector keeping track of how many times each vertex has been fired i.e. $v(i)$ is the number of times the ith vertex has been fired:

$$S_1 = S_0 + Lv. \tag{3.1}$$

Note that an inverse chip fire is the same as a -1 in the corresponding entry of v.

We can now use this framework to prove some of the facts that we asserted earlier. For states S and T, we define $S \sim_L T$ if there is a sequence of chip firing operations to obtain S from T.

Theorem 3.1.1. *The relation $\sim_L$ is an equivalence relation.*

Proof. The relation $\sim_L$ is reflexive since a sequence of no chip firing will not change the state. Using (3.1), let v be the zero vector, then $S_1 = S_0$ so $S_0 \sim_L S_0$. Suppose $S_0 \sim_L S_1$. Then we know that there is a vector v such that $S_1 = S_0 + Lv$. Rearranging this equation we see $S_0 = S_1 - Lv = S_1 + L(-v)$. Therefore the vector $-v$ gives the instructions on chip firing from S_1 to S_0, and $S_1 \sim_L S_0$. Thus, $\sim_L$ is symmetric. Now suppose $S_0 \sim_L S_1$ and $S_1 \sim_L S_2$. It follows from equation (3.1) that there are vectors v and w so $S_1 = S_0 + Lv$ and $S_2 = S_1 + Lw$. Substituting for S_1 in the second equation, we see $S_2 = (S_0 + Lv) + Lw = S_0 + L(v + w)$. Hence, $S_0 \sim_L S_2$, $\sim_L$ is transitive, and therefore an equivalence relation. □

Theorem 3.1.2. *For $Q(2)$ and states $S = (0, 0, 0, 0)$ and $T = (1, -1, 0, 0)$, there is no sequence of chip firings that can obtain one from the other i.e. $S \not\sim_L T$.*

Proof. Suppose there exists an integer vector v such that $T = S + Lv$. Notice that v must be an integer vector since we have only defined states and chip firing in term of integer amounts. Since S is the zero vector this means we must find v_1, v_2, v_3, v_4 that satisfy

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \tag{3.2}$$

Looking at the second and third rows we see $v_1 - 2v_2 + v_4 = -1$ and $v_1 - 2v_3 + v_4 = 0$. Subtracting the third from the second gives, $-2v_2 + 2v_3 = -1$, which we can solve for v_3 giving $v_2 = v_3 - \frac{1}{2}$. We have now reached a contradiction, since it is not possible for v_2 and v_3 to both be integers. Therefore $S \not\sim_L T$. $\square$

We could make similar arguments to show that all of the states of the square shown in Figure 3.1 are not equivalent by chip firing, but this process can quickly become tedious and does not give new insight into the problem. One key observation from the previous proof, however, is the restriction to integer states gives added structure and complexity to our problems. This will become a repeated theme, where instead of working over a field such as the rational or real numbers, we will work over integer rings so that we do not lose this added structure.

Now, for reasons which will become apparent later, let us restrict our attention to states whose entries sum to 0. We also introduce the binary operation of state addition where the chip value for any vertex in the output state is the sum of the chip values for the two input states. This can be thought of as superimposing two copies of the graph, each having its vertices labeled with their chip values, and adding the labels that fall on top of each other. From the vector point of view, state addition is equivalent to normal vector addition of state vectors. We can also extend this definition to apply to equivalence classes of states, where the sum of two classes is the class represented by the sum of the representatives of the summands. However, in order for this to make sense, the sum of two classes must be independent of the representatives chosen.

Theorem 3.1.3. *Addition of state classes is well-defined.*

Proof. Suppose $S_0 \sim_L S_1$ and $T_0 \sim_L T_1$. Then we can find integer vectors v and w so that $S_1 = S_0 + Lv$ and $T_1 = T_0 + Lw$. Then, $S_1 + T_1 = (S_0 + Lv) + (T_0 + Lw) = (S_0 + T_0) + L(v + w)$. Therefore $(S_0 + T_0) \sim_L (S_1 + T_1)$, so choice of representatives does not affect the equivalence class of the sum. $\square$

Now that we are armed with all the necessary tools, we can define the critical group of a graph.

Theorem 3.1.4. *The set of state equivalence classes for a graph Γ, where the sum of chips across all vertices is 0, with the binary operation of addition forms an abelian group.*

Proof. The class represented by the zero-state is an identity element for $K(\Gamma)$. Also for any state S, $S + (-S) = 0$, so the class containing $-S$ is an additive inverse for the class containing S. Since, vector addition is associative and commutative, it follows that the addition of states and therefore state classes also have these properties. Therefore $K(\Gamma)$ is an abelian group. □

The group described by Theorem 3.1.4 is defined to be the **critical group of** Γ, denoted $K(\Gamma)$.

We have already encountered the critical group for the square graph. Recall that $(0,0,0,0), (1,-1,0,0), (-1,1,0,0), (1,0,-1,0)$ are each representatives for different equivalence classes. We will now identify the critical group. Consider $(1,-1,0,0) + (-1,1,0,0) = (0,0,0,0)$. We see that these elements are additive inverses. Also $(1,0,-1,0) + (1,0,-1,0) = (2,0,-2,0) \sim_L (0,0,0,0)$, so it is an element of order 2. We can similarly see that $(1,-1,0,0)$ and $(-1,1,0,0)$ are both order 4. If we assume that these four elements make up the whole critical group (a fact that we will prove later) then we can see that $K(Q(2)) \cong \mathbb{Z}/4\mathbb{Z}$.

The critical group of a graph is an important graph invariant. It is often used to distinguish between different graphs, since isomorphic graphs have the same critical group. Also the critical group is related to the eigenvalues of the Laplacian matrix and this relationship can be exploited to demonstrate a variety of different graph properties [5]. For these reasons, calculating the critical groups of graphs is an important area of research in algebraic graph theory. As we have seen from the relatively simple example with the square, trying to calculate the critical group of a graph can become complicated very quickly. We will therefore introduce another method of calculating the critical group of a graph, using the Laplacian matrix.

In a linear algebra class, one is likely to encounter the reduced row echelon form (RREF) of a matrix. This is one example of a canonical form of a matrix. By looking at the RREF of a matrix, one can easily determine its rank, nullity, and whether it has an inverse. A standard process for finding RREF is Gauss–Jordan elimination, in which the operations of row addition and scalar multiplication are used to simplify the matrix into the desired form. When working over a field such as the reals or rationals, it is always possible to get this canonical form for L where all entries are either ones or zeroes. However, since we are working with integer matrices, not all of the entries of the Laplacian will be units that have a multiplicative inverse in the integers. In order to preserve the information and structure from working over integers, we will use a different canonical form called Smith normal form (SNF).

We say a matrix with integer entries, A, is **unimodular** if $det(A) = \pm 1$. For any integer matrix M, we can find unimodular integer matrices U and V, so that $S = UMV$ is a diagonal matrix. In this case we say that S is a diagonal form of M. We can write $S = diag[\alpha_1, \alpha_2, ..., \alpha_n]$. If it is the case that each a_i divides a_{i+1} while also allowing for some zeros at the end, then this is the **Smith normal form** of M, denoted $SNF(M)$. For any integer matrix, we can find the SNF, using an algorithm similar to that for RREF involving row and column operations. These operations are equivalent to a change in basis that is described by the matrices U and V. This algorithm can be programmed in a computer and works well for small matrices. However, for large matrices, such as the adjacency and Laplacian matrices for large-dimensional hypercubes, the calculations for this algorithm become prohibitively computationally expensive, so other methods are required.

Once the SNF of the Laplacian of Γ is known, we can use this information to find $K(\Gamma)$. By the structure theorem for finitely generated modules over a PID, we can decompose any finitely generated module into a direct sum of quotients using the elementary divisors [1]. In our case, the critical group of a graph is a finite abelian group, where the elementary divisors are the diagonal entries of the SNF of the graph's Laplacian. By the structure theorem for finitely generated modules over a PID, we can decompose the critical group of a graph into a direct sum of quotients of the form $\mathbb{Z}/a_i\mathbb{Z}$ where a_i are the entries in the SNF (or any diagonal form) of L. Note that for connected graphs there is always a single copy of 0 in the SNF of the Laplacian. This corresponds to a copy of $\mathbb{Z}$ in the group decomposition. However, since the critical group is finite, we can ignore this factor and only focus on the non-zero entries in the diagonal. This is equivalent to us demanding that the sum of the chips on all vertices of our graph was 0. For abelian groups such as the critical group, the subgroup of elements of finite order is called the **torsion subgroup**.

