The Sandpile Group of Subset Intersection Graphs

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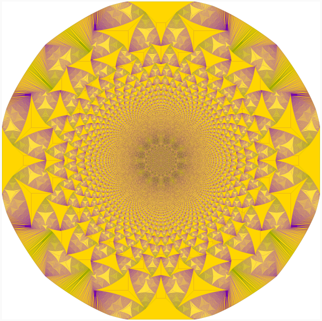
September 9, 2023

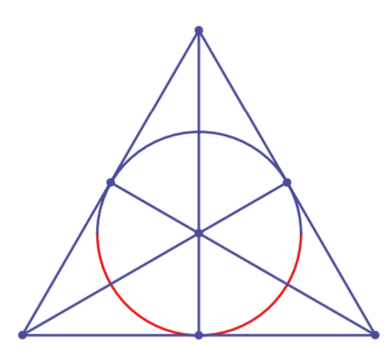
Introduction



- **Cellular automaton** (grid of cells, each of which has a state which changes with time)
- Piles of sand that topple
- Sandpile group can be derived from this model

Credit: Wikimedia Commons (`https://commons.wikimedia.org/wiki/File:Sandpile_on_infinite_grid,_3e7_grains.png`)

APPLICATIONS: t-DESIGNS



$2-(7,3,1)$ design

- $t-(V,k,\lambda)$**:** Family of k-subsets, called blocks, of a V-set, so any t-subset of the V-set is contained in exactly λ blocks
- Applications in **design of experiments** in statistics

Credit: Handbook of Combinatorial Designs

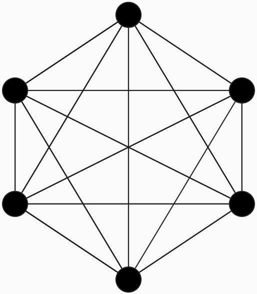
APPLICATIONS: ERROR-CORRECTING CODES

- Helps correct errors in data transmission
- **Association schemes** derived from graphs like ours help determine bounds of how efficiently data is corrected

Credit: NASA
(`https://www.nasa.gov/mission_pages/juno/images/index.html`)

Background

GRAPHS



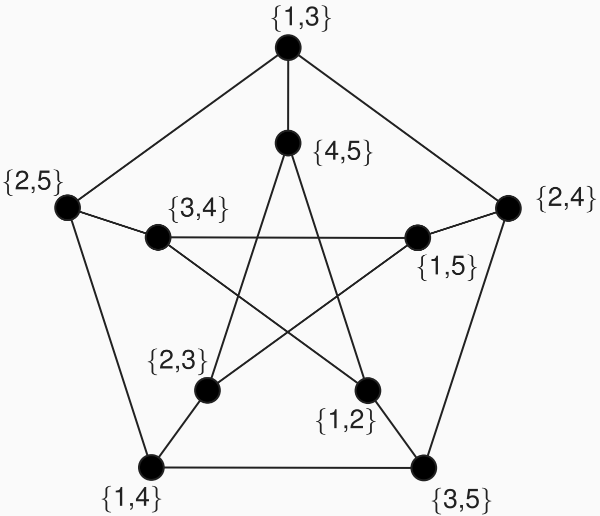
A **graph** Γ consists of a set of **vertices** $\mathscr{V}$ and a set of **edges** $\mathscr{E}$.

An edge $e = \{v_0, v_1\}$ is a pair of elements from the vertex set $\mathscr{V}$.

For an **intersection graph** $\Gamma(n, k, \ell)$:

- $\mathscr{V}$ is composed of size k **subsets of an** n **element set**
- $\mathscr{E}$ is determined by how much the subsets overlap (ℓ is the **size of the intersection** between adjacent subsets)

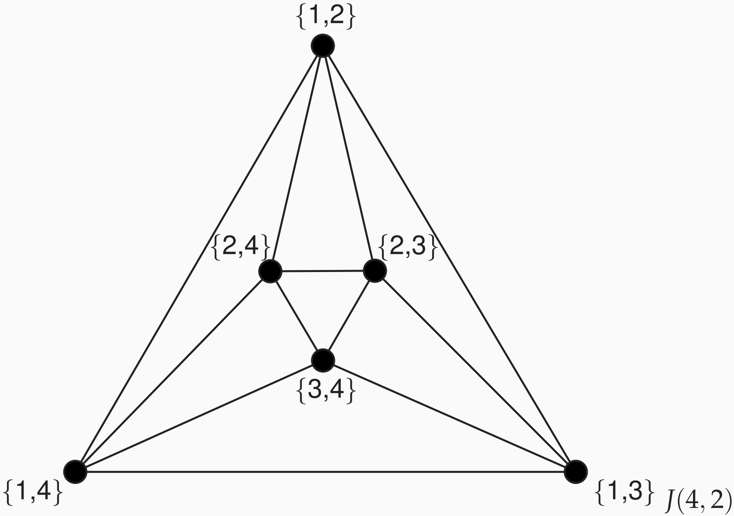
KNESER GRAPHS: $\ell = 0$



$KN(5, 2)$

edges connect sets that are disjoint

JOHNSON GRAPHS, $\ell = k - 1$



edges connect sets that have an intersection of size $k-1$

The Adjacency Matrix of a Graph

- $n \times n$ matrix where n is the number of vertices
- A vertex V is **adjacent** to another vertex W when there is an edge between V and W
- $A(i,j) = 1$ if and only if i and j are adjacent, otherwise is 0

$$A = \begin{array}{c} \\ \{1,2\} \\ \{1,3\} \\ \{1,4\} \\ \{2,3\} \\ \{2,4\} \\ \{3,4\} \end{array} \begin{array}{c} \begin{array}{cccccc} \{1,2\} & \{1,3\} & \{1,4\} & \{2,3\} & \{2,4\} & \{3,4\} \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Adjacency matrix for $J(4,2)$

