Chapter 1

A Brief Description

We first begin with a brief treatment of some subjects covered in an Elementary Differential Equations course that we assume you have taken. Then we describe some qualitative properties of differential equations that we are going to study in this book. Many of the descriptions here will be done with the geometric and physical arguments to help you see why certain qualitative properties are plausible and why sometimes we pursue a qualitative analysis rather than solving differential equations analytically or numerically. This will give you an opportunity to become familiar with the objective and terminology of qualitative analysis in a somewhat familiar setting.

1.1 Linear Differential Equations

To provide a background for our discussions, let's begin with some examples.

Example 1.1.1 To mathematically model the population growth of, say, a university, the simplest assumption we can make is to assume that the population grows at a rate proportional (with a proportional constant k) to its current population of that year. For example, k may be 0.05, which means the population grows 5% per year. If we use t for the time and $x(t)$

for the population at time t, and if we know the population at a time, say t_0, to be x_0, (for example, the population is $x_0 = 18,000$ in year $t_0 = 2001$), then we can set up the following equation

$$x'(t) = kx(t), \quad x(t_0) = x_0. \qquad \spadesuit \tag{1.1}$$

Eq. (1.1) is an equation involving the derivative of an unknown function $x(t)$ that we want to solve. Therefore, we define an **ordinary differential equation** as an equation involving derivatives of an unknown function with one variable.

The **order** of an ordinary differential equation is the highest derivative of the unknown function that appears in the equation. For example, Eq. (1.1) is a first-order ordinary differential equation.

A **solution** of an ordinary differential equation is a function that satisfies the ordinary differential equation. $x(t_0) = x_0$ in Eq. (1.1) is referred to as an **initial condition**, an **initial value**, or an **initial data**, and Eq. (1.1) is also called an **initial value problem**.

Sometimes, we only consider $t \geq t_0$ in Eq. (1.1) because we are only concerned with the development in the future time of t_0. Next, $x(t)$ in Eq. (1.1) is a number, thus we say that Eq. (1.1) is a **differential equation in** $\Re$ (or $\Re^1$), where $\Re = (-\infty, \infty)$. We also say that Eq. (1.1) is a **scalar equation**.

Since the study in this book doesn't involve partial differential equations, sometimes we will use "differential equations" or just "equations" to mean "ordinary differential equations."

The **direction field** consisting of direction vectors (or slope vectors) for Eq. (1.1) with $k > 0$ is given in **Figure 1.1**.

For Eq. (1.1), $x(t) \equiv 0$ is a solution (with its initial value being zero), and is called a **constant solution**. Otherwise, we assume $x(t) \neq 0$ such that Eq. (1.1) can be written as

$$\frac{x'(t)}{x(t)} = k. \tag{1.2}$$

Now, we can use the method of **separation of variables** to solve Eq. (1.2). That is, we separate the variables x and t and write Eq. (1.2) as

$$\frac{1}{x}dx = kdt,$$

then solve

$$\int \frac{1}{x}dx = \int kdt,$$

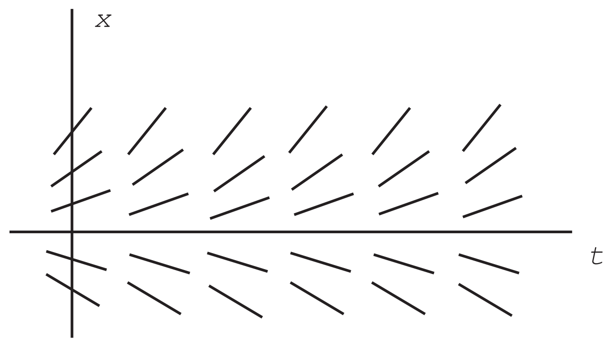

Figure 1.1: Direction field of Eq. (1.1) with $k > 0$

and obtain

$$\ln |x| = kt + C.$$

Finally, we derive the solution of Eq. (1.1), given by

$$x(t) = x_0 e^{k(t-t_0)}. \tag{1.3}$$

For the solution given in (1.3), we have the pictures in **Figure 1.2** and **Figure 1.3**.

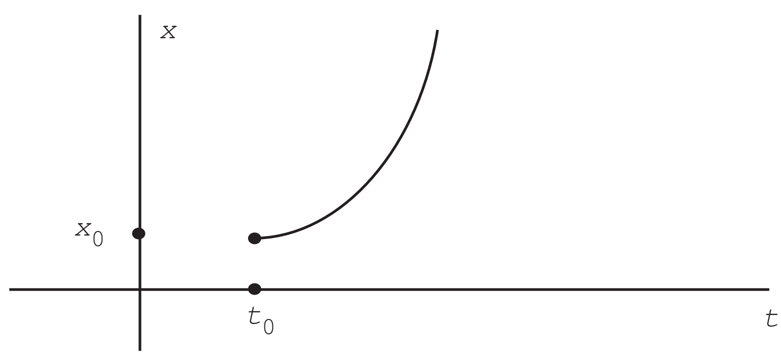

Figure 1.2: Solutions of Eq. (1.1) with $k > 0$

Accordingly, we say that in Eq. (1.1), the population **grows exponentially** when $k > 0$, and the population **decays exponentially** when $k < 0$.

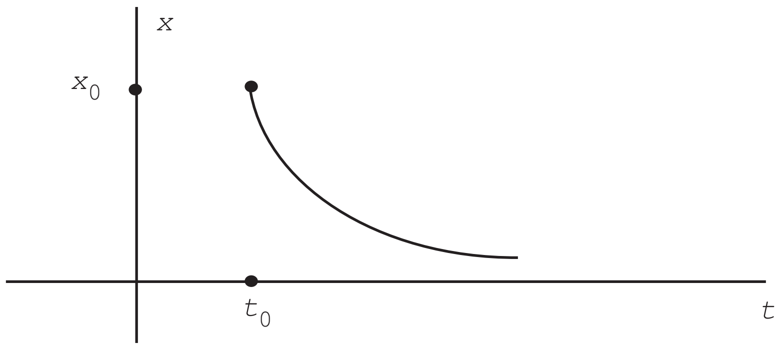

Figure 1.3: Solutions of Eq. (1.1) with $k < 0$

Now, if we do not solve Eq. (1.1), but, instead, start at (t_0, x_0) and **flow** along the directions of the direction field in Figure 1.1 as t increases, then the curve obtained, see **Figure 1.4**, matches well with the picture of solutions in Figure 1.2. The point of view of regarding solutions as curves flowing in a direction field will be very useful for understanding some results in differential equations, especially when we study the geometric aspects of differential equations.

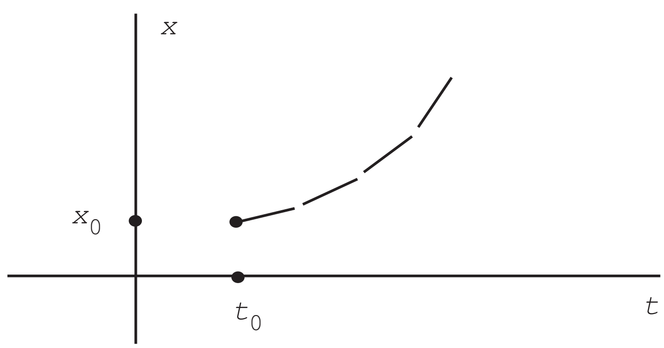

Figure 1.4: A curve obtained using the direction field in Figure 1.1

In Eq. (1.1), if we take t_0 to be 0, (for example, treat year 2001 as year 0), then Eq. (1.1) becomes

$$x'(t) = kx(t),\ x(0) = x_0, \tag{1.4}$$

and the solution is now given by

$$x(t) = x_0 e^{kt}.$$

Consider Eq. (1.1). If we assume further that other factors, such as those from the environment, are also involved in the population growth, then we may replace Eq. (1.1) by

$$x'(t) = kx(t) + f(t), \quad x(t_0) = x_0, \tag{1.5}$$

for some continuous "factor" function $f(t)$.

In some applications, the proportional constant k may change with the time t, thus in those cases we need to replace k by a continuous function in t, say $k(t)$. Then Eq. (1.5) becomes

$$x'(t) = k(t)x(t) + f(t), \quad x(t_0) = x_0. \tag{1.6}$$

Sometimes, other forms of differential equations are also encounted in applications, as in the following examples.

Example 1.1.2 (Restricted population growth) In many applications, it is assumed that the population ($x(t)$) does not exceed some number C, called the **carrying capacity** of the environment; it is also assumed that the population grows at a rate proportional (with a constant k) to the difference between C and the population at that time. Then $x(t)$ satisfies

$$x'(t) = k[C - x(t)], \quad x(t_0) = x_0. \qquad \spadesuit \tag{1.7}$$

Example 1.1.3 (Newton's law of cooling) Newton's law of cooling states that the temperature of a subject ($T(t)$) changes at a rate proportional (with a constant k) to the difference between the temperature of the subject and the temperature of the surrounding medium (T_m). Then we have

$$T'(t) = k[T_m - T(t)], \quad T(t_0) = T_0. \qquad \spadesuit \tag{1.8}$$

Eq. (1.7) and Eq. (1.8) are of the same form, so we only need to look at Eq. (1.7). Now, $x(t) = C$ is a constant solution. If $x(t) \neq C$, then, using separation of variables, we solve

$$\frac{1}{C - x} dx = k dt, \tag{1.9}$$

then the solution of Eq. (1.7) is given as

$$x(t) = C - [C - x_0]e^{-k(t-t_0)}, \quad t \geq t_0. \tag{1.10}$$

The direction field and the picture of the solutions for Eq. (1.7) are given in **Figure 1.5** and **Figure 1.6**. Again, they match well.

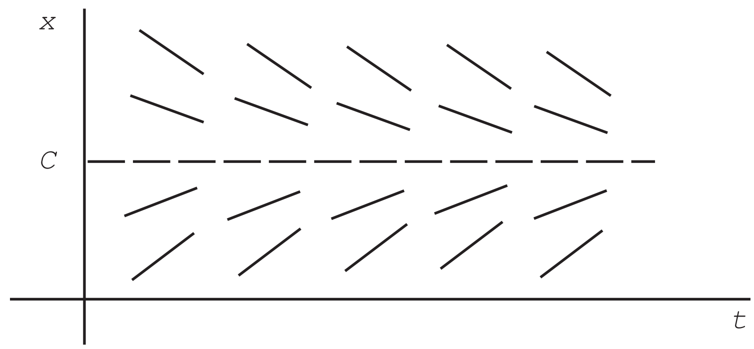

Figure 1.5: Direction field of Eq. (1.7) with $k > 0$

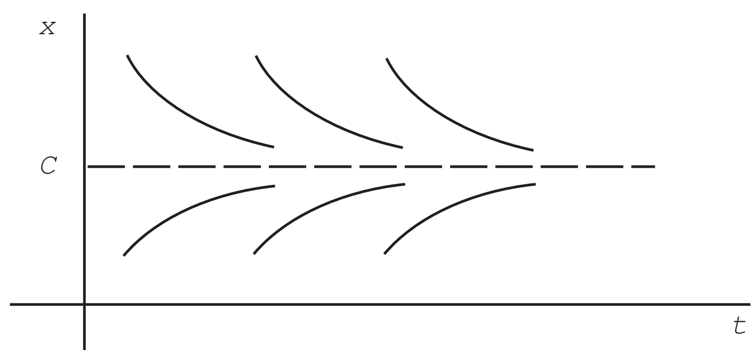

Figure 1.6: Solutions of Eq. (1.7) with $k > 0$

Eq. (1.7) and Eq. (1.8) are solved as above. However, we point out that they can also be formulated as Eq. (1.5) or Eq. (1.6), which has more applications. For example, Eq. (1.8) can be formulated as $T'(t) = -kT(t) + kT_m$.