Often this is used to give an equivalent definition of $K(\Gamma)$ as follows. Let $L(\Gamma) : \mathbb{Z}^V \to \mathbb{Z}^V$ be the map described by the Laplacian matrix of Γ. Then

$$coker(L) = \mathbb{Z}^V/Im(L) = \mathbb{Z} \oplus K(\Gamma)$$

Here we define $K(\Gamma)$ to be the torsion subgroup of the cokernel of L. Since $Im(L)$ contains all possible ways to chip fire on the graph, the quotient $\mathbb{Z}^V/Im(L)$ identifies all states in the same chip firing equivalence class and thus recovers our original definition of $K(\Gamma)$. Using our new definition, however, we can easily see that $SNF(L)$ gives the information for a decomposition of $K(\Gamma)$. In fact, it is not necessary to to use SNF, and any diagonal form will give an equivalent decomposition. This equivalence can be shown using the Chinese remainder theorem.

The Laplacian for $Q(2)$ is small enough to calculate the SNF either by hand or computer. We share the result below:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -2 & -3 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} L(Q(2)) \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, taking the diagonal entries of the left side,

$$\begin{aligned} coker(L(Q(2))) &\cong \mathbb{Z}/(1)\mathbb{Z} \oplus \mathbb{Z}/(1)\mathbb{Z} \oplus \mathbb{Z}/(4)\mathbb{Z} \oplus \mathbb{Z}/(0)\mathbb{Z} \\ &\cong \mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}. \end{aligned}$$

So, as we claimed earlier

$$K(Q(2)) \cong \mathbb{Z}/4\mathbb{Z}.$$

We can also define a similar invariant using the adjacency matrix. The **Smith group** for a linear mapping $A : \mathbb{Z}^n \to \mathbb{Z}^n$ is $coker(A) = \mathbb{Z}^n/Im(A)$. We denote the Smith group as $S(A)$. For adjacency matrices, this group does not have as nice of an interpretation involving operations on the vertices of the graph, but it is still a useful graph invariant to study. Also, it is important to note that unlike L, which for connected graphs always has a free rank one less than the dimension of the domain, the free rank of A varies depending on the graph. As such, we cannot always decompose $S(A)$ as a direct sum of a finite group and $\mathbb{Z}$ as we could with $S(L) = \mathbb{Z} \oplus K(\Gamma)$.

The adjacency matrix for $Q(2)$ has SNF of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and therefore,

$$S(A(Q(2))) \cong \mathbb{Z}^2$$

3.2 Graph Spectra

The **(ordinary) spectrum** of Γ is the set of eigenvalues of $A(\Gamma)$ with their multiplicities. We similarly define the **Laplacian spectrum** of a graph using the matrix $L(\Gamma)$. Graph spectra can be related to many different graph properties. In particular, for certain classes of graphs such as hypercube graphs, there is a connection between the spectrum and the Smith group [5].

Here we will state some results about the spectra of hypercubes that will become useful when we begin to calculate the Smith groups and critical groups.

Theorem 3.2.1. *The matrix $A(Q(n))$ has eigenvalues $-n+2l$ with multiplicity $\binom{n}{l}$ for each integer $0 \leq l \leq n$.*

Proof. Let I be a vertex of $Q(n)$. We use the subset point of view so $I \subset [n]$. Define v_I to be the 2^n-vector where $v_I(J) = (-1)^{|I \cap J|}$. We examine the entry of Av_I corresponding to subset J.

$$\begin{aligned} Av_I(J) &= A(J)^T \cdot v_I \\ &= \sum_{X \subset [n]} A(J) v_I(X) \\ &= \sum_{X \sim J} v_I(X) \\ &= \sum_{i=1}^{n} v_I(J \cup \{i\} \setminus J \cap \{i\}) \\ &= \sum_{i=1}^{n} (-1)^{|I \cap (J \cup \{i\} \setminus J \cap \{i\})|} \end{aligned}$$

We see that the only terms that are left are those that correspond to the sets that are adjacent to J, which we can write as the symmetric difference of J with a singleton. We now notice that if $i \notin I$, then $|I \cap J| = |I \cap (J \cup \{i\} \setminus J \cap \{i\})|$. Similarly, if $i \in I$, then $|I \cap J| = |I \cap (J \cup \{i\} \setminus J \cap \{i\})| \pm 1$. We can therefore split our sum into two parts using these cases, so

$$\begin{aligned} \sum_{k=1}^{n} (-1)^{|I \cap (J \cup \{i\} \setminus J \cap \{i\})|} &= |I|(-1)^{|I \cap J| \pm 1} + (n - |I|)(-1)^{|I \cap J|} \\ &= -|I|(-1)^{|I \cap J|} + (n - |I|)(-1)^{|I \cap J|} \\ &= (n - 2|I|)(-1)^{|I \cap J|} \\ &= (n - 2|I|) v_I(J) \end{aligned}$$

It follows that v_I is an eigenvector with eigenvalue $n - 2|I|$. Since there are $\binom{n}{l}$ choices for I, where $|I| = l$, this is a lower bound on the multiplicity of each eigenvalue. However, adding together all of the multiplicities from the v_I eigenvectors, we see that this accounts for all possible eigenvectors and therefore our lower bounds are in fact the multiplicities for each eigenvalue. □

Theorem 3.2.2. *The matrix $L(Q(n))$ has eigenvalues $-2n+2l$ with multiplicity $\binom{n}{l}$ for each integer $0 \le l \le n$.*

Proof. Using the previous result, we can diagonalize A as $A = WA'W^{-1}$ where A' is the diagonal matrix containing the eigenvalues of A. Consider $WLW^{-1} = W(A - nI)W^{-1} = A' - nI$. This means that W will also diagonalize L, and the spectrum of L is the same as that for A subtracting n from each eigenvalue. □

One important connection between the spectrum of a graph and its critical group is described by Kirchhoff's Matrix Tree Theorem which relates the order of the critical group to the determinant of the Laplacian matrix and also to the product of its eigenvalues. We state the theorem below.

Theorem 3.2.3 ([1])**.** *Let $\Gamma(V, E)$ be a simple connected graph with $|V| = n$, Laplacian L, and Laplacian spectrum $|\lambda_1| \ge |\lambda_2| \ge ... \ge |\lambda_n| = 0$. Denote a* *reduced Laplacian, that is the Laplacian with row i and column j removed, as $\bar{L}$. Then,*

$$|K(\Gamma)| = (-1)^{i+j} det(\bar{L}) = \frac{1}{n} \prod_{k=1}^{n-1} |\lambda_k|$$

Since we know the spectra of hypercube graphs, we can therefore deduce the order of their critical groups.

Corollary 3.2.3.1.

$$|K(Q(n))| = \frac{1}{2^n} \prod_{l=0}^{n-1} (2n - 2l)^{\binom{n}{l}}$$

3.3 The Critical Group of Hypercube Graphs

When describing the SNF of a matrix it is often useful to tackle the problem one prime at a time. This is accomplished by working over the p-localized integers $\mathbb{Z}_{(p)}$ instead of $\mathbb{Z}$. In this ring it is possible to divide by any integer that is not divisible by p. This allows us to simplify the problem while not losing all of the structure given by integer matrices. Once the SNF is known for all primes the results can be combined to give a full description, since the p-elementary divisors of a matrix when considered over $\mathbb{Z}$ are the same as those over $\mathbb{Z}_{(p)}$ [10].

For primes $p > 2$, we can deduce the critical group directly from eigenvalue information. This is a result of Bai [1] that we will state below.

Theorem 3.3.1 (Bai [1]). *For any odd prime number p, the Sylow p-group of the critical group of the n-cube, $Syl_p(K(Q(n)))$ has the following expression:*

$$Syl_p(K(Q(n))) \cong Syl_p\Big(\prod_{k=1}^{n} ((\mathbb{Z}/k\mathbb{Z})^{\binom{n}{k}}\Big)$$

Bai also confirmed a conjecture of Reiner about the 2-part of $Syl_2(K(Q(n))$. This result was obtained by finding a generating function for the number of occurrences of $\mathbb{Z}/2\mathbb{Z}$ using a generalized form of the Laplacian matrix [1].