The Degree Matrix of a Graph

- The **degree** of a vertex V is the number of vertices adjacent to V.
- $D_{ij} = \begin{cases} \text{deg}(i) & j = i \\ 0 & j \neq i \end{cases}$

$$D = \begin{array}{c} \\ \{1,2\} \\ \{1,3\} \\ \{1,4\} \\ \{2,3\} \\ \{2,4\} \\ \{3,4\} \end{array} \begin{array}{c} \begin{array}{cccccc} \{1,2\} & \{1,3\} & \{1,4\} & \{2,3\} & \{2,4\} & \{3,4\} \end{array} \\ \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \end{array}$$

Degree matrix for $J(4,2)$

- The **Laplacian matrix** L is defined as the difference between the degree and adjacency matrices: $L = D - A$

$$L = \begin{array}{c} \\ \{1,2\} \\ \{1,3\} \\ \{1,4\} \\ \{2,3\} \\ \{2,4\} \\ \{3,4\} \end{array} \begin{array}{c} \begin{array}{cccccc} \{1,2\} & \{1,3\} & \{1,4\} & \{2,3\} & \{2,4\} & \{3,4\} \end{array} \\ \begin{pmatrix} 4 & -1 & -1 & -1 & -1 & 0 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ -1 & -1 & 4 & 0 & -1 & -1 \\ -1 & -1 & 0 & 4 & -1 & -1 \\ -1 & 0 & -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & -1 & -1 & 4 \end{pmatrix} \end{array}$$

Laplacian matrix for $J(4,2)$

Adjacency:

- For any graph $\Gamma(n,k,\ell)$, we can find its adjacency matrix A
- A has eigenvalues $\mu_0, \mu_1, \cdots, \mu_k$
- The multiplicity m_j of any eigenvalue μ_j is $m_j = \binom{n}{j} - \binom{n}{j-1}$

Laplacian:

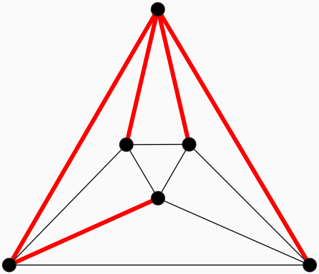
- For any graph $\Gamma(n,k,\ell)$, we can find its Laplacian matrix L
- L has eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_k$
  - Each vertex of $\Gamma(n,k,\ell)$ has degree d
  - $\lambda_j = d - \mu_j$
- The multiplicity m_j of any eigenvalue λ_j is $m_j = \binom{n}{j} - \binom{n}{j-1}$

Sandpile Groups

Biggs showed that diagonalizing the Laplacian matrix returns the sandpile group of a graph.

The **sandpile group** of a graph is a finitely generated abelian group which corresponds with several things:

- Its order is the number of spanning trees in the graph.
- The equivalence classes of chip-firing configurations on a graph are the elements of its sandpile group.



Smith Normal Form

Reducing a matrix M to its Smith Normal Form returns a diagonal matrix where each entry can be divided by the previous.

This is done by performing row and column operations on the original matrix.

$$\text{SNF}(M) = R \begin{bmatrix} 2 & 4 & -8 & 0 \\ 1 & 0 & -2 & 2 \\ 12 & -4 & -2 & 4 \\ 10 & -4 & -2 & 0 \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 48 \end{bmatrix}$$

However, finding the SNF of a large matrix using a computer either takes too long or runs out of memory before completion.

Subsets and Standard Objects

Set:

$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Subset:

$\overline{135}$

Standard Object:

$\overline{z_1\ z_2\ \ldots\ z_i\ \ldots\ z_k}\ \forall i\ z_i \geq 2 \cdot i$

$\overline{246}$

Examples:

Standard Objects:

$\overline{3}$, $\overline{259}$, $\overline{6789}$

NOT Standard Objects:

$\overline{1}$, $\overline{23}$, $\overline{2458}$

Our Work

Our Goal:

Diagonalize the Laplacian of subset intersection graphs to find the Sandpile group

$n = 5$
$k = 2$

Standard Objects

2-Subsets

	$\emptyset$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{24}$	$\overline{25}$	$\overline{34}$	$\overline{35}$	$\overline{45}$
$\overline{12}$	1	1	0	0	0	0	0	0	0	0
$\overline{13}$	1	0	1	0	0	0	0	0	0	0
$\overline{14}$	1	0	0	1	0	0	0	0	0	0
$\overline{15}$	1	0	0	0	1	0	0	0	0	0
$\overline{23}$	1	1	1	0	0	0	0	0	0	0
$\overline{24}$	1	1	0	1	0	1	0	0	0	0
$\overline{25}$	1	1	0	0	1	0	1	0	0	0
$\overline{34}$	1	0	1	1	0	0	0	1	0	0
$\overline{35}$	1	0	1	0	1	0	0	0	1	0
$\overline{45}$	1	0	0	1	1	0	0	0	0	1