Now, let's solve Eq. (1.5). We note that when $f = 0$, Eq. (1.5) becomes Eq. (1.1), whose solution is given by $x_0 e^{k(t-t_0)}$. Then we use the method of **variation of parameters**, that is, we determine the conditions on an unknown function $C(t)$ such that $C(t)e^{k(t-t_0)}$ is a solution of Eq. (1.5). This leads to

$$C'(t)e^{k(t-t_0)} + C(t)ke^{k(t-t_0)} = kC(t)e^{k(t-t_0)} + f(t),$$

hence,

$$C(t) = x_0 + \int_{t_0}^{t} e^{-k(s-t_0)} f(s)ds.$$

Therefore, we obtain the solution of Eq. (1.5), given by the **variation of parameters formula**

$$\begin{aligned} x(t) &= e^{k(t-t_0)}\Big[x_0 + \int_{t_0}^{t} e^{-k(s-t_0)} f(s)ds\Big] \\ &= e^{k(t-t_0)}x_0 + \int_{t_0}^{t} e^{k(t-s)} f(s)ds. \end{aligned} \tag{1.11}$$

If we define $T(t) = e^{kt}$, then (1.11) can be written as

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-s)f(s)ds. \tag{1.12}$$

Note that $T(t) = e^{kt}$ satisfies the following property:

(S1). $T(0) = 1$,

(S2). $T(t)T(s) = T(t+s),\ t,\ s \geq 0.$

In some literature, this property is called the "**semigroup property**."

To solve Eq. (1.6), we can use the idea of variation of parameters again (see an exercise) and let $f = 0$ first and derive the solution $x_0 e^{\int_{t_0}^{t} k(s)ds}$. We then determine the conditions on $C(t)e^{\int_{t_0}^{t} k(s)ds}$ from

$$C'(t)e^{\int_{t_0}^{t} k(s)ds} + C(t)k(t)e^{\int_{t_0}^{t} k(s)ds} = k(t)C(t)e^{\int_{t_0}^{t} k(s)ds} + f(t),$$

and obtain the solution of Eq. (1.6), given by another **variation of parameters formula**

$$\begin{aligned} x(t) &= e^{\int_{t_0}^t k(s)ds}\Big[x_0 + \int_{t_0}^t e^{-\int_{t_0}^s k(h)dh} f(s)ds\Big] \\ &= e^{\int_{t_0}^t k(s)ds} x_0 + \int_{t_0}^t e^{\int_s^t k(h)dh} f(s)ds. \end{aligned} \tag{1.13}$$

In this case, if we define $U(t,s) = e^{\int_s^t k(h)dh}$, then (1.13) can be written as

$$x(t) = U(t,t_0)x_0 + \int_{t_0}^t U(t,s)f(s)ds. \tag{1.14}$$

Now, $U(t,s) = e^{\int_s^t k(h)dh}$ satisfies the following property:

(E1). $U(t,t) = 1,\ t \geq t_0$,

(E2). $U(t,r)U(r,s) = U(t,s),\ t_0 \leq s \leq r \leq t$.

This property is called the "**evolution system property**" in some literature.

Some higher order differential equations can also be treated in a similar way. One example is given below.

Example 1.1.4 Consider the second-order differential equation

$$x''(t) + a_1 x'(t) + a_2 x(t) = f(t).$$

Besides using the characteristic equations, we can define

$$x_1(t) = x(t),\ x_2(t) = x'(t),$$

then,

$$\begin{cases} x_1'(t) = x'(t) = x_2(t), \\ x_2'(t) = x''(t) = -a_2 x(t) - a_1 x'(t) + f(t) = -a_2 x_1(t) - a_1 x_2(t) + f(t). \end{cases} \tag{1.15}$$

Thus, writing in matrix and vector notations, we obtain

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}. \tag{1.16}$$

Eq. (1.16) is called a **differential equation in** $\Re^2$, which looks like Eq. (1.5) when k in Eq. (1.5) is regarded as a 2×2 matrix and x as a vector in $\Re^2$. ♠

For equations in $\Re^2$, solutions should be viewed in the (t, x_1, x_2) space, and the direction field should be drawn in the (x_1, x_2) space, as in the following example.

Example 1.1.5 Consider

$$\begin{cases} x_1' &= x_2, \\ x_2' &= -x_1. \end{cases} \tag{1.17}$$

We find that $x_1(t) = \sin t$ and $x_2(t) = \cos t$ is a solution, and the picture in the (t, x_1, x_2) space is shown in **Figure 1.7**. The direction field of Eq. (1.17) is shown in **Figure 1.8**.

(To get Figure 1.8, you can check a few points. For example, at the point $[x_1, x_2]^T = [0, 1]^T$ (here T means the transpose, so $[0, 1]^T$ is a 2×1 vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in $\Re^2$), the direction in the field is $[x_1', x_2']^T = [x_2, -x_1]^T = [1, 0]^T$; at the point $[x_1, x_2]^T = [1, 1]^T$, the direction in the field is $[x_2, -x_1]^T = [1, -1]^T$; at the point $[x_1, x_2]^T = [1, 0]^T$, the direction in the field is $[x_2, -x_1]^T = [0, -1]^T$, and so on. Thus it goes like a circle in the clockwise direction. Again, the picture of the solution $x_1(t) = \sin t$, $x_2(t) = \cos t$ and the direction field in Figure 1.8 match well.) ♠

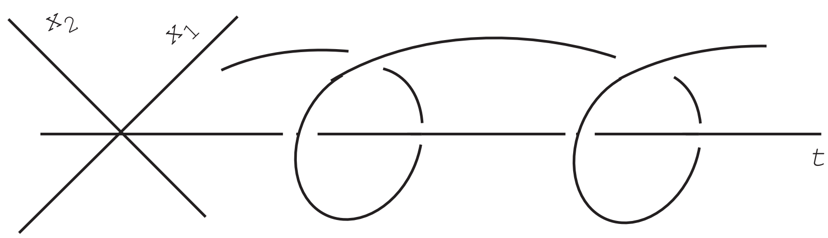

Figure 1.7: Solutions of Eq. (1.17) in the (t, x_1, x_2) space

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Observe that when the right-hand side of Eq. (1.5) or Eq. (1.6) is regarded as a function in (t, x), the term kx or $k(t)x$ involving x is **linear in** x. Thus, in this sense, Eq. (1.5) and Eq. (1.6) are called **linear differential equations**. When k and $k(t)$ in Eq. (1.5) and Eq. (1.6) are regarded as $n \times n$ matrices and x as an $n \times 1$ vector, Eq. (1.5) and Eq. (1.6) are called

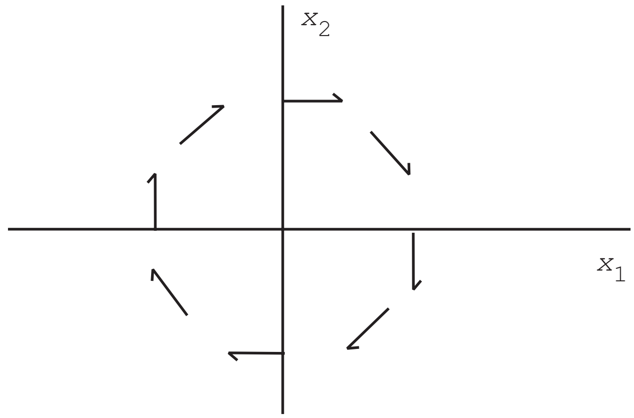

Figure 1.8: Direction field of Eq. (1.17) in the (x_1, x_2) space

linear differential equations in $\Re^n$. For equations in $\Re^n$, $n \geq 3$, if the t direction is also added, then we end up with something that is at least of four dimensions; hence, we lose geometric view because humans can only see objects of at most three dimensions. In those cases, especially in determining how close a solution is to the zero solution (when the zero is a solution), or in determining the distance from a point on the solution curve to the t-axis, we simply draw pictures in a plane, and treat $\Re^n$ as one dimensional, or use the vertical direction to denote the distance of a solution to the zero solution (t-axis, when the zero is a solution), as shown in **Figure 1.9**.

We will see in Chapter 3 that the solution formulas (1.12) and (1.14), as well as the semigroup and evolution system properties derived for one-dimensional equations are also valid for all linear differential equations in $\Re^n, n \geq 1$, and thereby constitute a complete and elegant theory for linear differential equations.

1.2 The Need for Qualitative Analysis

Our problems are almost solved, at least for finding solutions, if we only need to deal with linear differential equations. However, we are living in a complex world, and in most applications, such as those in biology, chemistry, and physics, we have to deal with **nonlinear differential equations**, that is, the differential equations where the terms involving the unknown function

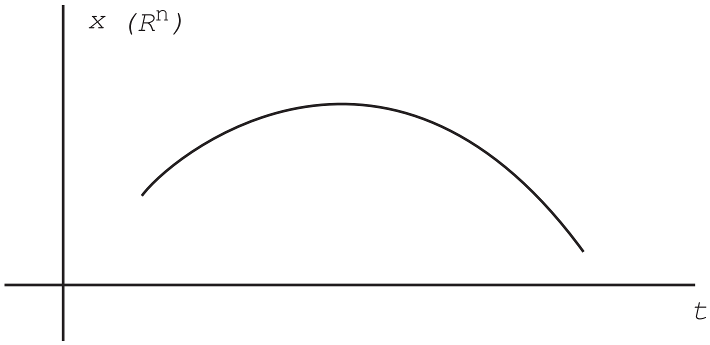

Figure 1.9: A solution in the $\Re \times \Re^n$ space viewed in the (t, x) plane where $\Re^n$ is treated as the x direction

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x are not linear in x, as in the following examples.