Theorem 3.3.2 (Bai [1]). *Let a_n denote the number of occurrences of $\mathbb{Z}/2\mathbb{Z}$ in $Syl_2(K(Q(n)))$.*

$$a_n = 2^{n-2} - 2^{\lfloor \frac{n-2}{2} \rfloor}$$

However, for higher powers of 2, the problem remains open. We will investigate possible methods for solving this problem in the following chapters.

3.4 The Smith Group of Hypercube Graphs

A complete description of the Smith group of hypercube graphs was discovered by Sin, Chandler, and Xiang in [8]. This problem as well can be divided into odd and even p, as well as cases for odd and even n. We will review the main results of their work and introduce some of the tools used by them that will be helpful to us later.

For odd primes p, A defined over $\mathbb{Z}_{(p)}$ is equivalent to its diagonal matrix with eigenvalues. Therefore the p part of the eigenvalues of A give the decomposition for $Syl_p(S(A))$. This is a result from [11]. The proof utilizes the fact that $Q(n)$ is a Cayley graph for the group $(\mathbb{Z}/2\mathbb{Z})^n$ and shows that certain matrices, which are derived from the character table and can be used to diagonalize A, are also unimodular.

Also, when n is odd, all eigenvalues of A are odd, so $det(A)$ is odd, and therefore the diagonal matrix with eigenvalues is a diagonal form which gives the whole Smith group.

We now focus our attention to the remaining part of the problem: to describe $Syl_2(S(A))$ for even n. One important step is to change the basis of the map described by A to the **monomial basis**. Generally we imagine A mapping a vertex to the sum of vertices adjacent to it. However we can re-imagine the domain of A, $\mathbb{Z}^V = \{f : V \to \mathbb{Z}\}$ to be functions that label the vertices with integers. There are many such functions. For example, working with $Q(2)$, we may define $f(v) = f(a_1, a_2) = 3a_1 - a_2$. This f would then assign an integer label of 3 to the vertex, $(1, 0)$. We now need to determine what A does to a given labeling function. We see that the map defined by A works as follows:

$$A(f(a_1, ..., a_n)) = \sum_{i=1}^{n} f(a_1, ..., 1 - a_i, ..a_n).$$

When the adjacency matrix composed with a function acts on a vertex, it gives that vertex a labeling of the sum of the original function applied to the neighbors of the vertex. In our example from before, $A(f(1,0)) = f(0,0)+f(1,1) = 0+2 = 2$. We then notice that any $f \in \mathbb{Z}^V$ can also be thought of as a restriction of polynomials over $\mathbb{Z}[X_1, X_2, ...X_n]$. Restriction of the domain of $\mathbb{Z}[X_1, X_2, ...X_n]$ from $\mathbb{Z}^n$ to $\{0, 1\}^n$ is a module homomorphism from $\mathbb{Z}[X_1, X_2, ...X_n]$ to $\mathbb{Z}^V$. We see that in $\mathbb{Z}^V$, for any i, $X_i^2 = X_i$. As such the kernel of the restriction is generated by polynomials of the form $X_i^2 - X_i$. Therefore we have

$$\mathbb{Z}^V \cong \mathbb{Z}[X_1, X_2, ...X_n]/(X_i^2 - X_i \text{ for } 1 \leq i \leq n).$$

We can therefore see a new basis for our module is the set of monomials, that is elements of the form $X_I = \prod_{i \in I} X_i$ for $I \subseteq [n]$. We can also see that

$$A(X_I) = \sum_{i \in I}(X_{I \setminus \{i\}} - X_I) + \sum_{i \notin I} X_I = (n - 2|I|)X_I + \sum_{J \subset I, |J|+1=|I|} X_J.$$

With respect to the monomial basis, we can rewrite the adjacency matrix as follows

$$\tilde{A} = \left(\begin{array}{c|c|c|c|c|c} nI & W_{0,1} & 0 & 0 & ... & 0 \\ \hline 0 & (n-2)I & W_{1,2} & 0 & ... & 0 \\ \hline 0 & 0 & (n-4)I & W_{2,3} & ... & 0 \\ \hline ... & ... & ... & ... & ... & ... \\ \hline 0 & 0 & 0 & 0 & ... & -nI \end{array}\right)$$

where $W_{t,k}$ is the subset inclusion matrix for t-subsets into k-subsets defined as follows. Let $W_{t,k}$ be the $\binom{n}{t} \times \binom{n}{k}$ matrix with rows and columns indexed by the t-subsets and k-subsets of $[n]$, respectively. If a t-subset is a subset of a k-subset then the corresponding entry in $W_{t,k}$ is 1 and if otherwise then the corresponding entry is 0.

These subset inclusion matrices have a known diagonal form discovered by Wilson in [27]. In their paper, Chandler, Sin, and Xiang use an argument involving the monomial basis, row operations, and the known diagonal form of W to find a diagonal form for $\tilde{A}$ that gives the Smith group for A. For details see [8], but we include the final result here. Note that this result gives the entire SNF, not just the 2-part even though that was all left to discover.

Theorem 3.4.1 (Chandler, Sin, and Xiang [8]). *For even n, $A(Q(n))$ has a diagonal form with entries $0 \leq k \leq \frac{n}{2}$ each with multiplicity $\binom{n}{\frac{n}{2}-k}$*

Chapter 4

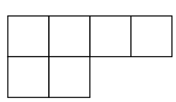
Representation Theory of Symmetric Groups

In the previous chapter, we examined various methods to calculate the critical group of a graph using the eigenvalues of the Laplacian matrix. However, these methods so far have not been sufficient to determine the 2-part of the critical group for hypercube graphs. While this problem remains unsolved, it is the opinion of the author that a solution can be found using the representation theory of symmetric groups. As such, we will spend this chapter reviewing important concepts from representation theory. In the remaining chapters, we will then examine how representation theory can solve a similar problem relating to a diagonal form of a certain class of matrices and share a framework to extend these tools for use in the hypercube problem.

Much of the theory for this section comes from *The Representation Theory of the Symmetric Groups* by G.D. James [21].

4.1 Partitions, Tableaux, and Tabloids

We say $\lambda = (\lambda_1, \lambda_2, ...)$ is a **partition** of n if each λ_j is a non-negative integer, $\lambda_j \geq \lambda_{j+1}$ and $\sum_{i=1}^{\infty} \lambda_i = n$. Given a partition λ, we can form a **diagram** for λ, denoted $[\lambda]$ as an array of left-aligned boxes with the j-th row of the array having λ_j boxes. We can then assign each of the integers from 1 to n to one of the n boxes without repeats to turn the diagram, $[\lambda]$, into a **λ-tableau**.



(a) $[\lambda]$

2	4	3	6
1	5		

(b) Example λ-tableau

Figure 4.1: $\lambda = (4, 2)$

The symmetric group on n elements, which we will denote G_n, has a natural action on tableaux. When an element from G_n acts on a λ-tableau it permutes the integer labels on the boxes. The subgroup of permutations that keep the rows of t, a λ-tableau, fixed set wise is called R_t, the **row stabilizer** of t. We say two tableaux, say t and s, are row equivalent if there is some $\sigma \in R_t$, so $s = \sigma t$. Row equivalence is an equivalence relation, and the equivalence classes formed by this relation are called **tabloids**. The tabloid from tableau t is denoted $\{t\}$. We can think of tabloids as tableaux where the order of elements in each row does not matter. It is often useful to identify m-subsets of $[n]$ with a tabloid. If $2m \leq n$ then we can identify an m-subset with the second row of an $(n-m, m)$ tabloid where the numbers in the second row are the elements of the subset in question. If $2m > n$, we can similarly identify the m-subset with the first row of an $(m, n-m)$-tabloid. In this case the second row of the tabloid will be the complement of the m-subset.

4.2 Specht Modules

With the natural action of G_n, formal linear combinations of λ-tabloids over the field F form an FG_n-module. We call this module a **permutation module** for the partition λ and denote it M^λ. This module is a cyclic module, and any λ-tabloid will generate all of M^λ. We will now examine some submodules of M^λ.