CONJUGATING L BY P

Recall: $L = D - A$

$$\begin{aligned} U_L &= P^{-1}LP \\ &= P^{-1}(D - A)P \\ &= P^{-1}DP - P^{-1}AP \\ &= D - P^{-1}AP \\ &= D - U_A \end{aligned}$$

Standard Object Inclusion Matrix

$$n = 5$$

$$\widetilde{W_{1,2}} = \begin{array}{c} \\ \overline{2} \\ \overline{3} \\ \overline{4} \\ \overline{5} \end{array} \begin{array}{c} \begin{array}{ccccc} \overline{24} & \overline{25} & \overline{34} & \overline{35} & \overline{45} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

Theorem:

$$U_A = \left(\begin{array}{c|c|c|c|c} f_{0,0}I & f_{0,1}\widetilde{W_{0,1}} & f_{0,2}\widetilde{W_{0,2}} & \cdots & f_{0,k}\widetilde{W_{0,k}} \\ \hline 0 & f_{1,1}I & f_{1,2}\widetilde{W_{1,2}} & \cdots & f_{1k}\widetilde{W_{1,k}} \\ \hline 0 & 0 & \ddots & \ddots & \vdots \\ \hline 0 & 0 & 0 & \ddots & f_{k-1,k}\widetilde{W_{k-1,k}} \\ \hline 0 & 0 & 0 & 0 & f_{k,k}I \end{array}\right)$$

$$f_{b,a} = c_{b,a} - \sum_{w=0}^{b-1} \binom{b}{w} f_{w,a},$$

$$c_{b,a} = \binom{k-b}{\ell-b}\binom{n-a-k+b}{k-a-\ell+b}.$$

Generalizing the Entries of U_L

Corollary:

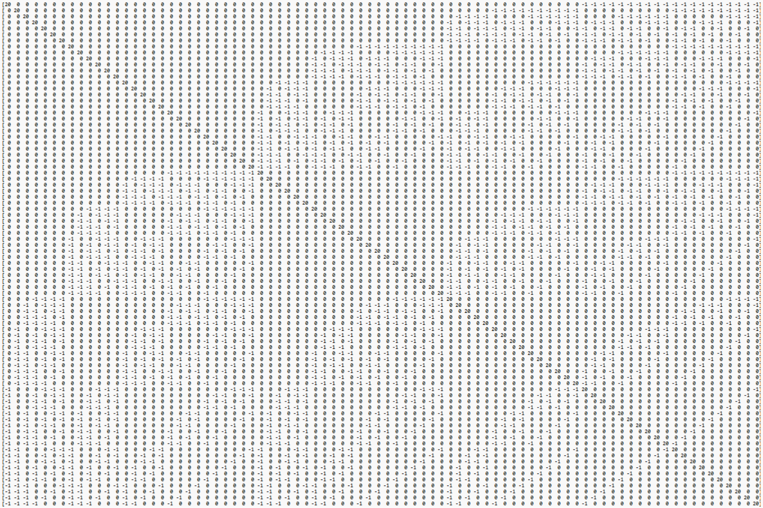
$$U_L = \left(\begin{array}{c|c|c|c|c}
\left(\binom{n-k}{k-l} - f_{0,0}\right)I & -f_{0,1}\widetilde{W_{0,1}} & -f_{0,2}\widetilde{W_{0,2}} & \cdots & -f_{0,k}\widetilde{W_{0,k}} \\ \hline
0 & \left(\binom{n-k}{k-l} - f_{1,1}\right)I & -f_{1,2}\widetilde{W_{1,2}} & \cdots & -f_{1,k}\widetilde{W_{1,k}} \\ \hline
0 & 0 & \ddots & \ddots & \vdots \\ \hline
0 & 0 & 0 & \ddots & -f_{k-1,k}\widetilde{W_{k-1,k}} \\ \hline
0 & 0 & 0 & 0 & \left(\binom{n-k}{k-l} - f_{k,k}\right)I
\end{array}\right)$$