Example 1.2.1 (Logistic equation) The exponential growth of $x' = kx$, which was studied around 1800, was used by some economists to argue that human misery is inevitable because population grows exponentially fast and supplies cannot keep up. In 1845, the Belgian mathematician P. Verhulst argued that to get better models, the proportional constant k of $x' = kx$ should be replaced by $C - x$, where C is the carrying capacity; and proposed the following equation,

$$x'(t) = ax(t)[C - x(t)], \quad x(t_0) = x_0, \ \ t \geq t_0, \tag{2.1}$$

where a and C are positive constants. The model is used to accommodate the situations that when the population $x(t)$ is small, the rate $x'(t) \approx aCx(t)$, thus the population grows exponentially; when $x(t)$ approaches C but is still less than C, the rate $x'(t)$ decreases and is still positive, thus the population is still growing but at a slow rate; finally, when $x(t)$ is large enough $(x > C)$, the rate $x'(t) < 0$, therefore the population decreases. These can be seen from the graph of the function $f(x) = ax[C - x] = -ax^2 + aCx$ in **Figure 1.10**. Verhulst called the solution curves of Eq. (2.1) "logistic curves," from a Greek word meaning "skilled in computation." Nowadays, equations of the form of Eq. (2.1) are called "logistic equations." After a change in function $\frac{x}{C} \to x$, Eq. (2.1) can be replaced by

$$x'(t) = rx(t)[1 - x(t)], \quad x(t_0) = x_0, \ \ t \geq t_0, \tag{2.2}$$

where $r = aC$. (Some analysis of Eq. (2.1) and Eq. (2.2) will be given later.) ♠

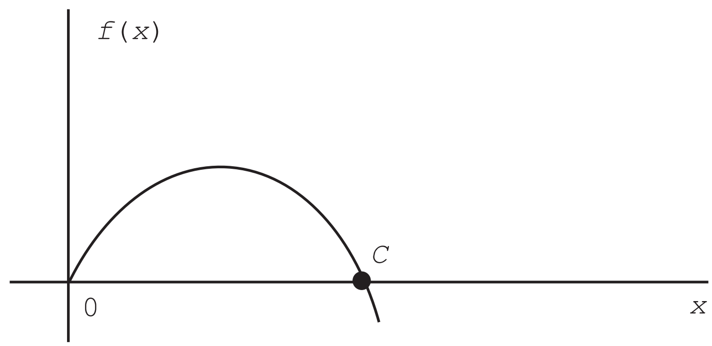

Figure 1.10: The graph of the function $f(x) = ax[C - x] = -ax^2 + aCx$

Example 1.2.2 Let's look at Eq. (1.6) again, but now assume that the function $f(t)$ is also determined by the unknown function x. Then we need to replace $f(t)$ by a function $f(t, x)$ that may be nonlinear in x, such as $f(t, x) = \sin(tx)$. Now we have

$$x'(t) = k(t)x(t) + \sin(tx(t)). \quad ♠ \tag{2.3}$$

Example 1.2.3 (Lotka-Volterra competition equation) Lotka-Volterra competition equation states that

$$\begin{cases} x_1' = \beta_1 x_1(K_1 - x_1 - \mu_1 x_2), \\ x_2' = \beta_2 x_2(K_2 - x_2 - \mu_2 x_1), \\ \quad x_1(0) \geq 0, \ \ x_2(0) \geq 0, \end{cases} \tag{2.4}$$

where $\beta_i,\ K_i,\ \mu_i,\ i = 1, 2$, are positive constants and $x_1(t),\ x_2(t)$ are two populations. If the populations x_1 and x_2 grow and decay independently of each other, then the constants μ_1 and μ_2 will not appear in Eq. (2.4), resulting in two independent differential equations where each is of the form of a logistic equation. However, if the two populations compete for a shared limited resource (space or a nutrient, for example), and each interferes with the other's utilization of it, then the growth or decay of one population will affect the well-being or fate of the other one. Now μ_1 and μ_2 will appear in

Eq. (2.4), and this explains why Eq. (2.4) is proposed. For detailed studies in this area, see for example, May [1973] and Smith [1974]. ♠

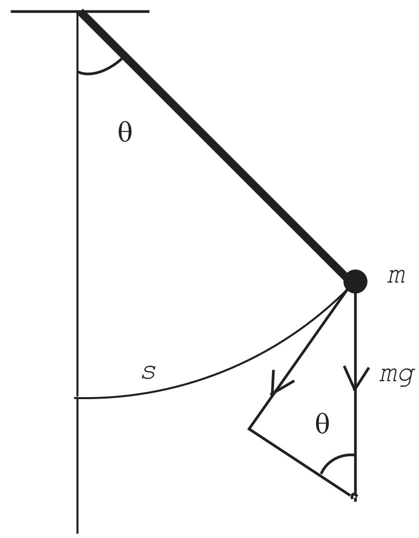

Figure 1.11: Motion of a simple pendulum

Example 1.2.4 (A simple pendulum) Figure 1.11 shows a rigid simple pendulum of length l oscillating around the vertical downward position.

It is assumed that the mass of the rod of the pendulum is negligible with respect to the mass m at the end of the pendulum. Let $\theta = \theta(t)$ be a function in the time variable t measuring the angle formed by the pendulum and the vertical downward direction, and let s be the arc length in the figure formed by the path of the end of the pendulum starting from the vertical downward position. Then $s = s(t)$ is a function in t and $s'(t)$ is the velocity of the end of the pendulum along the arc s (or in a direction tangential to the arc s). Now, the gravity of the pendulum is mg, hence, from the small triangle in the figure, the component of gravity in the direction tangential to the arc s is

$$-mg\sin\theta,$$

(the minus sign is needed because, for example, when the pendulum moves away, the force of gravity will try to drag the pendulum back). Next, assume that the damping or resistance is linear and is in the opposite direction of the velocity $s'(t)$, given in the form of

$$-\tau s'(t)$$

for some constant $\tau \geq 0$. Then, from Newton's second law of motion, which says, in this case, that

$$\begin{aligned}&(m)(\text{the tangential acceleration})\\&= \text{the tangential component of the gravitational force}\\&\quad + \text{the damping},\end{aligned}$$

we derive

$$ms''(t) = -mg\sin\theta(t) - \tau s'(t). \tag{2.5}$$

Since $\frac{s}{\theta} = \frac{2\pi l}{2\pi}$, we get $s = l\theta$. Hence Eq. (2.5) becomes

$$ml\theta''(t) = -mg\sin\theta(t) - \tau l\theta'(t). \tag{2.6}$$

Simplifying, we get the motion of a **simple pendulum**, given by the following differential equation,

$$\theta''(t) + k\theta'(t) + q\sin\theta(t) = 0, \tag{2.7}$$

where $k \geq 0$, $q > 0$ are constants, with k related to a damping term, and $\theta(t)$ measures the angle formed by the pendulum and the vertical downward position. ♠

In general, for the equations in physics governed by Newton's second law of motion, such as the motions concerning oscillations, the following type of second-order differential equations

$$x'' + f(t, x, x')x' + g(x) = p(t) \tag{2.8}$$

are subjects of intensive studies. Here, f usually represents a damping or friction term, such as k in Eq. (2.7); g represents a restoring force, such as $q\sin\theta(t)$ in Eq. (2.7); and p is an externally applied force.

Eq. (2.8) includes the famous **Lienard-type equations**

$$x'' + f(x)x' + g(x) = 0, \tag{2.9}$$

where $xg(x) > 0$ if $x \neq 0$, which includes the well-known **van der Pol equation**

$$x'' + (x^2 - 1)x' + x = 0, \tag{2.10}$$

named after Lienard [1928] and van der Pol [1927] for their important contributions in the analysis of the equations and their applications concerning sustained oscillations, the modeling of the voltage in a triode circuit and also the human heartbeat.

In Examples 1.2.1–1.2.4, the unknown function x **appears nonlinearly**, such as x^2 and $\sin(tx)$, thus those differential equations are called **nonlinear differential equations**. Certain nonlinear differential equations can be **solved analytically**, meaning that the formulas for solutions can be derived analytically. For example, the logistic equation in Example 1.2.1 can be solved analytically by rewriting the equation as

$$\frac{x'}{ax[C-x]} = 1, \quad ([C-x]x \neq 0),$$

then, using separation of variables and partial fractions, one obtains (see an exercise)

$$x(t) = \frac{Cx_0}{x_0 + [C - x_0]e^{-aC(t-t_0)}}, \quad t \geq t_0. \tag{2.11}$$

However, **most nonlinear differential equations, such as those in Examples 1.2.2–1.2.4, cannot be solved analytically, that is, no formulas for solutions are available.** (Try it to see why.)

Now, the question is: **What do we do for general nonlinear differential equations?** It is true that in most applications, differential equations are handled by numerical approximations with the help of powerful computers, and evidently courses in numerical methods are very popular nowadays. Students who have taken such courses may want to use numerical approximation methods to approximate solutions of nonlinear differential equations they cannot solve analytically.

But wait a minute and think about this: If we do not even know that a solution exists in the first place, then what are we approximating? Another question to ask is: Suppose an approximation gives one solution, and we then use a different way to make an approximation, are we sure that we will get the same solution? If we don't get the same solution, then which solution do we want to use in order to explain the physical situation that we are modeling using the differential equation?

A further question to consider is that to determine some asymptotic properties (properties of solutions for large time variable t), even though numerical solutions can be carried out to **suggest** certain properties, they are obtained through discretization on finite intervals. They reveal certain properties that are only valid for the limited numerical solutions on finite intervals and, therefore, cannot be used to **prove** the properties on the whole interval of all solutions of the original differential equations.

These questions and remarks give the reason why, besides learning some numerical methods, we also need a theory on **qualitatively analyzing differential equations**, that is, deriving certain properties qualitatively without solving differential equations analytically or numerically.

1.3 Description and Terminology

To get some basic idea of what do we mean by **qualitative analysis** or **qualitative theory**, let's look at the following analysis of the logistic equation (2.1).

Look at Figure 1.10. We see that $f(x) = ax[C - x] = 0$ has two roots: $x = 0$ and $x = C$. Now, define $x_1(t) = 0$, $x_2(t) = C$, $t \geq t_0$, then $x_1(t)$ and $x_2(t)$ are both constant solutions of Eq. (2.1) (with their corresponding initial values). Since x_1 and x_2 are constants, or will "stay put" for all $t \geq t_0$, they are also called **steady solutions**, **critical points**, or **equilibrium points**.

Furthermore, assume $x(t) > 0$ is another solution (we need $x \geq 0$ to represent the population). Then we have $x' = ax[C - x] > 0$ if $0 < x < C$; and $x' = ax[C - x] < 0$ if $x > C$. Therefore, on the x-axis, the solutions with initial values in $(0, C)$ will "**flow**" monotonously to C from the left-hand side of C as t increases; and solutions with initial values bigger than C will flow monotonously to C from the right-hand side of C as t increases, see **Figure 1.12**. (Think of a basketball that is rolling on the ground.)