Similarly to how we defined the row-stabilizer of a tableau, we can also define the j**-column stabilizer** of t, C_t^j, to be the subgroup of G_n containing the elements which fix each of the first j-columns of the tableau set-wise and fix the remaining elements point-wise. If j is greater than or equal to the number of entries in the second row, then we say $C_t^j = C_t$ and call it the **(full) column stabilizer** of t.

We can then define the j**-signed column sum** of t in the group algebra FG_n as

$$\kappa_t^j = \sum_{\sigma \in C_t^j} (-1)^\sigma \sigma$$

where $(-1)^\sigma$ is the signum of σ defined to be 1 when σ is an even permutation and -1 when σ is odd.

Given a tableau t we can define e_t^j, the j**-polytabloid** for t as

$$e_t^j = \kappa_t^j \{t\}.$$

Note that even though j-polytabloids are sums of tabloids, they are defined in terms of tableaux. It is possible for two different tableaux in the same tabloid to give different j-polytabloids. When $j \geq \lambda_2$, we can omit the j and simply call $e_t^j = e_t$ a polytabloid.

The submodule of M^λ generated by j-polytabloids is called a j**-Specht module**, S_j^λ. The j-Specht module generated by polytabloids is simply called a **Specht module**, denoted S^λ.

Over a field of characteristic 0, Specht modules give all ordinary irreducible representations of G_n. However, this is not always the case for fields of other characteristics. If we are looking for representations of G_3, then there are 3 different partitions we could use, $\lambda_1 = (3), \lambda_2 = (2,1)$, and $\lambda_3 = (1,1,1)$.

We first examine S^{λ_1}. Since this partition only has one non-zero entry, the tableaux in M^{λ_1} have only one row. Therefore as tabloids they are all equivalent, and the column stabilizer for any tableau is trivial. This means $S^{\lambda_1} = M^{\lambda_1}$. Then the action of any permutation on this tabloid is trivial and acts as an identity, so S^{λ_1} corresponds to the trivial representation.

Looking at λ_2, we can see that S^{λ_2} is spanned by $\overline{3}-\overline{1}$ and $\overline{2}-\overline{1}$, where $\overline{i_1...i_n}$ denotes a tabloid with two rows where the second row contains $i_1, ..., i_n$. We can see that the only element of G_3 that acts like an identity element on both of these spanning elements is the identity of G_3. Therefore, this representation has a trivial kernel, and is a faithful representation of G_3.

Finally, we examine S^{λ_3}. Since any λ_3-tableau has only one column, the column stabilizer for any tableau is G_3. This means that S^{λ_3} is spanned by a single polytabloid that is a sum of all the tabloids. If we say this polytabloid is e_t, then the coefficient of a tabloid is 1 if it came from an even permutation of t and -1 if it came from an odd permutation of t. Therefore, if an even permutation acts on S^{λ_3} it will act as an identity element, and an odd permutation will change the sign of each tabloid in the sum. This means S_3^{λ} is the sign representation for G_3.

Notice that when identifying the Specht modules as representations, each time we picked a basis and saw how G_n acted on those basis elements. While one advantage of the module point of view over the matrix point of view for representations is that it is not necessary to choose a basis, often it will be useful to think in terms of basis elements. For Specht modules, there is a standard basis.

Define a **standard tableau** to be a tableau with entries increasing moving to the right and down. If a tabloid contains a standard tableau then it is called a **standard tabloid**. We also say e_t is a **standard polytabloid** if t is standard. The set of standard polytabloids for a partition, λ, forms the standard basis for S^{λ}. A proof for this fact can be found in Chapter 8 of [21].

The dimension of a Specht module is independent of the ground field F. One can calculate the dimension of S^{λ} using the Hook-length formula.

Theorem 4.2.1 ([21])**.** *The dimension of a Specht module,*

$$dim(S^{\lambda}) = \frac{n!}{\prod_i h_i}$$

where h_i denotes the hook lengths for each box in $[\lambda]$.

For example, to find the dimension of $S^{(3,2,1)}$, we first find the hook-lengths for each box in the diagram. This means we count the number of boxes in a particular box's "hook", those being the box itself, the boxes to the right in the same row, and those below in the same column. The hook lengths for all boxes in $[(3,2,1)]$ are shown below.

$$\begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 3 & 1 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Figure 4.2: Hook lengths for $(3, 2, 1)$

Using these numbers and the Hook-length Formula we calculate

$$dim(S^{(3,2,1)}) = \frac{6!}{(5)(3)(3)(1)(1)(1)} = 16.$$

As a special case of this formula we can calculate the dimension of $S^{(n-m,m)}$, where $(n-m, m)$ is a partition, i.e. $m \leq \frac{n}{2}$.

Theorem 4.2.2.

$$dim(S^{(n-m,m)}) = \binom{n}{m} - \binom{n}{m-1}$$

Proof. For the boxes in the second row have hook-lengths $m, m-1, ..., 1$, since they only have boxes to the right and themselves to count. Boxes in the first row fall into two cases. The first m count $n-m$ boxes in their row plus one more below. The final $n-2m$ count only themselves and the boxes to their right. Therefore the hook lengths are $n-m+1, n-m, ..., n-2m+2, n-2m, n-2m-1, ..., 1$. Using the Hook-length formula, we see

$$\begin{aligned} dim(S^{(n-m,m)}) &= \frac{n!(n-2m+1)}{(n-m+1)!m!} \\ &= \frac{n!(n-2m+1)}{(n-m)!m!(n-m+1)} \\ &= \binom{n}{m}\left(\frac{n-2m+1}{n-m+1}\right) \\ &= \binom{n}{m} - \binom{n}{m}\frac{m}{n-(m-1)} \\ &= \binom{n}{m} - \binom{n}{m-1} \end{aligned}$$

$\square$

Moving forward we will focus on partitions of the form $(n-m, m)$ due to their correspondence with subsets. Many results we describe are still applicable to more general Specht modules.

We stated earlier that when $char(F) = 0$ the Specht modules give all the irreducible representations of G_n. However, when $char(F) > 0$ we still know the structure of Specht modules. Under mild conditions $S^{(n-m,m)}$ has a unique maximal submodule $S^{(n-m,m)} \cap S^{(n-m,m)\perp}$. We denote the quotient of $S^{(n-m,m)}$ with its unique maximal submodule as $D^{(n-m,m)}$. The appearance of $D^{(n-m,m)}$ as a quotient in the filtration of $S^{(n-m,m)}$ is determined only by divisibility properties of n and m.

Let a and b be integers with a base p expansion as follows.

$$a = a_0 + a_1 p + ... a_r p^r$$

$$b = b_0 + b_1 p + ... b_s p^s$$

where $a_r, b_s \neq 0$ and $0 \leq a_i, b_i < p$ for each i. We then say that a contains b to base p if $s < r$ and for each i, $b_i = 0$ or $b_i = a_i$.

Now define $f_p(n,m)$ to be 1 when $n+1$ contains m to base p and 0 otherwise.

Theorem 4.2.3 (James [21]). *Suppose $S^{(n-m,m)}$ is a Specht module and $j < m$. The multiplicity of $D^{(n-j,j)}$ as a factor of $S^{(n-m,m)}$ is $f_p(n-2j, m-j)$.*

This theorem is very powerful, because it allows us to fully describe Specht modules over any field. This is useful, because as we will see permutation modules have a sequence of submodules where successive quotients are isomorphic to Specht modules.

4.3 A Specht Module Series

Using j-Specht modules, we can form a sequence of submodules of $M^{(n-m,m)}$. Denoting the j-Specht Module for the partition $(n-m,m)$ as $S^{(n-m,j)(n-m,m)}$ we form the following series.

$$\begin{aligned} M^{(n-m,m)} &= S^{(n-m,0)(n-m,m)} \\ &\supseteq S^{(n-m,1)(n-m,m)} \\ &\supseteq S^{(n-m,2)(n-m,m)} \\ &\supseteq ... \\ &\supseteq S^{(n-m,m)(n-m,m)} \\ &= S^{(n-m,m)} \end{aligned}$$

This series has the property that quotients of successive submodules are isomorphic to Specht modules.

Theorem 4.3.1 (James [21]). *Let $S^{(n-m,j)(n-m,m)}$ and $S^{(n-m,j+1)(n-m,m)}$ be j-Specht modules. Then,*

$$S^{(n-m,j)(n-m,m)} / S^{(n-m,j+1)(n-m,m)} \cong S^{(n-j,j)}$$

Looking forward, we will use permutation modules as the domain and codomain for the maps described by matrices. Then we will use this submodule structure to find algebraic invariants of the matrices.