$$f_{b,a} = c_{b,a} - \sum_{w=0}^{b-1} \binom{b}{w} f_{w,a},$$

$$c_{b,a} = \binom{k-b}{\ell-b}\binom{n-a-k+b}{k-a-\ell+b}.$$

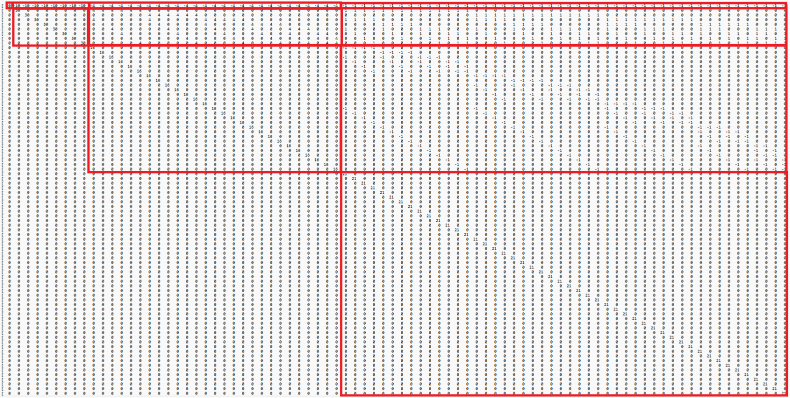
Example $KN(9,3) = \Gamma(9,3,0)$

$L =$



EXAMPLE $KN(9,3) = \Gamma(9,3,0)$

$P^{-1}LP =$



Example $KN(9,3) = \Gamma(9,3,0)$

$$P^{-1}LP = \left(\begin{array}{c|ccc|ccc|ccc}
0 & -10 & \cdots & -10 & -4 & \cdots & -4 & -1 & \cdots & -1 \\
\hline
 & 30 & & & & & & & & \\
0 & & \ddots & & & 4\widetilde{W_{1,2}} & & & 1\widetilde{W_{1,3}} & \\
 & & & 30 & & & & & & \\
\hline
 & & & & 16 & & & & & \\
0 & & 0 & & & \ddots & & & -1\widetilde{W_{2,3}} & \\
 & & & & & & 16 & & & \\
\hline
 & & & & & & & 21 & & \\
0 & & 0 & & & 0 & & & \ddots & \\
 & & & & & & & & & 21
\end{array}\right)$$

Example: $n = 5, \widetilde{E}_1 \widetilde{W}_{1,2} \widetilde{E}_2^{-1} = \widetilde{D}_{1,2}$

$$
\begin{aligned}
&= \widetilde{E}_1 \widetilde{W}_{1,2} \widetilde{E}_2^{-1} \\
&= \widetilde{E}_1 \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \widetilde{E}_2^{-1} \\
&= \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\
&= \widetilde{D}_{1,2}
\end{aligned}
$$

Diagonalizing Inclusion Matrices of Standard Objects

- $\widetilde{W}_{r,s}$ is a inclusion matrix
  - Rows indexed by standard objects of size r
  - Columns indexed by standard objects of size s
- $\widetilde{E}_j$ is a unimodular inclusion matrix
  - Rows indexed by a selection of standard objects of size $\leq j$
  - Columns indexed by all standard objects of size j
- $\widetilde{D}_{r,s} =$ diag$(\binom{s-j}{r-j}^{\widetilde{e}_j}, j = 0, 1, \cdots r)$
  - $\widetilde{e}_j = \binom{n}{j} - \binom{n}{j-1} - (\binom{n}{j-1} - \binom{n}{j-2})$

Theorem: There exist $\widetilde{E}_r$ and $\widetilde{E}_s$ such that $\boxed{\widetilde{E}_r \widetilde{W}_{r,s} \widetilde{E}_s^{-1} = \widetilde{D}_{r,s}}$

More visually...

$$\widetilde{E}_r \widetilde{W}_{r,s} \widetilde{E}_s^{-1} = \widetilde{D}_{r,s} = \begin{pmatrix} \underbrace{\binom{s-0}{r-0}}_{e_0} & & & \\ & \underbrace{\binom{s-1}{r-1}}_{e_1} & & \\ & & \ddots & \\ & & & \underbrace{\binom{s-r}{r-r}}_{e_j} \end{pmatrix}$$

Combining the $\widetilde{E}_j$'s

$$E = \left(\begin{array}{c|c|c|c} \widetilde{E}_0 & 0 & \dots & 0 \\ \hline 0 & \widetilde{E}_1 & \dots & 0 \\ \hline \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & \widetilde{E}_k \end{array}\right)$$

PUTTING U_L INTO A NEW FORM...

$$EU_LE^{-1} =$$

$$= \left(\begin{array}{c|c|c|c} \widetilde{E}_0 & 0 & \dots & 0 \\ \hline 0 & \widetilde{E}_1 & \dots & 0 \\ \hline \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & \widetilde{E}_k \end{array}\right) \left(\begin{array}{c|c|c|c} \lambda_0 I & f_{0,1}\widetilde{W}_{0,1} & \dots & f_{0,k}\widetilde{W}_{0,k} \\ \hline 0 & \lambda_1 I & \dots & f_{1,k}\widetilde{W}_{1,k} \\ \hline \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & \lambda_k I \end{array}\right) \left(\begin{array}{c|c|c|c} \widetilde{E}_0^{-1} & 0 & \dots & 0 \\ \hline 0 & \widetilde{E}_1^{-1} & \dots & 0 \\ \hline \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & \widetilde{E}_k^{-1} \end{array}\right)$$

$$= \left(\begin{array}{c|c|c|c} \lambda_0 \widetilde{E}_0 & f_{0,1}\widetilde{E}_0\widetilde{W}_{0,1} & \dots & f_{0,k}\widetilde{E}_0\widetilde{W}_{0,k} \\ \hline 0 & \lambda_1 \widetilde{E}_1 I & \dots & f_{1,k}\widetilde{E}_1\widetilde{W}_{1,k} \\ \hline \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & \lambda_k \widetilde{E}_k I \end{array}\right) \left(\begin{array}{c|c|c|c} \widetilde{E}_0^{-1} & 0 & \dots & 0 \\ \hline 0 & \widetilde{E}_1^{-1} & \dots & 0 \\ \hline \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & \widetilde{E}_k^{-1} \end{array}\right)$$

$$= \left(\begin{array}{c|c|c|c} \lambda_0 I & f_{0,1}\widetilde{D}_{0,1} & \dots & f_{0,k}\widetilde{D}_{0,k} \\ \hline 0 & \lambda_1 I & \dots & f_{1,k}\widetilde{D}_{1,k} \\ \hline \dots & \dots & \dots & \dots \\ \hline 0 & 0 & \dots & \lambda_k I \end{array}\right)$$