Accordingly, we have the picture in **Figure 1.13**, which tells us very roughly what the solutions should look like.

The concavity in Figure 1.13 is determined based on the increasing or decreasing of $x'(t)$. For example, when $0 < x(t) < \frac{C}{2}$, $x(t)$ increases as t increases (because in Figure 1.12, x moves to the right on the x-axis when $0 < x < C$); hence, $x'(t) = ax(t)[C - x(t)]$ increases in t (because now $f(x) = ax[C - x]$ increases in Figure 1.12), thus the function $x(t)$ is concave up. When $\frac{C}{2} < x(t) < C$, $x(t)$ increases in t; hence, $x'(t) = ax(t)[C -$

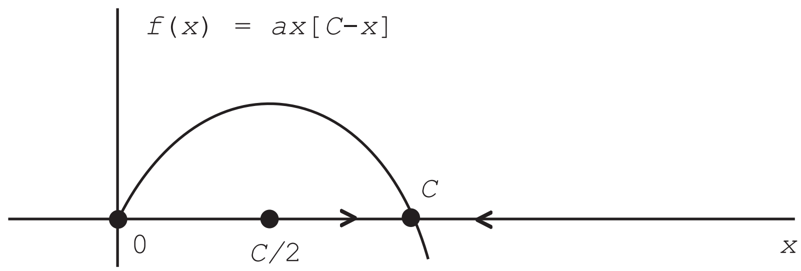

Figure 1.12: Flows of the solutions of a logistic equation on the x-axis

$x(t)]$ decreases in t (because now $f(x) = ax[C - x]$ decreases), therefore the function $x(t)$ is concave down. The case for $x(t) > C$ can also be determined similarly, (see an exercise).

Of course, in this special case, Eq. (2.1) can be solved and the formula (2.11) for the solutions can be used to check these properties. But we do want to point out that it is much easier to determine the concavity of the solutions from Figure 1.12 than by looking at the formula (2.11). To see why, you should take the second derivative of $x(t)$ given in (2.11) and then see how difficult it is to determine its signs, (see an exercise).

In Figure 1.12, $x_1(t) = 0$ "sends other solutions away," hence it is called a **repeller** or **source**, or we say that **the critical point (or the constant solution)** $x_1 = 0$ **is unstable**. However, $x_2(t) = C$ **attracts** other solutions, thus it is called a **sink** or an **attractor**, or we say that **the critical point (or the constant solution)** $x_2 = C$ **is stable**. Articles can be found, for example, in Krebs [1972] and Murray [1989], indicating that the logistic equation (2.1) provides a good match for the experiments done with colonies of bacteria, yeast, or other simple organisms in conditions of constant food supply, climate, and with no predators. However, results of experiments done with fruit flies, flour beetles, and other organisms that have complex life cycles are more complex and do not match well with the logistic equation, because other facts are involved, including age structure and the time-delay effect.

In the above, we derived certain properties, including stabilities, of the solutions of the logistic equation (2.1) without solving it. These properties are called **qualitative properties** because they only tell us the certain **ten-**

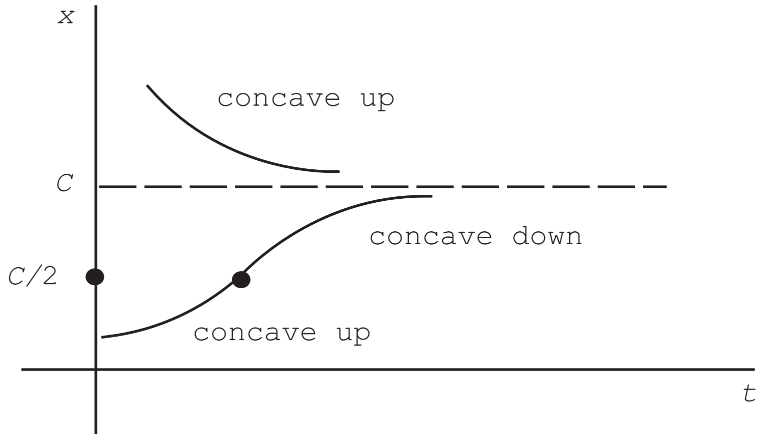

Figure 1.13: A rough sketch of the solutions of a logistic equation according to the flows

dency of how the solutions will behave, and no quantitative information is given. However, in many applications in sciences, the qualitative properties of the equations are the only things we care about, especially when analytical solutions are not available. Therefore, it is very valuable to learn some qualitative analysis of differential equations, as it adds to your knowledge of the subject and helps you get well equipped with tools useful in your applications of differential equations in your future studies and careers.

The first qualitative property we will study is existence and uniqueness theory. This can be used to verify that some differential equations have solutions and these solutions are uniquely determined without solving the differential equations analytically. This theory will also provide a foundation for numerical approximation methods. Based on existence and uniqueness of solutions, we will study other qualitative properties in this book, such as bifurcation, chaos, stability, boundedness, and periodicity, as we explain in the following.

Example 1.3.1 (Euler's buckling beam) A famous example in physics used to introduce the notion of bifurcation is Euler's buckling beam studied by Euler [1744]. If a small weight is placed on the top of a beam shown in **Figure 1.14**, then the beam can support the weight and stay vertical. When the weight increases a little, the position of the beam will change a

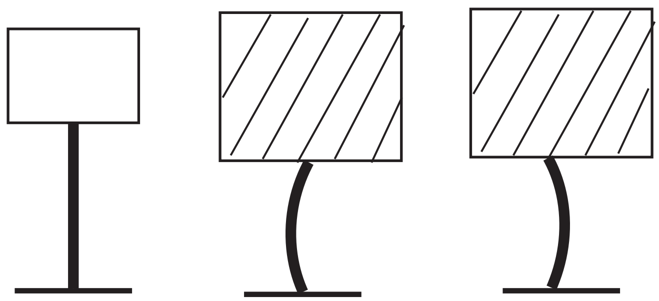

Figure 1.14: Euler's buckling beam

little and remain nearly vertical. Accordingly, this change in position of the beam is called a **quantitative change**. However, if you keep increasing the weight, then there will be a moment that the beam cannot take it any more and will **buckle**, or there is a critical value such that when the weight increases beyond that value the beam will buckle, see Figure 1.14.

Now, the difference is that the beam had undergone a **qualitative change**: from nearly vertical to a buckling position. And with some symmetry assumption, the beam can buckle in all directions.

Therefore, we find that for some systems, when some **parameters**, such as the weight here, are varied and pass some critical values, the systems may experience some abrupt changes, or undergo some qualitative changes. These qualitative changes are generally called **bifurcations**, and the parameter values at which bifurcations occur are called **bifurcation points** or **bifurcation values**. Euler's buckling beam will be analyzed in some detail in Chapter 6 (Bifurcation), where a differential equation describing the motion of the beam will be given, and the bifurcation value, called **Euler's first buckling load**, will be calculated. ♠

Let's look at one more example, which can also explain why the word "bifurcation" is used.

Example 1.3.2 Consider the scalar differential equation

$$x' = \mu - x^2, \tag{3.1}$$

where $\mu \in \Re$ is a parameter. If $\mu < 0$, then Eq. (3.1) has no critical point (that is, $x' = \mu - x^2 = 0$ has no solution), or the curve $y = \mu - x^2$ will not intersect the x-axis, see **Figure 1.15**.

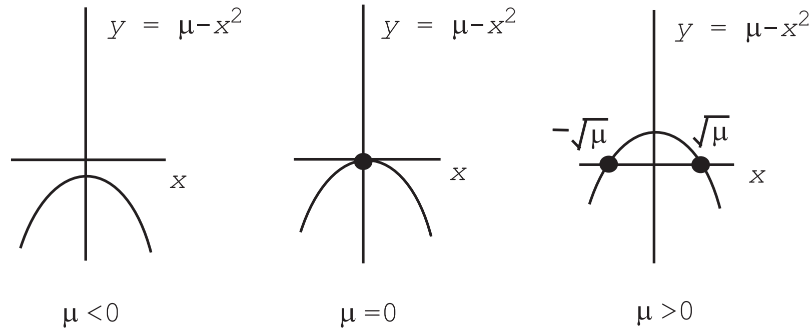

Figure 1.15: Graph of $y = \mu - x^2$ and the critical points of $x' = \mu - x^2$

When μ increases to 0 from below, the graph of $y = \mu - x^2$ moves up and intersects the x-axis when $\mu = 0$, in which case, one critical point appears at $x = 0$. If μ continues to increase, then $\mu > 0$ and hence the graph of $y = \mu - x^2$ will cross the x-axis and then two critical points appear at

$$x = \sqrt{\mu} \quad \text{and} \quad x = -\sqrt{\mu}. \tag{3.2}$$

Now, if we treat μ as the independent variable and treat the corresponding critical point x (if any) as a function of μ, and graph those functions in one (μ, x) plane, then we get **Figure 1.16**, from which we find that for $\mu < 0$, there is no critical point (or function $x(\mu)$ is not defined for $\mu < 0$); however, when μ increases and passes 0, then suddenly, **two branches** of critical points appear according to $x = \sqrt{\mu}$ and $x = -\sqrt{\mu}$, or a "**bi**"-furcation takes place. This explains why the word "bifurcation" is used. In Eq. (3.1), the total number of critical points is also a qualitative property of the system, therefore, when the parameter μ is varied and passes 0, the system undergoes a qualitative change: the number of critical points changes from 0 to 2. Thus, we say that for Eq. (3.1), when the parameter μ is varied, a bifurcation occurs at the bifurcation value $\mu = 0$. ♠

Example 1.3.3 Let x_0 be any fixed number in $[0, 1]$ and consider a recursion relation

$$x_1 = r \sin \pi x_0,\ x_2 = r \sin \pi x_1,\ \cdots,\ x_{m+1} = r \sin \pi x_m, \quad m = 0, 1, 2, \cdots,$$

where $r \in [0, 1]$ is regarded as a parameter. Let's do it with $x_0 = 0.5$, $r = 0.6$,

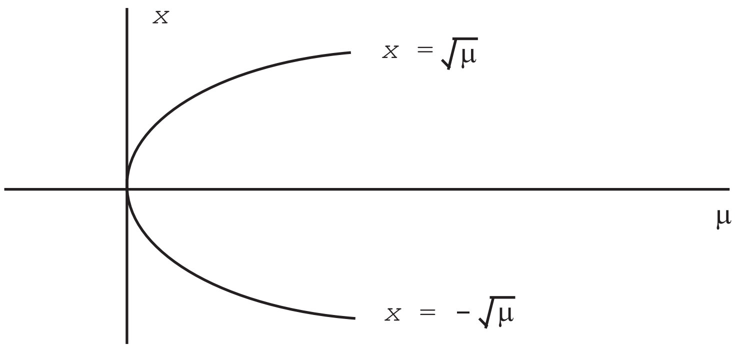

Figure 1.16: The branching of two sets of critical points determined by $x = \pm\sqrt{\mu}$, or a bifurcation takes place

and use a software called Maple, then the code

```
x[0]:=0.5;
for i from 1 by 1 to 100 do
x[i]:=evalf(0.6*sin(Pi*x[i-1]));
od;
```

gives the following result (note that we use $x[m]$ for x_m because it is what you will see in Maple).

x[1] = .6, x[2] = .570633, x[3] = .585288, x[4] = .578590, x[5] = .581804, x[6] = .580294, x[7] = .581011, x[8] = .580672, x[9] = .580833, x[10] = .580757, x[11] = .580793, x[12] = .580776, x[13] = .580784, x[14] = .580780, x[15] = .580782, x[16] = .580781, x[17] = .580781, x[18] = .580781, x[19] = .580781, x[20] = .580781, ······
x[81] = .580781, x[82] = .580781, x[83] = .580781, x[84] = .580781, x[85] = .580781, x[86] = .580781, x[87] = .580781, x[88] = .580781, x[89] = .580781, x[90] = .580781, x[91] = .580781, x[92] = .580781, x[93] = .580781, x[94] = .580781, x[95] = .580781, x[96] = .580781, x[97] = .580781, x[98] = .580781, x[99] = .580781, x[100] = .580781.