Chapter 5

p-ranks of Incidence Matrices

In this section we will show how the representation theory from the previous chapter can be used to solve problems related to diagonal form. In particular we will derive information related to a diagonal form for the matrices $W_{m,k}$ which first appeared in Chapter 2. This form was first discovered by Wilson in [27], but we will follow a proof from Joliffe in [22] which utilizes representation theory.

5.1 p-rank for $W_{m,k}$

Recall that a **subset inclusion matrix** for $[n]$, $W_{m,k}(n)$, is the matrix with rows indexed by m-subsets of $[n]$ and columns indexed by k-subsets of $[n]$ where $W_{m,k}(i,j) = 1$ if the subset corresponding to the jth column contains the subset corresponding to the ith row, and $W_{m,k}(i,j) = 0$ otherwise. We can think of $W_{m,k}$ as encoding a module homomorphism $\psi_{k,m} : M^{(n-k,k)} \to M^{(n-m,m)}$ where, thinking of the second row of a tabloid as a subset, a k-subset is sent to the sum of the m subsets which it contains. Note that here we are using the correspondence between subsets and tabloids, so we assume $2m < 2k \leq n$. However, we may also use the fact that $W^T_{m,k} = W_{n-m,n-k}$ to solve the problem in general.

This map acts very nicely on j-polytabloids. We state a few results below.

Theorem 5.1.1. *If $j \leq i$, then*

$$\psi_{k,i}(e^j_t) = \sum_{\{s\}} e^j_s$$

where s are the tableaux obtained from moving $k-i$ of the last $k-j$ entries in the second row of t to the end of the first row. In the case when $i = j$,

$$\psi_{k,j}(e^j_t) = e^j_{t'}$$

where t' is the tableau obtained by moving the last $k-j$ entries of the second row of t to the end of the first row.

Proof. We have

$$\begin{aligned}\psi_{k,j}(e_t^j) &= \psi_{k,j}(\kappa_t^j\{t\}) \\ &= \psi_{k,j}(\sum_{\sigma\in C_t^j}(-1)^\sigma\sigma\{t\}) \\ &= \sum_{\sigma\in C_t^j}(-1)^\sigma\sigma\psi_{k,j}(\{t\}) \\ &= \sum_{\sigma\in C_t^j}\Big((-1)^\sigma\sigma\sum_{\{s\}}\{s\}\Big)\end{aligned}$$

where {s} are the $(n-j,j)$-tabloids where the second row is a subset of the second row of $\{t\}$. If one of the first j elements of the second row of $\{t\}$ is in the first row of $\{s\}$, then the transposition in C_t^j that swaps it with the number directly above it in t, say τ, acts as an identity on $\{s\}$ since it does not set-wise change the first row. Therefore applying the signed column sum to {s}, we see that the term with this transposition (which is odd) will cancel with the identity permutation. Furthermore, any $\sigma\in C_t^j$ will cancel with $\tau\sigma$. Therefore, the only tabloids that survive are those whose generating tableau do not contain any of the first j entries of the second row of t in their first row. Since these tableaux, which we denote s', have the same first j columns as t, $C_t^j = C_{s'}^j$. Thus,

$$\begin{aligned}\psi_{k,j}(e_t^j) &= \sum_{\sigma\in C_t^j}(-1)^\sigma\sigma\sum_{\{s\}}\{s\} \\ &= \sum_{\{s'\}}\sum_{\sigma\in C_{s'}^j}(-1)^\sigma\sigma\{s'\} \\ &= \sum_{\{s'\}}e_{s'}^j\end{aligned}$$

When $i=j$, there is only one such $\{s'\}$. We may call its generating tableau t', and it is obtained by moving the last $k-j$ entries of the second row of t to the end of the first row.

□

Theorem 5.1.2. *If $i<j$, then*

$$\psi_{k,i}(e_t^j) = 0.$$

Proof. Similarly to the previous theorem we see

$$\psi_{k,i}(e_t^j) = \sum_{\sigma\in C_t^j}(-1)^\sigma\sigma\sum_{\{s\}}\{s\}$$

However, since $i < j$, we are guaranteed the situation from the previous theorem where one of the first j elements of the second row of t is on the first row of s. Therefore it will cancel out in the sum. In this case, we do not have any terms that survive, so

$$\psi_{k,i}(e_t^j) = 0.$$

□

We can see that the proofs for the previous two theorems are basically identical, and the inclusion map very nicely sends j-polytabloids to other j-polytabloids. Only if j is too large, then everything cancels out and j-polytabloids are mapped to 0. This fact allows us to use the Specht Series from 4.3 to aid in our understanding of $\psi_{k,m}$, by examining the image when the domain is restricted to j-Specht modules. By theorem 5.1.2, if $j > m$, then $S^{(n-k,j)(n-k,k)}$ is in the kernel of $\psi_{k,m}$. Also by theorem 5.1.1, we see that if $0 \leq j \leq m$, then the image of $S^{(n-k,j)(n-k,k)}$ is contained in $S^{(n-m,j)(n-m,m)}$. This allows us to create a filtration of $Im(\psi_{k,m})$ where each level of the filtration takes the form $P_{k,m}^j = Im(\psi_{k,m}) \cap S^{(n-m,j)(n-m,m)}$ for $0 \leq j \leq m$. We see

$$Im(\psi_{k,m}) = P_{k,m}^0 \supseteq P_{k,m}^1 \supseteq ... \supseteq P_{k,m}^m \supseteq \{0\}.$$

We summarize this situation with the following diagram.

$$\begin{array}{ccc}
M^{(n-k,k)} & \xrightarrow{\psi_{k,m}} & P_{k,m}^0 \subseteq M^{(n-m,m)} \\
\cup & & \cup \\
S^{(n-k,1)(n-k,k)} & \xrightarrow{\psi_{k,m}} & P_{k,m}^1 \subseteq S^{(n-m,1)(n-m,m)} \\
\cup & & \cup \\
\vdots & & \vdots \\
\cup & & \cup \\
S^{(n-k,m)(n-k,k)} & \xrightarrow{\psi_{k,m}} & P_{k,m}^m \subseteq S^{(n-m,m)} \\
\cup & & \cup \\
S^{(n-k,m+1)(n-k,k)} & \xrightarrow{\psi_{k,m}} & \{0\}
\end{array}$$

Figure 5.1: $\psi_{k,m}$ applied to each level of a Specht series

Now we will use this filtration to find the p-rank of $\psi_{k,m}$, that is the rank of the mapping when the base field for the permutation modules has a characteristic of p.

To accomplish this, we will use theorem 4.3.1 which states that quotient of successive j-Specht modules $S^{(n-m,j)(n-m,m)}$ and $S^{(n-m,j+1)(n-m,m)}$ is isomorphic to $S^{(n-j,j)}$. Looking at the map $\psi_{m,j} : S^{(n-m,j)(n-m,m)} \to S^{(n-j,j)}$, theorem 5.1.1 and theorem 5.1.2 tell us that $ker(\psi_{m,j}) = S^{(n-m,j+1)(n-m,m)}$ and $Im(\psi_{m,j}) = S^{(n-j,j)}$. Not only does this prove 4.3.1, it also shows that by applying a second inclusion map at each level of our diagram, it is possible to calculate the rank of $\psi_{k,m}$ in terms of the dimensions of submodules of various $S^{(n-j,j)}$, which have known dimension. However, this is made easier when we notice $P^j_{k,m}/P^{j+1}_{k,m}$ will always either be isomorphic to $S^{(n-j,j)}$ or the zero module, a fact that we will prove in theorem 5.1.3. This is amazingly true even though Specht modules are not always irreducible in finite fields.