Example $KN(9,3) = \Gamma(9,3,0)$, Before Using E

$$P^{-1}LP = \left(\begin{array}{c|ccc|ccc|ccc}
0 & -10 & \cdots & -10 & -4 & \cdots & -4 & -1 & \cdots & -1 \\ \hline
 & 30 & & & & & & & & \\
0 & & \ddots & & & 4\widetilde{W_{1,2}} & & & 1\widetilde{W_{1,3}} & \\
 & & & 30 & & & & & & \\ \hline
 & & & & 16 & & & & & \\
0 & & 0 & & & \ddots & & & -1\widetilde{W_{2,3}} & \\
 & & & & & & 16 & & & \\ \hline
 & & & & & & & 21 & & \\
0 & & 0 & & & 0 & & & \ddots & \\
 & & & & & & & & & 21
\end{array}\right)$$

Example $KN(9,3) = \Gamma(9,3,0)$, After Using E

$$EU_LE^{-1} =$$

$$\left[\begin{array}{c|cccc|ccccccc|cccccccccc}
0 & -10 & 0 & \cdots & 0 & -4 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 30 & 0 & \cdots & 0 & 8 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & & \vdots & 0 & 4 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & 30 & 0 & \cdots & 0 & 4 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 16 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & \ddots & & & & & \vdots & 0 & -2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & \ddots & & & \vdots & 0 & \cdots & 0 & -2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & \ddots & & \vdots & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 & \vdots & & & & & \ddots & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 16 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 21 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & 0 & \ddots & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & \ddots & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & \ddots & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & \ddots & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 21
\end{array}\right]$$

Final Diagonalization Steps

$k = 2$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc}
\lambda_0 & -f_{0,1} & 0 & \cdots & 0 & -f_{0,2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & \lambda_1 & 0 & \cdots & 0 & -2f_{1,2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & & \vdots & 0 & -f_{1,2} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & & & & \vdots \\
0 & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 & -f_{1,2} & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & \ddots & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & \ddots & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_2
\end{array}\right]
$$

$k = 2$

$$\begin{bmatrix} \lambda_0 & -f_{0,1} & -f_{0,2} \\ 0 & \lambda_1 & -2f_{1,2} \\ 0 & 0 & \lambda_2 \end{bmatrix} \sim \begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix}$$

$k = 2$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc}
y_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & y_2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \lambda_1 & & \vdots & 0 & -f_{1,2} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & & & & \vdots \\
0 & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 & -f_{1,2} & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & y_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & \lambda_2 & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & \ddots & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_2
\end{array}\right]
$$

$k = 2$

$$U_L \sim \left[\begin{array}{c|cccc|ccccccc}
y_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & y_2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \lambda_1 & & \vdots & 0 & -f_{1,2} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & & & & \vdots \\
0 & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 & -f_{1,2} & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & y_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & \lambda_2 & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & \lambda_2 & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_2
\end{array}\right]$$

$k = 2$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc}
y_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & y_2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \lambda_1 & & \vdots & 0 & -f_{1,2} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & & & & \vdots \\
0 & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 & -f_{1,2} & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & y_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & \lambda_2 & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & \lambda_2 & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_2
\end{array}\right]
$$

$k = 2$

$$\begin{bmatrix} \lambda_1 & -f_{1,2} \\ 0 & \lambda_2 \end{bmatrix} \sim \begin{bmatrix} y_4 & 0 \\ 0 & y_5 \end{bmatrix}$$

$k = 2$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc}
y_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & y_2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & {\color{red}y_4} & & \vdots & \vdots & & & & & & \vdots \\
\vdots & \vdots & & {\color{red}\ddots} & 0 & \vdots & & & & & & \vdots \\
0 & 0 & \cdots & 0 & {\color{red}y_4} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & y_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & {\color{red}y_5} & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & {\color{red}\ddots} & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & {\color{red}y_5} & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & \lambda_2 & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_2
\end{array}\right]
$$

$k = 2$

$$U_L \sim \left[\begin{array}{c|cccc|ccccccc}
y_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & y_2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & y_4 & & \vdots & \vdots & & & & & & \vdots \\
\vdots & \vdots & & \ddots & 0 & \vdots & & & & & & \vdots \\
0 & 0 & \cdots & 0 & y_4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & y_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & y_5 & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & y_5 & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & \lambda_2 & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_2
\end{array}\right]$$

$k = 2$ Final Result

Theorem:

Now that we have the matrix in diagonal form, we can see that the sandpile group of any intersection graph $\Gamma(n, 2, \ell)$ is

$$K(\Gamma) = \mathbb{Z}_{y_1} \oplus \mathbb{Z}_{y_2} \oplus \mathbb{Z}_{y_3} \oplus \mathbb{Z}_{y_4}^{m_1 - 1} \oplus \mathbb{Z}_{y_5}^{m_1 - 1} \oplus \mathbb{Z}_{\lambda_2}^{m_2 - m_1}$$