Accordingly, we find that

$$x_m \to .580781 \quad \text{as} \quad m \to \infty.$$

Or you can imagine that when you say "Order!" the numbers x_m will listen to you and order themselves to approach .580781.

Next, let's still use $x_0 = 0.5$ but replace $r = 0.6$ with $r = 0.77$ in the above code, then we get

x[1] = .77, x[2] = .509210, x[3] = .769677, x[4] = .509794, x[5] = .769635, x[6] = .509871, x[7] = .769629, x[8] = .509881, x[9] = .769628, x[10] = .509882, x[11] = .769628, x[12] = .509883, x[13] = .769628, x[14] = .509883, x[15] = .769628, x[16] = .509883, x[17] = .769628, x[18] = .509883, x[19] = .769628, x[20] = .509883, $\cdots\cdots$
x[81] = .769628, x[82] = .509883, x[83] = .769628, x[84] = .509883, x[85] = .769628, x[86] = .509883, x[87] = .769628, x[88] = .509883, x[89] = .769628, x[90] = .509883, x[91] = .769628, x[92] = .509883, x[93] = .769628, x[94] = .509883, x[95] = .769628, x[96] = .509883, x[97] = .769628, x[98] = .509883, x[99] = .769628, x[100] = .509883.

In this case, the numbers x_m will "pile up" at the two values

$$\{.509883, \quad .769628\},$$

or the sequence $\{x_m\}$ repeats each of the two values after every two iterations, in which case the set of the two values $\{.509883, .769628\}$ looks like a "**cycle**" with period 2.

Finally, let $x_0 = 0.5$ and $r = 0.9$, then we get

x[1] = .9, x[2] = .278115, x[3] = .690053, x[4] = .744288, x[5] = .647712, x[6] = .804821, x[7] = .517917, x[8] = .898574, x[9] = .281945, x[10] = .696955, x[11] = .733141, x[12] = .669193, x[13] = .775825, x[14] = .582725, x[15] = .869776, x[16] = .358013, x[17] = .811936, x[18] = .501337, x[19] = .899992, x[20] = .278136, ...
x[81] = .899132, x[82] = .280447, x[83] = .694267, x[84] = .737523, x[85] = .660844, x[86] = .787522, x[87] = .557134, x[88] = .885540, x[89] = .316696, x[90] = .754849, x[91] = .626626, x[92] = .829721, x[93] = .458815, x[94] = .892477, x[95] = .298264, x[96] = .725220, x[97] = .683961, x[98] = .753835, x[99] = .628681, x[100] = .827452.

Now, the placement of those numbers are so complex and unpredictable that no matter how loud you shout "Order!!" nobody will listen! So you may want to say "It is chaotic!" If you do, then you are right, because that is exactly the word we are going to use to describe the situation. Of course, you probably want to ask "what does this have to do with differential equations?" We will explain in Chapter 7 that the recursion relation $x_{m+1} = r \sin \pi x_m$, or in general $x_{m+1} = f(x_m)$, defines a "difference equation," or a "map," which is a discrete-time version of a differential equation. ♠

Example 1.3.3 indicates that for some differential equations, the behavior of solutions are very complicated and showing "no orders," therefore, they are generally described as **chaos**. Think about how strange things are in Example 1.3.3 because the maps $x_{m+1} = 0.6 \sin \pi x_m$, $x_{m+1} = 0.77 \sin \pi x_m$, and $x_{m+1} = 0.9 \sin \pi x_m$ look "almost the same," so how could the small difference in the coefficients of 0.6, 0.77, and 0.9 make the sequences $\{x_m\}$ of the iterations behave so differently? This is, in fact, the key to understand bifurcation and chaos: When parameters are different, the corresponding systems could behave completely differently.

Besides the above discrete maps, solutions of continuous systems (that is, differential equations) can also be chaotic. Especially for differential equations in $\Re^n$, $n \geq 3$, solutions are moving in space and could get twisted and twisted and become complex and strange. A famous equation is given by the **Lorenz system**,

$$\begin{cases} \frac{dx}{dt} = 10(y - x), \\ \frac{dy}{dt} = 28x - y - xz, \\ \frac{dz}{dt} = xy - (8/3)z, \end{cases} \tag{3.3}$$

in a milestone paper of Lorenz [1963] (in fact, the paper was reprinted in SPIE Milestone Series, 1994). The system was used to model the weather forecast (see Chapter 7 for some details). Despite of its innocuous looks, the numerical experiments of Lorenz [1963] showed that the solutions of Eq. (3.3) behave in a very complex and strange fashion. For example, the (x, z) plane projection of a three-dimensional solution of the Lorenz system is given in **Figure 1.17**.

The solution in Figure 1.17 does not intersect itself in $\Re^3$, so the crossings in Figure 1.17 are the result of projection in $\Re^2$. Here, the solution will cruise a few circuits on one side, then suddenly moves to the other side and cruises a few circuits, and then suddenly moves back $\cdots$. This process will continue forever, such that the solution will wind around the two sides infinitely many times without ever settling down. The solution also moves around the two sides in an unpredictable fashion. Lorenz showed with numerical experiments that the system (3.3) has an attractor whose properties are so strange and

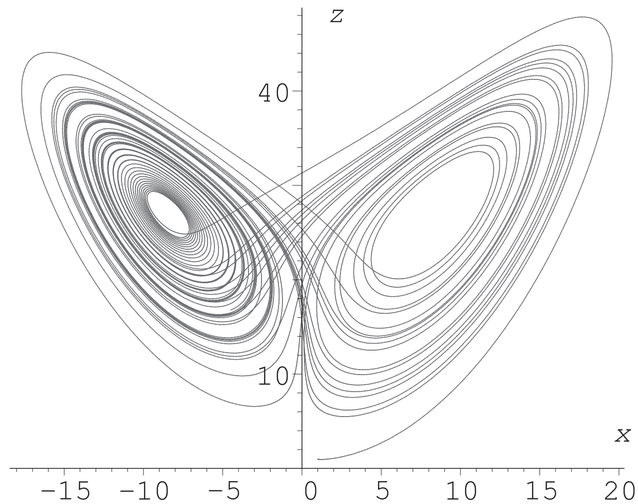

Figure 1.17: The (x, z) plane projection of a three-dimensional solution of the Lorenz system (3.3)

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complex that it is called a **strange attractor**, a very important subject in the study of chaos.

However, we also need to point out that solutions of differential equations in $\Re^2$ do behave in an "orderly" or "predictable" fashion, due to another milestone result: the Poincaré-Bendixson theorem in $\Re^2$, to be studied in Chapter 8.

For the subject on stability, let's look at the following examples.

Example 1.3.4 Consider the scalar differential equation $x'(t) = 0,\ x(t_0) = x_0$. The solution is given by $x(t) = x_0,\ t \geq t_0$, see **Figure 1.18**.

In particular, $\phi(t) = 0,\ t \geq 0$, is a solution (with the initial value being zero). Now, for any $t_0 \geq 0$ and any other initial value x_0 that is close to ϕ, the corresponding solution $x(t) = x_0$ will stay close to ϕ for $t \geq t_0$. ♠

According to the situation in Example 1.3.4, we say that the **solution** ϕ **is stable**, which we define as follows for a general solution ϕ that may be nonzero:

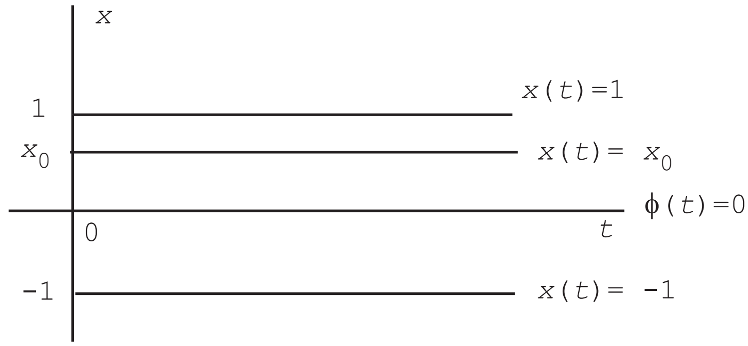

Figure 1.18: Solutions of $x'(t) = 0$ concerning stability

A solution ϕ defined on $[t_\phi, \infty)$ is said to be **stable** *if for any $\varepsilon > 0$ and any $t_0 \geq t_\phi$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ ($\delta(\varepsilon, t_0)$ means δ is determined by ε and t_0; typically $\delta \leq \varepsilon$), such that if the initial value x_0 satisfies $|x_0 - \phi(t_0)| \leq \delta$, then the corresponding solution $x(t)$ starting from t_0 satisfies $|x(t) - \phi(t)| \leq \varepsilon$ for $t \geq t_0$.* See **Figure 1.19**.

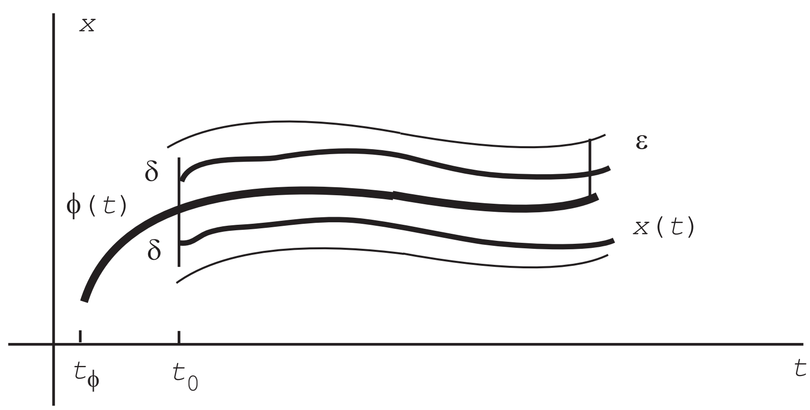

Figure 1.19: Definition of a stable solution ϕ

In physics applications, a solution ϕ may represent certain behavior or property of some physical experiment, and x_0 may be the initial measurement of certain quantity. For example, in some experiment, we need to put one gallon of acid initially to create a certain reaction, which inevitably involves some errors in measurements or approximations. That is, in applications, real data always have some inherent uncertainty, and initial values taken from real data are never known precisely. Now, ϕ being stable means that the corresponding behavior or property is stable in the sense that a small change in initial measurement will result in a small change in the behavior or property for future time. This idea can be seen further in the following example.