Theorem 5.1.3. *Define $L^{(n-j,j)} = P^j_{k,m}/P^{j+1}_{k,m}$. If $p|\binom{k-j}{m-j}$ then $L^{(n-j,j)} \cong \{0\}$. Otherwise, $L^{(n-j,j)} \cong S^{(n-j,j)}$.*

Proof. Since $\psi_{k,m}(e^j_t) \in P^j_{k,m}$, we know $\psi_{m,j}(\psi_{k,m}(e^j_t)) \in L^{(n-j,j)}$. We see

$$\psi_{m,j}(\psi_{k,m}(e^j_t)) = \psi_{m,j}\Big(\sum_{\{s\}} e^j_s\Big)$$

where $\{s\}$ are the $(n-m,m)$-tabloids formed by moving $k-m$ entries from the last $k-j$ entries of the second row of t to the end of the first row. By theorem 5.1.1, since the first j entries on the second row are the same for each s, $\psi_{m,j}$ maps each of the e^j_s to $e^j_{s'} \in S^{(n-j,j)}$. There are $\binom{k-j}{m-j}$ different $\{s\}$ in the sum, so

$$\begin{aligned}\psi_{m,j}\Big(\sum_s e^j_s\Big) &= \sum_s \psi_{m,j} e^j_s \\ &= \sum_{\{s\}} e^j_{s'} \\ &= \binom{k-j}{m-j} e^j_{s'}\end{aligned}$$

Therefore, as long as $p \not| \binom{k-j}{m-j}$, any $e^j_t \in S^{(n-j,j)}$ is also in the image of $\psi_{m,j}$, so $L^{(n-j,j)} = S^{(n-j,j)}$.

If $p|\binom{k-j}{m-j}$, then this argument does not work, and it is not yet enough to show that $L^{(n-j,j)}$ is the zero module. To show this, we move our attention from j-polytabloids to examining how this composition of inclusion maps acts on tabloids (or 0-polytabloids).

Let $\{t\}$ be an $(n-k,k)$-tabloid. With $\{s\}$ defined as before, $\psi_{k,m}(\{t\}) = \psi_{k,m}(e^0_t) = \sum_{\{s\}} e^0_s = \sum_{\{s\}}\{s\}$. Now, we may examine where $\psi_{m,j}$ maps this sum. We see,

$$\begin{aligned}
\psi_{m,j}(\psi_{k,m}(\{t\})) &= \psi_{m,j}(\sum_{\{s\}\subseteq\{t\}} \{s\}) \\
&= \sum_{\{s\}\subseteq\{t\}} \psi_{m,j}(\{s\}) \\
&= \sum_{\{s\}\subseteq\{t\}} \sum_{\{r\}\subseteq\{s\}} \{r\} \\
&= \binom{k-j}{m-j} \sum_{\{r\}\subseteq\{t\}} \{r\}
\end{aligned}$$

Thus, the image of any tabloid is mapped by $\psi_{m,j}$ to $\binom{k-j}{m-j}$ times the sum of some $(n-j,j)$-tabloids. Therefore if $p|\binom{k-j}{m-j}$, $L^{(n-j,j)} \cong \{0\}$. □

We are now ready to calculate the p-rank of $W_{m,k}$, which will be the dimension of the image of $\psi_{k,m}$. Before that, however, we will update the diagram from before to include the maps into the quotients for reference.

$$\begin{array}{ccccc}
M^{(n-k,k)} & \xrightarrow{\psi_{k,m}} & P^0_{k,m} \subseteq M^{(n-m,m)} & \xrightarrow{\psi_{m,0}} & L^{(n)} \subseteq S^{(n-0,0)} \\
\cup & & \cup & & \\
S^{(n-k,1)(n-k,k)} & \xrightarrow{\psi_{k,m}} & P^1_{k,m} \subseteq S^{(n-m,1)(n-m,m)} & \xrightarrow{\psi_{m,1}} & L^{(n-1,1)} \subseteq S^{(n-1,1)} \\
\cup & & \cup & & \\
\vdots & & \vdots & & \\
\cup & & \cup & & \\
S^{(n-k,m)(n-k,k)} & \xrightarrow{\psi_{k,m}} & P^m_{k,m} \subseteq S^{(n-m,m)} & \xrightarrow{\psi_{m,m}} & L^{(n-m,m)} \subseteq S^{(n-m,m)} \\
\cup & & \cup & & \\
S^{(n-k,m+1)(n-k,k)} & \xrightarrow{\psi_{k,m}} & \{0\} & &
\end{array}$$

Figure 5.2: Mapping each level of $Im(\psi_{k,m})$ into a quotient

Theorem 5.1.4.

$$rank_p(W_{m,k}) = \sum_{p\nmid\binom{k-j}{m-j}} \binom{n}{j} - \binom{n}{j-1}$$

Proof.

$$\begin{aligned} rank_p(W_{m,k}) &= dim(Im(\psi_{k,m})) \\ &= \sum_{j=0}^{m} dim(L^{(n-j,j)}) \\ &= \sum_{p \nmid \binom{k-j}{m-j}} dim(S^{(n-j,j)}) \\ &= \sum_{p \nmid \binom{k-j}{m-j}} \binom{n}{j} - \binom{n}{j-1} \end{aligned}$$

□

For example, consider $W_{1,2}(4)$, the subset inclusion matrix for 1-subsets contained in 2-subsets of a 4 element set. We can calculate $\binom{k-j}{m-j}$ for $0 \leq j \leq 1$. For $j = 0$, $\binom{k-j}{m-j} = \binom{2-0}{1-0} = 2$, and for $j = 1$, $\binom{k-j}{m-j} = \binom{2-1}{1-1} = 1$. This means the only time $W_{1,2}(4)$ will not have full rank is over a field of characteristic 2, since 2 is the only prime that divides either of these numbers. From the theorem, we see $rank_2(W_{1,2}) = \binom{4}{1} - \binom{4}{0} = 3$. We compare this result to the SNF of the matrix below and note that it agrees with our results.

$$SNF(W_{1,2}(4)) = diag_{4\times 6}[1, 1, 1, 2]$$

It is important to note that our result does not by itself, give a diagonal form for $W_{m,k}$. This is due to the fact that the p-rank only tells us the number of 1's in the diagonal form of the matrix when elements are in a field of characteristic p. It is insufficient to determine how many elements divisible by higher powers of p appear in a diagonal form. One advantage of Wilson's result in [27], is a formula for a full diagonal form, rather than only p-ranks. However, in the next chapter we will discuss a method for using representation theory to find a full diagonal form for Laplacian matrices of hypercube graphs. We omit the method from this chapter, since Wilson's formula gives an easy way to calculate a diagonal form for $W_{k,m}(n)$.

Theorem 5.1.5 (Wilson's Formula [27])**.** *$W_{k,m}$ has a diagonal form with entries $\binom{k-j}{m-j}$ for $0 \leq j \leq k$, each with multiplicity $\binom{n}{j} - \binom{n}{j-1}$.*

The reader should note the similarities between the binomial coefficients appearing in this formula, and those from our result on p-ranks.

Chapter 6

The Critical Group Revisited

We now turn our attention back to the hypercube. In particular we will build a framework to describe the 2-part of the critical groups of hypercube graphs in terms of Specht modules. While we do not yet have a formula for this missing part of the critical group, we will provide some illustrative examples to demonstrate how representation theory may be the key to solving this problem.

6.1 $L(Q_n)$ as a Module Homomorphism

In the previous chapter we calculated p-ranks of $W_{m,k}$ by examining the module homomorphism it encodes. Here we will take a similar step. However, in this case, the domain of this homomorphism will not be a permutation module, but rather a direct sum of permutation modules, corresponding to all subsets of an n-element set. Define $\mathcal{M}^n = \bigoplus_{i=0}^{n} M^{(n-i,i)}$. Then the module homomorphism described by $L(Q_n)$ is $\Lambda : \mathcal{M}^n \to \mathcal{M}^n$, and $\Lambda(\{t\}) = -n\{t\} + \sum_s \{s\} + \sum_r \{r\}$, where s are the tableaux formed by moving an entry from the second row of t to the first and r are those formed from moving an entry from the first row of t to the second. We can also write for $(n-i,i)$-tabloid $\{t\}$, $\Lambda(\{t\}) = -n\{t\} + \psi_{i,i-1}(\{t\}) + \psi_{i,i+1}(\{t\})$.

Note that in our definition of $\mathcal{M}^n$ we include $M^{(n-i,i)}$ where $2i > n$. In these cases $(n-i,i)$ is not a true partition, since our original definition demands that the number of columns in a given row be monotonically decreasing. However, we can work around this issue by imagining swapping the first and second rows and using the first row of the tabloids to correspond to large subsets. Therefore when we use $M^{(n-i,i)}$ for $2i > n$ we are really thinking about $M^{(i,n-i)}$. However, for notational convenience we will primarily use the former. When $2i \leq n$, $M^{(n-i,i)}$ will denote the standard permutation module that we have used in previous chapters, with the second row of tabloids corresponding to subsets.