$k = 3$

$$U_L \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
\lambda_0 & -f_{0,1} & 0 & \cdots & 0 & -f_{0,2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -f_{0,3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & \lambda_1 & 0 & \cdots & 0 & -2f_{1,2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -3f_{1,3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & & \vdots & 0 & -f_{1,2} & 0 & \cdots & \cdots & \cdots & 0 & 0 & -f_{1,3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 & -f_{1,2} & 0 & \cdots & 0 & 0 & \cdots & 0 & -f_{1,3} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -3f_{2,3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & \ddots & & & & & \vdots & 0 & -2f_{2,3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & \ddots & & & \vdots & 0 & \cdots & 0 & -2f_{2,3} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & \ddots & & \vdots & 0 & \cdots & \cdots & 0 & -f_{2,3} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 & \vdots & & & & & \ddots & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -f_{2,3} & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & 0 & \ddots & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & \ddots & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & \ddots & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & \ddots & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_3
\end{array}\right]$$

$k = 3$

$$\begin{bmatrix} \lambda_0 & -f_{0,1} & -f_{0,2} & -f_{0,3} \\ 0 & \lambda_1 & -2f_{1,2} & -3f_{1,3} \\ 0 & 0 & \lambda_2 & -3f_{2,3} \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} \sim \begin{bmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & y_4 \end{bmatrix}$$

$k = 3$

$$U_L \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
y_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & y_2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \lambda_1 & & \vdots & 0 & -f_{1,2} & 0 & \cdots & \cdots & \cdots & 0 & 0 & -f_{1,3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & \ddots & 0 & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 & -f_{1,2} & 0 & \cdots & 0 & 0 & \cdots & 0 & -f_{1,3} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & y_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & \lambda_2 & & & & & \vdots & \vdots & -2f_{2,3} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & \lambda_2 & & & \vdots & 0 & \cdots & 0 & -2f_{2,3} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & \ddots & & \vdots & 0 & \cdots & \cdots & 0 & -f_{2,3} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 & \vdots & & & & & \ddots & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -f_{2,3} & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & y_4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & 0 & \lambda_3 & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & \lambda_3 & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & \ddots & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & \ddots & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_3
\end{array}\right]$$

$k = 3$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
y_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & y_2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & y_5 & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & \ddots & 0 & \vdots & & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & y_5 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & y_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & y_6 & & & & & \vdots & \vdots & \ddots & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & y_6 & & & \vdots & 0 & & & 0 & & & & & & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & \color{red}{\lambda_2} & & \vdots & 0 & \cdots & \cdots & 0 & \color{red}{-f_{2,3}} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \color{red}{\ddots} & 0 & \vdots & & & & & \color{red}{\ddots} & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \color{red}{\lambda_2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \color{red}{-f_{2,3}} & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & y_4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & 0 & y_7 & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & y_7 & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & \color{red}{\lambda_3} & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \color{red}{\ddots} & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & \color{red}{\lambda_3} & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_3
\end{array}\right]
$$

$k = 3$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
y_1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & y_2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & y_5 & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & \ddots & 0 & \vdots & & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & y_5 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & y_3 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & y_6 & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & y_6 & & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & y_8 & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 & \vdots & & & & & & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & y_8 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & y_4 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & 0 & y_7 & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & y_7 & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & \frac{\lambda_2\lambda_3}{y_8} & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & \frac{\lambda_2\lambda_3}{y_8} & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & \lambda_3 & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \lambda_3
\end{array}\right]
$$

$k = 3$ Final Result

Theorem:

Now that all entries are on the diagonal, we see that the general form for the sandpile group of $k = 3$ intersection graphs $\Gamma(n, 3, \ell)$ is

$$K(\Gamma) = \mathbb{Z}_{y_1} \oplus \mathbb{Z}_{y_2} \oplus \mathbb{Z}_{y_3} \oplus \mathbb{Z}_{y_4} \oplus \mathbb{Z}_{y_5}^{m_1-1} \oplus \mathbb{Z}_{y_6}^{m_1-1} \oplus \mathbb{Z}_{y_7}^{m_1-1} \oplus \mathbb{Z}_{y_8}^{m_2-m_1} \oplus \mathbb{Z}_{\frac{\lambda_2\lambda_3}{y_8}}^{m_2-m_1} \oplus \mathbb{Z}_{\lambda_3}^{m_3-m_2}$$

GENERAL k

The problem for general k is then reduced to finding the Smith group of a series of $k+1-j$ by $k+1-j$ matrices for $0 \leq j \leq k$:

$$N_j = \begin{bmatrix} \lambda_j & -\binom{1}{0}f_{j,j+1} & \cdots & -\binom{k-j}{0}f_{jk} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \lambda_{k-1} & -\binom{k-j}{k-1-j}f_{k-1,k} \\ 0 & \cdots & 0 & \lambda_k \end{bmatrix}$$

General k Final Result

Theorem:

For each j, we reduce N_j exactly $m_{k-j+1} - m_{k-j}$ times to Smith form.Then

$$K(\Gamma(n,k,\ell)) = \bigoplus_{j=0}^{k} S(N_j)^{m_j - m_{j-1}}.$$

where $S(N)$ denotes the **Smith group** of a matrix N.