Example 1.3.5 Consider again the motion of a simple pendulum given by

$$\theta''(t) + k\theta'(t) + q\sin\theta(t) = 0, \quad t \geq t_0, \tag{3.4}$$

where $k \geq 0$, $q > 0$ are constants, with k related to a damping term. If we place this pendulum in honey or any viscous fluid, and if the inertia term (related to θ'') is relatively small compared to the strong damping (related to $k\theta'$) of the viscous fluid, and if the angle $\theta(t)$ is also small, then we can neglect the $\theta''(t)$ term and approximate $\sin\theta$ with θ and then consider the differential equation

$$\theta'(t) = -(\frac{q}{k})\theta(t), \quad t \geq t_0, \tag{3.5}$$

where we have assumed $k > 0$ since a damping exists. The solution of Eq. (3.5) is given by

$$\theta(t) = \theta_0 e^{-\frac{q}{k}(t-t_0)}, \quad t \geq t_0,$$

and we have

$$\lim_{t\to\infty} \theta(t) = \lim_{t\to\infty} \theta_0 e^{-\frac{q}{k}(t-t_0)} = 0.$$

Now, the interpretation in physics is that $\phi(t) = 0$ is a solution corresponding to the steady state (downward vertical position), if the pendulum is moved slightly from the downward vertical position, then, due to the strong damping of the medium, the pendulum tends to the downward vertical position but will not cross the downward vertical position, that is, no oscillations will occur. See **Figure 1.20**.

That is, in this case, for any initial value θ_0 that is close to $\phi = 0$, the corresponding solution $\theta(t) = \theta_0 e^{-\frac{q}{k}(t-t_0)}$ will not only stay close to ϕ for $t \geq t_0$, but we also have $\lim_{t\to\infty} \theta(t) = \lim_{t\to\infty} \theta_0 e^{-\frac{q}{k}(t-t_0)} = 0 = \phi$, or

$$\lim_{t\to\infty} |\theta(t) - \phi(t)| = 0. \qquad \spadesuit \tag{3.6}$$

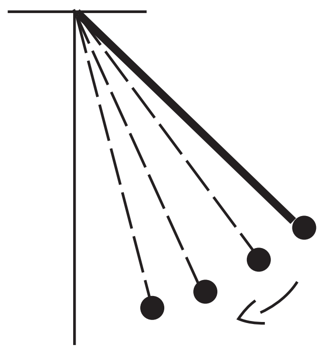

Figure 1.20: The motion of a pendulum going back to the downward vertical position without oscillations

In this sense, we say that for Eq. (3.5) of Example 1.3.5, the solution ϕ **attracts** other solutions, or the **solution ϕ is asymptotically stable**, see **Figure 1.21**.

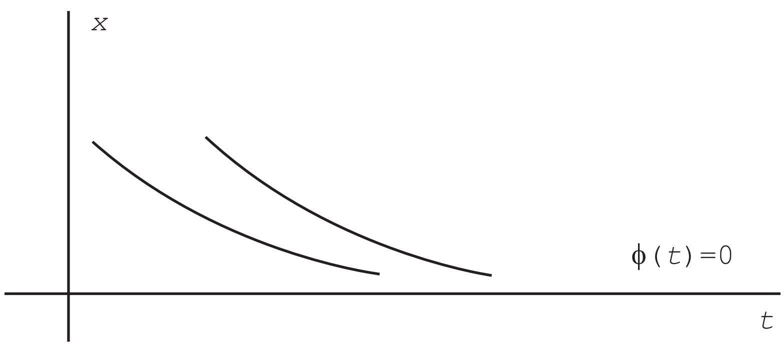

Figure 1.21: The solution $\phi = 0$ is asymptotically stable

A solution ϕ being asymptotically stable means, roughly, that ϕ is stable, and in addition, one has

$$\lim_{t\to\infty} |x(t) - \phi(t)| = 0,$$

where x is any solution whose initial value is close to ϕ. Evidently, $\phi = 0$ in

Example 1.3.4 is stable but not asymptotically stable, because solutions are given by constants there.

Example 1.3.6 Consider the scalar differential equation $x'(t) = 3x(t)$, $x(0) = x_0$. The solution is given by $x(t) = x_0 e^{3t}$, $t \geq 0$, see **Figure 1.22**.

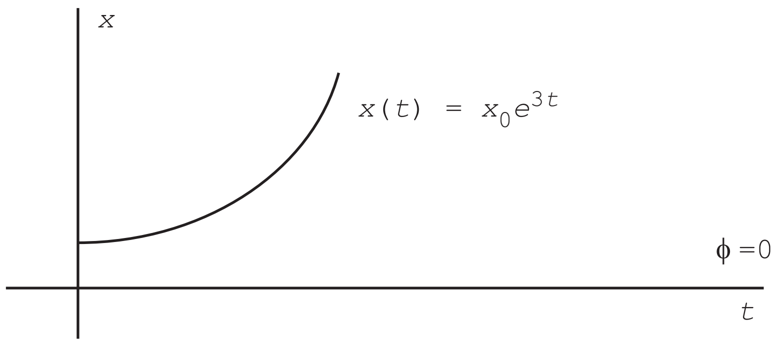

Figure 1.22: Solutions of $x'(t) = 3x(t)$ concerning instability

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Now, we also have $\phi(t) = 0$ as a solution. For any other initial value $x_0 \neq 0$, no matter how close it is to ϕ, the corresponding solution $x(t) = x_0 e^{3t}$ will not stay close to ϕ for $t \geq 0$. In this sense, we say that the **solution** ϕ **is unstable**. For example, $\phi = 0$ in the logistic equation (2.1) is unstable.

♠

Next, we examine boundedness properties. In Example 1.3.4, where the solutions are given by $x(t) = x_0$, $t \geq t_0$, if we specify a range B_1 first (that is, let $B_1 > 0$), then we are able to find a **bound** $B_2 > 0$ (typically $B_2 \geq B_1$) such that when an initial value x_0 is in the range of B_1 (that is, $|x_0| \leq B_1$), then we can use B_2 to bound or control the corresponding solution for $t \geq t_0$. In fact, in this case, we can take $B_2 = B_1$, such that for the solution $x(t) = x_0$, $t \geq t_0$,

$$|x_0| \leq B_1 \quad \text{implies} \quad |x(t)| = |x_0| \leq B_1 = B_2, \quad t \geq t_0.$$

Accordingly, we say that in Example 1.3.4, the **solutions are uniformly bounded**, which is defined as (see **Figure 1.23**):

The solutions of a differential equation are said to be **uniformly bounded** *if for any* $B_1 > 0$*, there exists a* $B_2 > 0$ *such that if an initial value* $|x_0| \leq B_1$*, then the corresponding solution starting from* t_0 *satisfies* $|x(t)| \leq B_2$ *for* $t \geq t_0$*.*

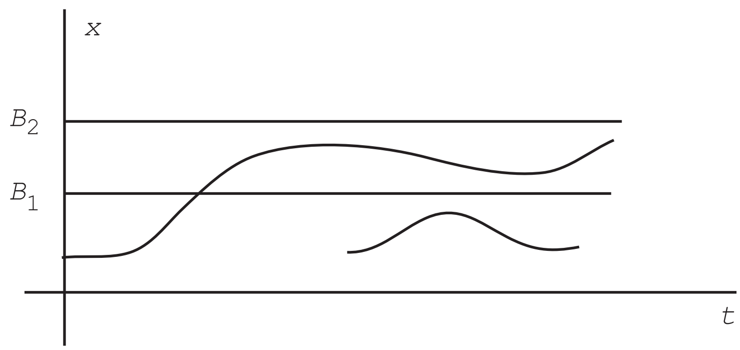

Figure 1.23: The solutions are uniformly bounded

The solutions of Eq. (3.5) in Example 1.3.5 are also uniformly bounded, but the solutions in Example 1.3.6 are not uniformly bounded.

Next, let's consider a related concept. We prescribe a bound B to begin with, and allow the initial value x_0 (at time t_0) to be in a range B_1 that is arbitrary, say, maybe $B_1 > B$. Now, in general, B cannot be used to bound the corresponding solution for $t \geq t_0$, because when $B_1 > B$, B cannot even be used to bound the initial value x_0 for which $|x_0| = B_1 > B$, see **Figure 1.24**.

Thus, it only makes sense to require that B can be used to bound the corresponding solution when t is large, say for example, when $t \geq t_0 + T$, where $T > 0$ is a constant, see **Figure 1.25**.

The requirement that "solutions be bounded by B when t is large" cannot be met by the solutions in Example 1.3.4, because the solution is given by $x(t) = x_0$ there, thus when $|x_0| = B_1 > B$, one has $|x(t)| = |x_0| = B_1 > B,\ t \geq t_0$. But this requirement can be met by the solutions of Eq. (3.5) in Example 1.3.5, because the solution is given by $x(t) = x_0 e^{-\frac{q}{k}(t-t_0)}$ there, thus when $|x_0| \leq B_1$,

$$|x(t)| = |x_0 e^{-\frac{q}{k}(t-t_0)}| \leq B_1 e^{-\frac{q}{k}(t-t_0)} \leq B, \quad t \geq t_0 + T,$$

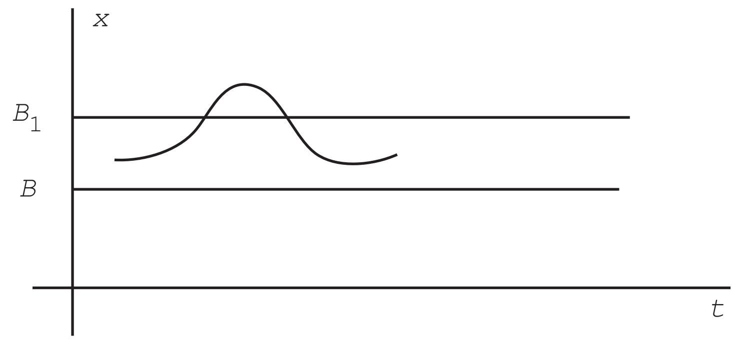

Figure 1.24: B cannot be used to bound the solutions on $[t_0, \infty)$

is true if we solve $T > 0$ in such a way that if $B_1 \le B$, then let $T = 1$ (or any positive number); if $B_1 > B$, then solve T from $B_1 e^{-\frac{q}{k}T} = B$ and obtain $T = -\frac{k}{q} \ln \frac{B}{B_1} > 0$.