Just as we did in the previous chapter, with a single permutation module, we also wish to create a series of submodules for $\mathcal{M}^n$. Luckily, we can use the known Specht series for each component of $\mathcal{M}^n$ to aid in this task. There are still many ways to form a descending sequence, but it is natural to consider the one in which each level corresponds to j-polytabloids. Below we give an example of this series for $n = 4$.

$$\begin{array}{ccccccccc}
M^{(4,0)} & \oplus & M^{(3,1)} & \oplus & M^{(2,2)} & \oplus & M^{(1,3)} & \oplus & M^{(0,4)} \\
 & & & & \cup & & & & \\
 & & S^{(3,1)} & \oplus & S^{(2,1)(2,2)} & \oplus & S^{(1,3)} & & \\
 & & & & \cup & & & & \\
 & & & & S^{(2,2)} & & & &
\end{array}$$

Figure 6.1: Filtration for $\mathcal{M}^4$

Note that each column is the normal Specht series for $M^{(n-i,i)}$, and each row corresponds to j-polytabloids with increasing j for smaller submodules. When $j > i$, there are no new j-polytabloids for $M^{(n-i,i)}$, so we end up with a triangular shaped diagram. Recall that when $i > n - i$, we have defined $M^{(n-i,i)} = M^{(i,n-i)}$. This is why both the first and last modules in the sum have only a trivial submodule in the next level and are thus omitted from the diagram.

Since in general for modules A and B with submodules A' and B' respectively, $\frac{A\oplus B}{A'\oplus B'} \cong \frac{A}{A'} \oplus \frac{B}{B'}$, we can use theorem 4.3.1 to find the quotients between each level of our filtration.

Theorem 6.1.1.

$$\frac{\bigoplus_{i=j}^{n-j} S^{(n-i,j)(n-i,i)}}{\bigoplus_{i=j+1}^{n-j-1} S^{(n-i,j+1)(n-i,i)}} \cong (S^{(n-j,j)})^{n+1-2j}$$

where $(S^{(n-j,j)})^k = \bigoplus_{l=1}^{k} S^{(n-j,j)}$.

We may map elements of $\mathcal{M}^n$ into a quotient module by applying the function $\psi_j = \bigoplus_{i=j}^{n-j} \psi_{i,j}$ to an element on the jth level of the filtration. The map ψ_j acts on each component of the direct sum separately, so when looking at the first level of the filtration, which corresponds to formal sums of subsets, only 0-subsets are sent to the first copy of $S^{(n-0,0)}$ in the quotient $(S^{(n-0,0)})^k$, only 1-subsets are sent to the second copy of $S^{(n-0,0)}$ in $(S^{(n-0,0)})^k$, 2-subsets to the third, and so on.

6.2 Higher Powers of 2

The representation theoretical method we have developed so far has only been able to identify the 2-rank of L. This is equivalent to the multiplicity of 1 in $SNF(L)$ when considered over integers modulo 2. However, it is already known from other methods that the 2-rank of $L(Q_n)$ is 2^{n-1} [1]. Our goal is to adapt the representation theoretical method to find the multiplicity of higher powers of 2 in $SNF(L)$.

For $\Lambda : \mathcal{M}^n \to \mathcal{M}^n$, define the following submodules for integers $i \geq 0$:

$$M_i = \{x \in \mathcal{M}^n : 2^i | \Lambda(x)\}$$

$$N_i = \left\{\frac{1}{2^i}\Lambda(x) : x \in M_i\right\}.$$

The M_i (not to be confused with permutation modules), form a descending sequence of submodules of the domain, with M_0 being the entire domain $\mathcal{M}^n$. On the other hand, the N_i form an increasing sequence of submodules of the co-domain, where $N_0 = Im(\Lambda)$. Eventually, the sequence of N_i's stabilizes to the purification of $Im(\Lambda)$, that being the smallest submodule containing $Im(\Lambda)$ that is a direct summand of $\mathcal{M}^n$. With $\bar{N_i}$ denoting N_i modulo 2, and e_i denoting the multiplicity of 2^i in $SNF(L)$ over $\mathbf{Z}_{(2)}$, we see [5]

$$dim(\bar{N_i}) = e_0 + e_1 + ... + e_i.$$

The following theorem will allow us determine values for e_i, using lower bounds for $dim(\bar{N_i})$. To state the theorem, define $v_p(x) = max\{i : p^i | x\}$.

Theorem 6.2.1. *Suppose Λ is the Laplacian map for Laplacian matrix L of a connected graph Γ with v vertices:*

$$\Lambda\colon \mathbb{Z}_{(p)}^{V(\Gamma)} \to \mathbb{Z}_{(p)}^{V(\Gamma)}.$$

Set $d = v_p(|K(\Gamma)|)$, where $K(\Gamma)$ is the critical group of Γ. Let e_i denote the multiplicity of p^i in the SNF of L. Suppose that we have

$$N_{a_0} \subseteq N_{a_1} \subseteq N_{a_2} \subseteq ... \subseteq N_{a_\ell}$$

with

$$0 = a_0 < a_1 < a_2 < \cdots < a_\ell$$

and corresponding lower bounds

$$0 \leq b_0 < b_1 < b_2 \cdots < b_\ell$$

satisfying:

1. $b_i \leq \dim \overline{N_{a_i}}$ *for* $0 \leq i \leq \ell$,
2. $b_\ell = v - 1$,

3. $\sum_{j=1}^{\ell} a_j(b_j - b_{j-1}) = d$.

Then

1. $e_i = 0$ *if* $i \notin \{0, a_1, a_2, \ldots, a_\ell\}$,

2. $e_0 = b_0$,

3. $e_{a_j} = b_j - b_{j-1}$ *for* $1 \leq j \leq \ell$.

Proof. Note that in the hypothesis of the theorem, the condition $b_\ell = v - 1$ implies that $N_{a_\ell} = P$, the purification of the image of Λ, $\dim \overline{N_{a_\ell}} = v - 1$, and $e_i = 0$ for $i > a_\ell$. We have

$$\begin{aligned}
d =& \sum_{i \geq 0} i e_i \\
=& \sum_{j=1}^{\ell} \sum_{a_{j-1} < i \leq a_j} i e_i && (6.1) \\
\leq& \sum_{j=1}^{\ell} a_j \sum_{a_{j-1} < i \leq a_j} e_i && (6.2) \\
=& \sum_{j=1}^{\ell} a_j (\dim \overline{N_{a_j}} - \dim \overline{N_{a_{j-1}}}) \\
=& - a_1 \dim \overline{N_0} + \sum_{j=1}^{\ell-1} (a_j - a_{j+1}) \dim \overline{N_{a_j}} + a_\ell \dim \overline{N_{a_\ell}} && (6.3) \\
\leq& - a_1 b_0 + \sum_{j=1}^{\ell-1} (a_j - a_{j+1}) b_j + a_\ell b_\ell && (6.4) \\
=& \sum_{j=1}^{\ell} a_j (b_j - b_{j-1}) \\
=& d,
\end{aligned}$$

and so equality must hold throughout. The equality of lines 6.1 and 6.2 implies that $e_i = 0$ unless $i \in \{0, a_1, a_2, \ldots, a_\ell\}$. Equality of lines 6.3 and 6.4 implies that $b_i = \dim \overline{N_{a_i}}$ for $0 \leq i \leq \ell$, and the conclusion of the theorem immediately follows. □

6.3 Working with the Monomial Basis

To simplify calculations, we will take advantage of the monomial basis which we first described in chapter 3 when working with adjacency matrices. Recall that with respect to the monomial basis

$$A(X_I) = \sum_{i \in I} (X_{I \setminus \{i\}} - X_I) + \sum_{i \notin I} X_I = (n - 2|I|) X_I + \sum_{J \subset I, |J|+1=|I|} X_J.$$

Suppose Q is the change of basis matrix such that A with respect to the monomial basis $\tilde{A} = Q^{-1}AQ$. We see that since $L = A - nI$,

$$\tilde{L} = Q^{-1}LQ = \tilde{A} - nI.$$

We therefore conclude that with respect to the monomial basis,

$$\Lambda(X_I) = -2|I|X_I + \sum_{J \subset I, |J|+1=|I|} X_J.$$

This change in basis simplifies our calculations, since it allows us to focus on only one direction of subset inclusion rather than the original two. Also, when we work modulo 2 the term corresponding to the input set will be killed, since it always has an even coefficient.