EXAMPLE $KN(9,3) = \Gamma(9,3,0)$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
0 & -10 & 0 & \cdots & 0 & -4 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 30 & 0 & \cdots & 0 & 8 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & & \vdots & 0 & 4 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & \ddots & \vdots & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & 30 & 0 & \cdots & 0 & 4 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 16 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & \ddots & & & & & \vdots & 0 & -2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & \ddots & & & \vdots & 0 & \cdots & 0 & -2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & \ddots & & \vdots & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 & \vdots & & & & & \ddots & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 16 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 21 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & 0 & \ddots & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & \ddots & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & \ddots & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & \ddots & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 21
\end{array}\right]
$$

Example $KN(9,3) = \Gamma(9,3,0)$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 2 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \ddots & & \vdots & 0 & 4 & 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & \ddots & \vdots & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & 30 & 0 & \cdots & 0 & 4 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 60 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & 16 & & & & & \vdots & 0 & -2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & \ddots & & & \vdots & 0 & \cdots & 0 & -2 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & \ddots & & \vdots & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 & \vdots & & & & & \ddots & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 16 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & 0 & 21 & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & \ddots & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & \ddots & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & \ddots & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 21
\end{array}\right]
$$

Example $KN(9,3) = \Gamma(9,3,0)$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & {\color{red}30} & & \vdots & 0 & {\color{red}4} & 0 & \cdots & \cdots & \cdots & 0 & 0 & {\color{red}1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & {\color{red}\ddots} & \vdots & \vdots & & {\color{red}\ddots} & & & & \vdots & \vdots & & {\color{red}\ddots} & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & {\color{red}30} & 0 & \cdots & 0 & {\color{red}4} & 0 & \cdots & 0 & 0 & \cdots & 0 & {\color{red}1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 60 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & {\color{red}16} & & & & & \vdots & \vdots & {\color{red}-2} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & {\color{red}\ddots} & & & & \vdots & \vdots & & {\color{red}\ddots} & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & {\color{red}16} & & & \vdots & 0 & \cdots & 0 & {\color{red}-2} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & \ddots & & \vdots & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 & \vdots & & & & & \ddots & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 16 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & 0 & {\color{red}21} & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & {\color{red}\ddots} & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & {\color{red}21} & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & \ddots & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & \ddots & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 21
\end{array}\right]
$$

Example $KN(9,3) = \Gamma(9,3,0)$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & \textcolor{red}{1} & & \vdots & 0 & \textcolor{red}{0} & 0 & \cdots & \cdots & \cdots & 0 & 0 & \textcolor{red}{0} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & \textcolor{red}{\ddots} & \vdots & \vdots & & \textcolor{red}{\ddots} & & & & \vdots & \vdots & & \textcolor{red}{\ddots} & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & \textcolor{red}{1} & 0 & \cdots & 0 & \textcolor{red}{0} & 0 & \cdots & 0 & 0 & \cdots & 0 & \textcolor{red}{0} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 60 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & \textcolor{red}{6} & & & & & \vdots & \vdots & \textcolor{red}{0} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \textcolor{red}{\ddots} & & & & \vdots & \vdots & & \textcolor{red}{\ddots} & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & \textcolor{red}{6} & & & \vdots & 0 & \cdots & 0 & \textcolor{red}{0} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & 16 & & \vdots & 0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 & \vdots & & & & & \ddots & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 16 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & 0 & \textcolor{red}{1680} & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & \textcolor{red}{\ddots} & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & \textcolor{red}{1680} & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & 21 & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & \ddots & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 21
\end{array}\right]
$$

Example $KN(9,3) = \Gamma(9,3,0)$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & 1 & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & \ddots & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 60 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & 6 & & & & & \vdots & \vdots & \ddots & & & & & & & & \vdots \\
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\vdots & \vdots & & & \vdots & \vdots & & & 6 & & & \vdots & 0 & & & 0 & & & & & & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & \color{red}{16} & & \vdots & 0 & \cdots & \cdots & 0 & \color{red}{-1} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \color{red}{\ddots} & 0 & \vdots & & & & & \color{red}{\ddots} & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \color{red}{16} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \color{red}{-1} & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & 0 & 1680 & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & 1680 & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & \color{red}{21} & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \color{red}{\ddots} & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & \color{red}{21} & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 21
\end{array}\right]
$$

Example $KN(9,3) = \Gamma(9,3,0)$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & 1 & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & \ddots & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 60 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & 6 & & & & & \vdots & \vdots & \ddots & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & 6 & & & \vdots & 0 & & & 0 & & & & & & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & \color{red}{1} & & \vdots & 0 & \cdots & \cdots & 0 & \color{red}{0} & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \color{red}{\ddots} & 0 & \vdots & & & & & \color{red}{\ddots} & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \color{red}{1} & 0 & \cdots & \cdots & \cdots & \cdots & 0 & \color{red}{0} & 0 & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & 0 & 1680 & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & 1680 & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & \color{red}{336} & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \color{red}{\ddots} & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & \color{red}{336} & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & \ddots & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 21
\end{array}\right]
$$