Accordingly, we say that for Eq. (3.5) in Example 1.3.5, the **solutions are uniformly ultimately bounded**, which is defined as:

The solutions of a differential equation are said to be **uniformly ultimately bounded** *if there is an (independent or generic) constant $B > 0$ such that for any $B_1 > 0$, there exists a $T > 0$ such that if an initial value $|x_0| \le B_1$, then the corresponding solution starting from t_0 satisfies $|x(t)| \le B$ for $t \ge t_0 + T$.* (See Figure 1.25.)

Therefore, the solutions in Example 1.3.4 and Example 1.3.6 are not uniformly ultimately bounded.

Notice the difference between uniform boundedness and uniform ultimate boundedness. In uniform boundedness, the bound B_2 can be chosen later after the initial range B_1 is fixed. However, in uniform ultimate boundedness, the bound B is fixed first, and the initial range B_1 can be chosen arbitrarily later and, of course, can be bigger than B, therefore, B may be used to bound the solutions only when t is large.

For the study of periodic solutions, consider the following examples.

Example 1.3.7 Consider the scalar differential equation

$$x''(t) + x(t) = 0. \tag{3.7}$$

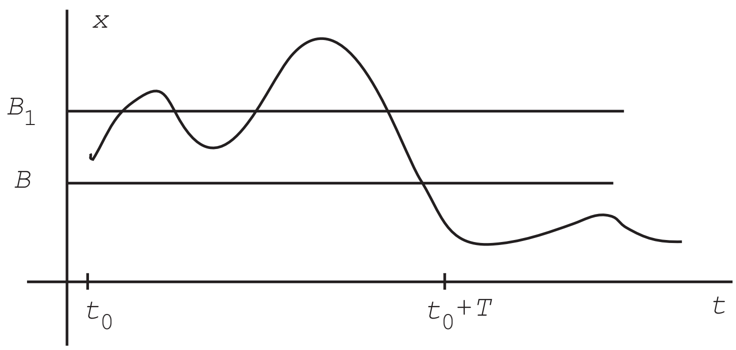

Figure 1.25: B may be used to bound the solutions on $[t_0 + T, \infty)$ for a large T

We find that $x(t) = \sin t$ is a solution. Now,

$$x(t + 2\pi) = x(t).$$

Thus we say that Eq. (3.7) has a **periodic solution** of period 2π. ♠

Example 1.3.8 For the scalar differential equation $x'(t) = 4$, the solutions are straight lines with slope 4, hence the equation has no periodic solution. In general, let's consider the scalar differential equation

$$x' = f(x), \tag{3.8}$$

where f is a continuous function in x. If $f(x) > 0$ (or $f(x) < 0$) for all x, then any solution $x(t)$ (if exists) is strictly increasing (or decreasing) in t, thus Eq. (3.8) has no periodic solutions. If $f(x) = 0$ has some real roots, for example, when the curve of $f(x)$ is given in **Figure 1.26**, then $x_1(t) = \alpha$ and $x_2(t) = \beta,\ t \geq t_0$, are two constant solutions, hence they are periodic solutions (with periods being any positive numbers). Now, x_1 and x_2 are the **only** periodic solutions of Eq. (3.8). Because, for example, if the initial value of a solution is from (α, β), then, similar to the analysis of the logistic equation (2.1), the solution will flow toward the critical point β and will never come back to where it started, thus it cannot be periodic. ♠

This geometric interpretation matches well with some experiments in physics. For example, consider Example 1.3.5 where a pendulum is placed

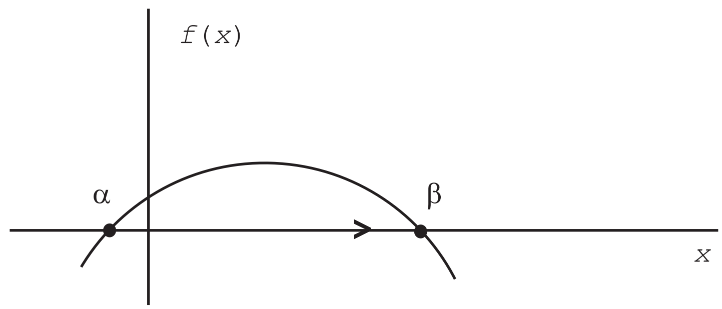

Figure 1.26: The graph of $f(x)$ showing no periodic solutions other than the constant solutions

in honey or any viscous fluid and the motion of the pendulum is approximated by the first-order differential equation (3.5), which is governed by Eq. (3.8). We find in Example 1.3.5 that no oscillations can occur there, that is, nonconstant periodic solutions do not exist there.

The analysis of Eq. (3.8) indicates that when studying periodic solutions, other forms of differential equations should be considered. For example, differential equations in $\Re^n$, $n \geq 2$, may have periodic solutions because now a solution may follow a "circle" and comes back to where it started.

Now, we have introduced some qualitative properties concerning existence and uniqueness of solutions, bifurcation, chaos, stability, boundedness, and periodicity that we will study in this book. Next, we briefly describe how to derive these properties for some simple differential equations. This will help you get ready to the ideas and methods that we will use in the rest of the book for general differential equations.

Example 1.3.9 For existence and uniqueness of solutions, let's consider the scalar differential equation

$$x'(t) = x(t), \quad x(0) = 1, \quad t \geq 0. \tag{3.9}$$

To get some idea of what to expect for general (nonlinear) differential equations, we define the right-hand side of Eq. (3.9) as $f(t, x) = x$, and consider the **Picard approximations** given by

$$\begin{cases} x_0(t) = x_0 = 1, \\ x_1(t) = x_0 + \int_0^t f(s, x_0(s))ds = 1 + \int_0^t x_0(s)ds = 1 + \int_0^t ds = 1 + t, \\ x_2(t) = x_0 + \int_0^t f(s, x_1(s))ds = 1 + \int_0^t x_1(s)ds = 1 + \int_0^t (1+s)ds, \\ \cdots \\ x_n(t) = x_0 + \int_0^t f(s, x_{n-1}(s))ds. \end{cases}$$

Now, an induction shows that

$$x_n(t) = 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!},$$

consequently, we have

$$\lim_{n\to\infty} x_n(t) = e^t,$$

and we can take a derivative to check that e^t is really a solution of Eq. (3.9). ♠

Thus, for a general differential equation, we will use the Picard approximations to define a sequence of functions on a certain interval. Then, we verify, under some conditions this sequence converges to a function that gives rise to a solution of the equation.

For bifurcations, we will demonstrate, using examples and geometrical analysis, that under certain circumstances, the implicit function theorem fails to apply, thus singularities may exist and some qualitative properties of solutions may change abruptly when some parameters are varied, such as the creation and disappearance of critical points, or the exchange of stabilities of critical points.

For chaos, we will look at the discrete maps and the Lorenz system, and discuss their qualitative property changes, such as the period-doubling bifurcation cascades and their routes to chaos.

For stability and boundedness, if the differential equations are linear, then the structure of solutions using the semigroup and evolution system properties, as given by the variation of parameters formulas (1.12) and (1.14), can be used to derive the properties. When **eigenvalues** are available, they can be used directly to derive the properties. For example, in Example 1.3.4, the eigenvalue is 0, thus $\phi = 0$ is stable but not asymptotically stable; in Eq. (3.5) of Example 1.3.5, the eigenvalue is $-\frac{q}{k} < 0$, thus $\phi = 0$ is asymptotically stable; and the eigenvalue in Example 1.3.6 is $3 > 0$, thus $\phi = 0$ is unstable.

Otherwise, for general nonlinear differential equations, to determine the stability properties of a solution ϕ (maybe nonzero in general), we define a function V (called a **Liapunov function**) that is related to the distance between another solution and the solution ϕ. (Typically, we make a transformation, after which ϕ is regarded as the zero solution.) Then we take a derivative (in time t) of V by plugging in the differential equation and argue, with certain conditions, that the derivative is $V' \leq 0$. This indicates that the distance of another solution and ϕ is decreasing, which may reveal the desired properties.

In fact, this idea is already used in the analysis of the logistic equation (2.1). For example, for the critical point $\phi = C$ there, when $x(t) > C$, the distance of $x(t)$ and ϕ is $V = x(t) - C$. Now $\frac{d}{dt}V = x'(t) = ax[C - x] < 0$, thus $x(t)$ flows to $\phi = C$ from the right-hand side. When $0 < x(t) < C$, the distance of $x(t)$ and ϕ is $V = C - x(t)$. Now $\frac{d}{dt}V = -x'(t) = -ax[C-x] < 0$, thus $x(t)$ flows to $\phi = C$ from the left-hand side. Therefore, the distance of $x(t)$ and $\phi = C$ is always decreasing and hence $\phi = C$ is stable, which is already obtained. As for the critical point $\phi = 0$ there, the distance of $x(t) > 0$ and ϕ is $V = x(t)$. Now, $\frac{d}{dt}V = x'(t) = ax[C - x] > 0$ for $x(t) \in (0, C)$, thus the distance of $x(t)$ and $\phi = 0$ is increasing and hence $\phi = 0$ is unstable, which is also already obtained.

To further demonstrate this idea, let's look at the following example.

Example 1.3.10 Consider the scalar differential equation

$$x'(t) = -t^2 x^3(t), \quad t \geq 0. \tag{3.10}$$

Now, $\phi = 0$ is a solution. To determine the stability of $\phi = 0$, we define

$$V(t, x) = [x - 0]^2 = x^2. \tag{3.11}$$

Let $x(t)$ be a solution of Eq. (3.10), then $V(t, x(t)) = x^2(t)$, which is related to the distance of the solution $x(t)$ and $\phi = 0$. Now, taking a derivative of $V(t, x(t))$ in t and plugging in Eq. (3.10), we obtain

$$\begin{aligned} \frac{d}{dt}V(t, x(t)) &= \frac{d}{dt}x^2(t) = 2x(t)x'(t) = 2x(t)[-t^2 x^3(t)] \\ &= -2t^2 x^4(t) < 0, \text{ if } t > 0, \ x(t) \neq 0. \end{aligned} \tag{3.12}$$

Thus we expect that $|x(t) - \phi(t)| = |x(t)| \to 0$ as $t \to \infty$ (which in fact is true using the so-called **Liapunov's method** that we will introduce later). ♠

In the analysis of the logistic equation (2.1) and Example 1.3.10, the key idea is that we can **obtain certain results without solving the differential equations**. Of course, in the special case of Example 1.3.10, we can actually solve Eq. (3.10) using separation of variables (see an exercise) and obtain that

$$x(t) = \frac{1}{\sqrt{\frac{2}{3}t^3 + c}} \longrightarrow 0, \quad t \to \infty, \tag{3.13}$$

where c is a positive constant.