6.4 $n = 3$ Example

In this section we will use the tools we have built to solve for the 2-part of the 3-cube. First we find a lower bound for $dim(N_0) = dim(Im(\Lambda))$. We accomplish this in the same way we found p-ranks in the previous chapter, by sending elements in $Im(\Lambda)$ into the quotients defined by the Specht series and keeping track of which elements survive.

We consider $\Lambda(e_t^j)$, where t is an $(n-i,i)$-tableau. We have

$$\begin{aligned}
\Lambda(e_t^j) &= \Lambda(\sum_{\sigma \in C_t^j} (-1)^\sigma \sigma\{t\}) \\
&= \sum_{\sigma \in C_t^j} (-1)^\sigma \sigma \Lambda(\{t\})) \\
&= \sum_{\sigma \in C_t^j} (-1)^\sigma \sigma(-2i\{t\} + \sum_{\{s\} \subset_{i-1} \{t\}} \{s\}) \\
&= -2ie_t^j + \sum_{s'} e_{s'}^j
\end{aligned}$$

where s' are the tableaux obtained from from moving one of the last $i-j$ entries of the second row of t to the end of the first row. Recall that if $2i > n$ we identify a $(n-i,i)$"tableau" with the $(i, n-i)$-tableau with rows swapped. In this case, s' will move an entry from the last $i-j$ boxes of the first row to the end of the second.

Now we apply ψ_j to identity the image in the jth level of the quotient, $\frac{\bigoplus_{i=j}^{n-j} S^{(n-i,j)(n-i,i)}}{\bigoplus_{i=j+1}^{n-j-1} S^{(n-i,j+1)(n-i,i)}} \cong (S^{(n-j,j)})^{n+1-2j}$. We see

$$\psi_j(\Lambda(e_t^j)) = (i-j)e_{s'}^j \oplus -2ie_{t'}^j$$

where s' is the tableau with the last $i-1-j$ entries of the second row of t are moved to the end of the first row, and t' is the tableau with the last $i-j$ entries of the second row moved of t moved to the end of the first.

Since modulo 2 the $-2ie^j_{t'}$ part of the summand will be killed, for $0 \le i \le n$ the ith copy of $S^{(n-j,j)}$ in $(S^{(n-j,j)})^{n+1-2j}$ will appear in $\psi_j(Im(\Lambda))$ if and only if $i-j$ is odd. This means that at least every other $S^{(n-j,j)}$ contributes to $\bar{N_0}$. In the $n=3$ case, $dim(\bar{N_0}) \ge 2dim(S^{(n-0,0)}) + dim(S^{(n-1,1)}) = 4$, so we take as a lower bound $b_0 = 4$. Here we note that it has been shown using other methods that $dim(\bar{N_0}) = 2^{n-1}$ [1]. This means that for odd n our representation theoretical bound is tight.

For N_1, we must find elements in $Im(\Lambda)$ divisible by 2. Trivially, twice any element in N_0 will be in N_1, so $dim(\bar{N_1}) \ge dim(\bar{N_0})$. However we also find

$$\Lambda(\{2,3\}+\{1,3\}+\{1,2\}) = -4(\{2,3\}+\{1,3\}+\{1,2\}) + 2(\{3\}+\{2\}+\{1\}).$$

So,

$$-2((\{2,3\}+\{1,3\}+\{1,2\}) + \{3\}+\{2\}+\{1\} \in N_1$$

We can quickly verify that this gives us something new. Applying ψ_0 with $\emptyset$ denoting the empty set, we see $\psi_0(-2((\{2,3\}+\{1,3\}+\{1,2\}) + \{3\}+\{2\}+\{1\}) = -6\emptyset \oplus 3\emptyset$, which when taken modulo 2 generates the copy of $S^{(n-0,0)}$ corresponding to 1-subsets in the quotient. So, $dim(\bar{N_1}) \ge dim(\bar{N_0}) + 1$, and we take $b_1 = 4+1 = 5$ as our lower bound.

After some experimenting, we see that we cannot find anything in N_2 that when considered modulo 2 gives us a new element different to those from N_1, so we move to N_3. Taking a $(1,2)$-"tableau" t corresponding to a 2-subset, let t' be the $(2,1)$-tableau where the last entry from the second row of t has been moved to the end of the first row.

$$\begin{aligned} \Lambda(3e^1_{t'} \oplus -2e^1_t) &= 3\Lambda(e^1_{t'}) \oplus -2\Lambda(e^1_t) \\ &= 3[0 \oplus -2e^1_{t'}] \oplus -2[e^1_{t'} \oplus -4et^1] \\ &= -8e^1_{t'} \oplus 8e^1_t \\ \implies -e^1_{t'} \oplus e^1_t \in N_3 & \end{aligned}$$

Applying ψ_1, we see

$$\psi_1(-e^1_{t'} \oplus e^1_t) = -e^1_{t'} \oplus e^1_{t'}.$$

Since we already have the copy of $S^{(n-1,1)}$ corresponding to 1-polytabloids of 1-subsets from N_0, the only summand we add is the copy of $S^{(n-1,1)}$ corresponding to 1-polytabloids of 2-subsets. The dimension of $S^{(n-1,1)}$ is 2, so this tells us $dim(\bar{N_3}) \ge dim(\bar{N_1}) + 2$. Thus we take as a lower bound $b_3 = 5+2 = 7$.

We now have enough information to apply theorem 6.2.1. From the Matrix Tree Theorem we know $|K(Q(3)| = 2^7 * 3$, so $d = v_2(|K(Q(3)|) = 7 = b_3$. Also,

$$\sum_{j=1}^{l} a_j(b_j - b_{j-1}) = 1(5-4) + 3(7-5) = 7 = d.$$

Therefore, $e_0 = 4, e_1 = 1$, and $e_2 = 2$. Using a computer, we calculate $SNF(L(Q(3))) = diag[1, 1, 1, 1, 2, 2^3, 2^3 \cdot 3, 0]$, and find our result agrees. Our calculation required us to find vectors in each of the N_i to find our lower bounds for $dim(\bar{N}_i)$. We picked these vectors by hand, but if a systematic way to choose these vectors that resulted in the correct lower bounds were found, then the problem would be solved in general.

6.5 Observations

To conclude, we will state some of the author's observations about the 2-part of $K(Q)$ that may be useful for those who wish to study it further. These observations also demonstrate how the representation theoretical approach is useful for solving this problem and offers insights missing from other approaches.

1. The number of 2's in $SNF(L(Q(n)))$ was shown by Bai to be $a_n = 2^{n-2} - 2^{\lfloor \frac{n-2}{2} \rfloor}$ for $n \geq 2$ [1]. For odd $n \geq 3$ this can be written as

$$\binom{n}{k} + \binom{n}{k-1} + \binom{n}{k-4} + \binom{n}{k-5} + ...$$

   where $k = \frac{n-3}{2}$ and the sum includes only the $\binom{n}{k-j}$ where $j = 0, 1$ modulo 4. This sum can also be written in terms of the dimensions of Specht modules as follows.

$$dim(S^{(n-k,k)}) + \sum_{j \geq 0} \sum_{1 \leq i \leq 3} (2 + j) dim(S^{(n-(k-i-3j),(k-i-3j))})$$

2. For $n = 7, 9, 11$ the same formula,

$$dim(S^{(n-k,k)}) + \sum_{j \geq 0} \sum_{1 \leq i \leq 3} (2 + j) dim(S^{(n-(k-i-3j),(k-i-3j))})$$

   this time with $k = \frac{n-7}{2}$, seems to predict the multiplicity of 4 in $SNF(L(Q_n))$. The author is hesitant to formally conjecture this is true for all odd $n \geq 7$, however, due to a lack of available data.

3. Should 1. and 2. hold, all but one $S^{(n-j,j)}$ in each level of the quotient would be included in N_2. Calculations on the multiplicities of higher powers of 2 for $n \leq 11$ indicate they can be written as sums of the irreducible components, $D^{(n-k,k)}$ of the remaining $S^{(n-j,j)}$.

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