Example $KN(9,3) = \Gamma(9,3,0)$

$$
U_L \sim \left[\begin{array}{c|cccc|ccccccc|cccccccccc}
1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 2 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & 0 & 1 & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & \ddots & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 60 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & 0 & 6 & & & & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & \ddots & & & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & 6 & & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & 1 & & \vdots & \vdots & & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & \ddots & 0 & \vdots & & & & & & & & & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\hline
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & 1680 & & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & \ddots & & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & 1680 & & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & 336 & & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & \ddots & & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & 336 & & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & 21 & & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & & & & & & \vdots & \vdots & & & & & & & & \ddots & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 21
\end{array}\right]
$$

Example $KN(9,3) = \Gamma(9,3,0)$

From this diagonal form, the sandpile group of KN(9,3) is

$$K(KN(9,3)) = \mathbb{Z}_2 \oplus \mathbb{Z}_{60} \oplus (\mathbb{Z}_6)^7 \oplus (\mathbb{Z}_{1680})^7 \oplus (\mathbb{Z}_{336})^{19} \oplus (\mathbb{Z}_{21})^{21}$$

Conclusion

Recap

Our results find sandpile group much faster than the computer.

Take $\Gamma(n, 3, 0)$, for example:

n	Us	Computer	n	Us	Computer
7	0.031	0.022	14	2.645	1:17.579
8	0.063	0.088	15	4.783	2:35.555
9	0.127	0.333	16	6.819	5:17.419
10	0.279	1.005	17	11.122	10:58.193
11	0.467	2.896	18	16.485	21:46.368
12	0.870	9.301	19	23.841	ERROR
13	1.563	26.505	20	34.001	ERROR

Even without a computer, our work shows that the sandpile group can be found for any subset intersection graph.

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ACKNOWLEDGEMENTS

- Josh Ducey and Brant Jones, our mentors
- Hala Nelson, for organizing this experience
- National Science Foundation, for supporting our research
- James Madison University, for providing an incredible environment to do our work
- Holly Bill, Elke Doby, Jacob Steger, and Daniel Trudell, for keeping things fun when math got hard
- JMU Department of Mathematics and Statistics

Questions

Error-Correcting Codes

- An (n, M, d) error-correcting code is a set of M vectors $u = u_1 \cdots u_n$ of 0s and 1s of length n (**codewords**) such that any two distinct codewords differ in at least d places. n is the **length** of the code, M is the **size**, and d is the **minimum distance**.
- A $(5, 2, 5)$ code: $\{00000, 11111\}$.
- A $(3, 4, 2)$ code: $\{000, 011, 101, 110\}$
- We want to find a bound of how large M can be for given values of n (small) and d (large)
- This bound describes efficiency of code performing linear optimization

- An **association scheme** with n classes consists of a finite set X of v points, together with $n+1$ symmetric relations $R_0, R_1, \cdots, R_n$ defined on X which satisfy
  - For every $x, y \in X$, $(x, y) \in R_i$ for exactly one i.
  - $R_0 = \{(x, x) : x \in X\}$ is the identity relation
  - If $(x, y) \in R_k$, the number of $z \in X$ such that $(x, z) \in R_i$ and $(y, z) \in R_j$ is a constant $p_{i,j,k}$ depending on i, j, k but not on the particular choice of x and y
- Two points x and y are i**th associates** if $(x, y) \in R_i$.
- Can be described by a complete graph (every pair of vertices is connected by a unique edge) with v vertices that correspond to elements of X with edges between vertices x and y labeled i if x and y are ith associates.

Hamming Association Scheme

- **Hamming Distance:** The number of places where two vectors differ
- X is the set of binary vectors of length n. Two vectors are adjacent if they are Hamming distance i apart

JOHNSON ASSOCIATION SCHEME (TRIANGULAR)

- X is all $\binom{V}{n}$ binary vectors of length V and weight n where V, n are fixed integers with $0 \leq n \leq V/2$. Two vectors u, v are ith associates if dist$(u, v) = 2i$ for $i = [n]$
- Johnson graph: distance-1 relation in the Johnson scheme
- Kneser graph: graph that describes being at the maximum distance in the Johnson scheme
- Applications to t-designs

- A $t-(V,k,\lambda)$ **design** is a family of k-subsets (blocks) of a V-set such that any t-subset of the V-set is contained in exactly λ blocks.

A Key Characteristic of Smith Normal Form

The ith entry of a matrix in Smith Normal Form is equal to the greatest common divisor of the ixi minors (η_i) of the original matrix divided by the gcd of the $(i-1)$x$(i-1)$ minors (η_{i-1}).

Take our matrix M from earlier: $\begin{bmatrix} 2 & 4 & -8 & 0 \\ 1 & 0 & -2 & 2 \\ 12 & -4 & -2 & 4 \\ 10 & -4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 48 \end{bmatrix}$

$$\eta_1 = \text{gcd}(1, 2, 4, 8, 10, 12) = 1$$
$$\eta_2 = \text{gcd}(4, 8, 16, 56, 92, 40, 32, 48, 76, 22, 20, 18) = 2$$
$$\eta_3 = \text{gcd}(72, 96, 168, 48, 56, 152, 80, 192, 304, 160, 16, 32) = 8$$
$$\eta_4 = \text{gcd}(384) = 384$$