This idea of deriving certain qualitative properties without solving the differential equations can also be found in applications in physics.

Example 1.3.11 Consider the scalar differential equation

$$u'' + g(u) = 0, \quad u = u(t) \in \Re, \tag{3.14}$$

where g is nonlinear, $g(0) = 0$, and satisfies some other conditions. (Note that Eq. (2.7) of a simple pendulum is a special case of Eq. (3.14) when $k = 0$, that is, damping is ignored.) Now, $u = 0$ is a constant solution or an equilibrium. In physics, we can think of $g(u)$ as the restoring force acting on a particle at a displacement u from the equilibrium $u = 0$, and of u' as the velocity of the particle. Then the potential energy at a displacement u from equilibrium is $\int_0^u g(s)ds$, and the kinetic energy is $\frac{1}{2}(u')^2$. Thus, the **total energy** is

$$V(t) = \frac{1}{2}[u'(t)]^2 + \int_0^{u(t)} g(s)ds. \tag{3.15}$$

Now, the **law of conservation of energy** in physics indicates that $V(t)$ is a constant, or $\frac{d}{dt}V(t) = 0$. Indeed, we have

$$\frac{d}{dt}V(t) = u'u'' + g(u)u' = u'[u'' + g(u)] = 0. \tag{3.16}$$

That is, without solving Eq. (3.14), we can define a function V in (3.15) and obtain that the total energy of Eq. (3.14) is a constant, or $\frac{d}{dt}V(t) = 0$. This shows the compelling connection of the method of using a Liapunov function in **mathematics** and the conservation of energy in **physics**. Later, we will verify that this V function for Eq. (3.14) is related to the distance of the solution u and the equilibrium $u = 0$, thus some properties can be derived in this direction, and applications in physics can be carried out. ♠

For periodicity, we see from Example 1.3.7 that Eq. (3.7) has a periodic solution of period 2π. Now, for a general differential equation, to find a periodic solution on interval $[0, \infty)$ of period, say for example $T > 0$, we need a solution $x(t)$ such that

$$x(t+T) = x(t), \quad t \geq 0.$$

In particular, when $t = 0$, we need

$$x(T) = x(0).$$

Accordingly, we define a **mapping** P such that if $x(t)$ is the unique solution corresponding to the initial value $x(0) = x_0$, then we let

$$P(x_0) = x(T),$$

see **Figure 1.27**.

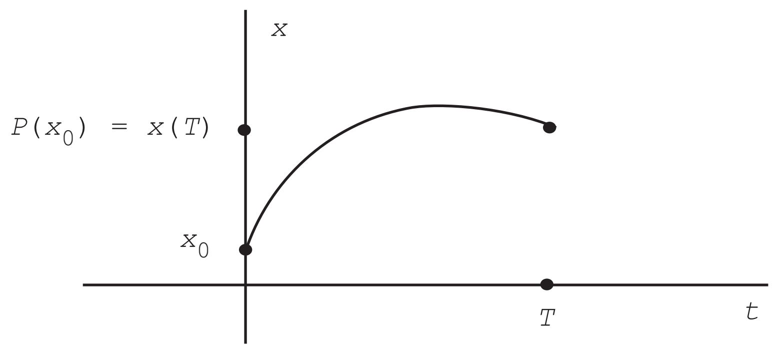

Figure 1.27: The mapping $P : P(x_0) = x(T)$

Notice that if P has a **fixed point**, that is, if there exists an $\overline{x}_0$ such that

$$P(\overline{x}_0) = \overline{x}_0,$$

then we obtain, for the unique solution $x(t)$ with the initial value $x(0) = \overline{x}_0$,

$$x(T) = P(\overline{x}_0) = \overline{x}_0 = x(0).$$

Based on this, other results can be used to verify that

$$x(t+T) = x(t), \quad t \geq 0,$$

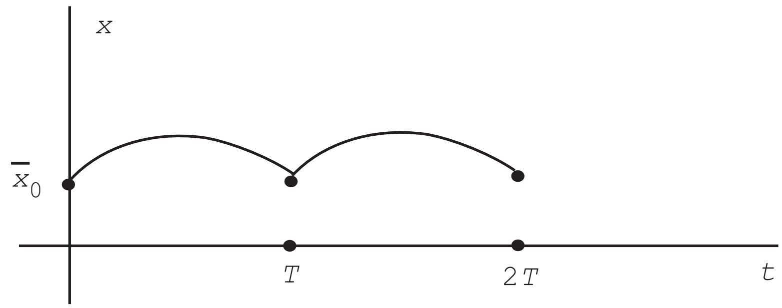

Figure 1.28: A T-periodic solution

therefore, $x(t)$ is a periodic solution of period T, see **Figure 1.28**.

Now, we have briefly described the qualitative properties that we will study in this book. We hope this will inspire your interest and curiosity enough to ask some questions, such as "are those concepts about stability and boundedness the same or what are their differences?" or "how could things like these be done for general nonlinear differential equations?" or even questions like "how bad could solutions of differential equations get?" In doing so, you will be motivated to study the following chapters.

Chapters 1–7 are for upper level undergraduate students, thus we only present the proofs for those results and theorems where some elementary arguments using calculus and linear algebra can be produced. (Notice that the subject on Fixed Point Method in Chapter 2 is optional.) Thus the proofs are accessible and will give these students a chance to use their knowledge in calculus and linear algebra to solve some problems before completing their undergraduate study. For other results whose proofs are too complex and/or involve other subjects not covered here, we do not prove them; instead, we argue their plausibility using geometric and physical interpretation.

Of course, if there are time constraints, the following results can be mentioned without detailed proofs: in Chapter 2, the proofs concerning existence and existence without uniqueness of solutions, the dependence on parameters and the maximal interval of existence; in Chapter 3, differential equations with periodic coefficients and Floquet theory; in Chapter 5, the proofs concerning Liapunov's method; in chapters 6–7, certain proofs concerning bifurcation and chaos.

Chapters 8–12, together with chapters 1–7, are for beginning graduate students. Therefore the treatment and proofs of some results are quite involved. Certain results and their analysis are closely related to current research in differential equations, (in fact, a few are taken from some recent research papers), thereby preparing you to access or even do some research in qualitative theory of differential equations.

One more thing we would like to point out is that chapters 6–12 are rather independent of each other and instructors may choose among them to best fit the last part of the course.

Chapter 1 Exercises

1. Derive (1.10) in Example 1.1.3.

2. Solve the following differential equations.

(a) $x'(t) = 5x(t)$, $x(1) = 4$.

(b) $x'(t) = -4x(t) + t$, $x(2) = 5$.

(c) $x'(t) = tx(t) + 7t$, $x(3) = 6$.

(The integration techniques, such as integration by parts or by substitutions should be used to complete the answers.)

3. Consider

$$x'(t) = k(t)x(t) + f(t), \quad x(t_0) = x_0,$$

for some continuous functions $k(t)$ and $f(t)$.

(a) Solve the equation when $f = 0$.

(b) Review the derivation of the variation of parameters formula by assuming that $C(t)e^{\int_{t_0}^{t} k(s)ds}$ is a solution and then find the formula for $C(t)$.

(c) Let

$$y(t) = e^{\int_{t_0}^{t} k(s)ds}\Big[x_0 + \int_{t_0}^{t} e^{-\int_{t_0}^{s} k(h)dh} f(s)ds\Big].$$

Find $y'(t)$ in terms of $y(t)$ and $f(t)$.

4. Show that

(a) $T(t) = e^{kt}$ satisfies the "semigroup property" (S1) and (S2).

(b) $U(t, s) = e^{\int_s^t k(h)dh}$ satisfies the "evolution system property" (E1) and (E2).

5. Draw the direction field for

(a) $x' = tx$.

(b) $x' = -x^2$.

(c) $x_1' = x_1^2,\ x_2' = x_2^2$.

(d) $x_1' = tx_1x_2,\ x_2' = -x_2^2$.

6. Sketch the function $f(x) = x^2 + x - 2$. Then sketch, on the x-axis, the flows of the solutions of $x' = f(x)$. Finally sketch the solutions in the (t, x)-plane.

7. Consider the logistic equation in Example 1.2.1.

(a) Determine the concavity of the solution $x(t)$ when $x(t) > C$ by using Figure 1.12.

(b) Check that $x_1(t) = 0,\ x_2(t) = C,\ t \geq 0$, are both constant solutions. Then solve for other solutions.

(c) For the solution given by (2.11), determine the concavity by using the second derivative.

8. Verify that the logistic equation (2.1) can be replaced by

$$x'(t) = rx(t)[1 - x(t)], \tag{3.17}$$

where $r = aC$.

9. Solve the equation in Example 1.3.10.

10. Show that $x(t) = \cos t$, $y(t) = \sin t$ satisfies $\{x'(t) = -y(t),\ y'(t) = x(t)\}$.

11. Consider

$$x'(t) = 2x(t),\ x(0) = 1, \tag{3.18}$$

and define

$$\begin{aligned} x_0(t) &= 1, \\ x_1(t) &= 1 + \int_0^t 2x_0(s)ds, \\ x_m(t) &= 1 + \int_0^t 2x_{m-1}(s)ds, \quad m = 2, 3, \cdots. \end{aligned}$$

(a) Use an induction to find a formula for $x_m(t)$.

(b) For $t \in \Re$ fixed, find $\lim_{m\to\infty} x_m(t)$.

(c) Define $x(t) = \lim_{m\to\infty} x_m(t)$. Check if $x(t)$ is a solution of Eq. (3.18).

12. Examine the change of the number of critical points for

(a) $x' = \mu + x^2$,

(b) $x' = \mu - x - e^{-x}$,

where μ is regarded as a parameter.

13. Start with any real number $x_0 \in (0, 1)$ and use a calculator or Maple to find $x_1, x_2, \cdots, x_m$ up to $m = 30$ for

(a) $x_{m+1} = \sin x_m$,

(b) $x_{m+1} = \cos x_m$,

(c) $x_{m+1} = 0.5 \sin \pi x_m$,

(d) $x_{m+1} = 0.76 \sin \pi x_m$,

(e) $x_{m+1} = 0.92 \sin \pi x_m$,

(f) $x_{m+1} = 0.94 \sin \pi x_m$,

(g) $x_{m+1} = 0.98 \sin \pi x_m$.

14. Find a V function for the equation

$$x'(t) = -t^4 x(t),$$

such that its derivative in t satisfies $V' \leq 0$.

15. Find all the periodic solutions of the scalar differential equation

$$x' = x[x - 1][x + 1].$$

Argue why they are all the periodic solutions for the equation.