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Preface

Why should we learn some qualitative theory of differential equations?

Differential equations are mainly used to describe the change of quantities or behavior of certain systems in applications, such as those governed by Newton's laws in physics.

When the differential equations under study are linear, the conventional methods, such as the Laplace transform method and the power series solutions, can be used to solve the differential equations analytically, that is, the solutions can be **written out** using formulas.

When the differential equations under study are nonlinear, analytical solutions cannot, in general, be found; that is, solutions cannot be written out using formulas. In those cases, one approach is to use **numerical approximations**. In fact, the recent advances in computer technology make the numerical approximation classes very popular because powerful software allows students to quickly approximate solutions of nonlinear differential equations and visualize, even in 3-D, their properties.

However, in most applications in biology, chemistry, and physics modeled by nonlinear differential equations where analytical solutions may be unavailable, people are interested in the questions related to the so-called **qualitative properties**, such as: will the system have at least one solution? will the system have at most one solution? can certain behavior of the system be controlled or stabilized? or will the system exhibit some periodicity? If these questions can be answered without solving the differential equations, especially when analytical solutions are unavailable, we can still get a very good understanding of the system. Therefore, besides learning some numerical methods, it is also important and beneficial to learn how to analyze some qualitative properties, such as the existence and uniqueness of solutions, phase portraits analysis, dynamics of systems, stability, bifurcations, chaos, boundedness, and periodicity of differential equations without solving them analytically or numerically.

This makes learning the qualitative theory of differential equations very valuable, as it helps students get well equipped with tools they can use when they apply the knowledge of differential equations in their future studies and careers. For example, when taking a numerical methods course, before a numerical approximation is carried out, the existence and uniqueness of solutions should be checked to make sure that there does exist one and only one solution to be approximated. Otherwise, how does one know what one is approximating? A related remark is that even though numerical solutions can be carried out to **suggest** certain properties, they are obtained through discretization on finite intervals. They reveal certain properties that are only valid for the limited numerical solutions on finite intervals and, therefore, cannot be used to **determine** the qualitative properties on the whole interval of all solutions of the original differential equations.

Based on the above remarks, we conclude that in order to have a more complete knowledge of differential equations, and be able to analyze differential equations without solving them analytically or numerically, we should learn some qualitative theory of differential equations.

To whom is this book written?

This book is written for upper level undergraduates (second undergraduate course in ODEs) and beginning graduate students. To be more specific, Chapters 1–7 are for upper level undergraduate students, where the basic qualitative properties concerning existence and uniqueness, structures of solutions, phase portraits, stability, bifurcation, and chaos are discussed. Chapters 8–12, together with Chapters 1–7, are for beginning graduate students, where some additional subjects on stability, dynamical systems, bounded and periodic solutions are covered.

Another reason for writing this book is that nowadays it is a popular trend for upper level undergraduate students and beginning graduate students to get involved in some research activity. Compared to other more abstract subjects in mathematics, qualitative analysis of differential equations is readily accessible to upper level undergraduates and beginning graduate students. It is a vast hunting field in which students will get an opportunity to combine and apply their knowledge in linear algebra, elementary differential equations, advanced calculus, and others to "hunt some prey."

Furthermore, the qualitative analysis of differential equations is on the border line of applied mathematics and pure mathematics, so it can attract students interested in either discipline.

How does this book differ from other ODE books?

It is often the case that in a book written at the graduate level or even at the upper undergraduate level, there are "jumps" in the reasonings or in the proofs, as evidenced by words like "obviously" or "clearly." However, inexperienced undergraduate and beginning graduate students need more careful and detailed guidance to help them learn the material and gain maturity on the subject.

In this book, I selected only the subjects that are of fundamental importance, that are accessible to upper level undergraduate and beginning graduate students, and that are related to current research in the field. Then, for each selected topic, I provided a complete analysis that is suitable for the targeted audience, and filled in the details and gaps which are missing from some other books. Sometimes, I produced elementary proofs using calculus and linear algebra for certain results that are treated in a more abstract frame in other books. Also, examples and reasons are given before introducing many concepts and results.

Therefore, this book is different from other ODE books in that it is more detailed, and, as the title of this book indicates, the level of this book is lower than most books for graduate students, and higher than the books for elementary differential equations. Moreover, this book contains many interesting pure and applied topics that can be used for one or two semesters.

What topics are covered in this book?

Chapter 1. A Brief Description. We first give a brief treatment of some subjects covered in an elementary differential equations course. Then we introduce some terminology and describe some qualitative properties of differential equations that we are going to study in this book. We use the geometric and physical arguments to show why certain qualitative properties are plausible and why sometimes we pursue a qualitative analysis rather than solving differential equations analytically or numerically. This will give readers an opportunity to become familiar with the objective and terminology of qualitative analysis in a somewhat familiar setting.

Chapter 2. Existence and Uniqueness. In Section 1, we use examples from applications to define general first order differential equations in $\Re^n$. In Section 2, we study existence and uniqueness of solutions, that is, we examine under what conditions a differential equation has solutions and how the solutions are uniquely determined, without solving the differential equation analytically. A condition called "Lipschitz condition" is utilized. In Section 3, we show under certain conditions that solutions are continuous and differentiable with respect to initial data and parameters. In Section 4, we determine structures of the maximal intervals of existence for solutions, and study properties of solutions with respect to the maximal intervals of existence. In Section 5 (which may be optional), we introduce the Fixed Point Method. We use the contraction mapping principle to derive existence and uniqueness of solutions if a local Lipschitz condition is satisfied. Then, when a local Lipschitz condition is not assumed, we use Schauder's second fixed point theorem to obtain existence of solutions, in which case, uniqueness is not guaranteed.

Chapter 3. Linear Differential Equations. In Section 1, we make some definitions concerning linear differential equations. In Section 2, we study general nonhomogeneous linear differential equations and obtain the fundamental matrix solutions and verify that they satisfy the "evolution system property." Then we derive the variation of parameters formula using the fundamental matrix solutions and observe what these solutions should look like. In Section 3, we look at equations with constant coefficients and examine detailed structure of solutions in terms of eigenvalues of the leading constant matrix, using the Jordan canonical form theorem. In addition we derive the Putzer algorithm that can be used to actually solve or compute solutions for equations with constant coefficients. This result will appeal to readers interested in computation. In Section 4, we look at equations with periodic coefficients and study Floquet theory, which allows us to transform equations with periodic coefficients into equations with constant coefficients. The results of Section 3 can then be applied to the transformed equations. The concept of Liapunov exponents is also briefly introduced in Section 4.

Chapter 4. Autonomous Differential Equations in $\Re^2$. In Section 1, we introduce the concept of dynamical systems, discuss possible trajectories in phase planes for two-dimensional autonomous equations, and outline the relationship between nonlinear differential equations and their linearizations. In Section 2, we provide a complete analysis for linear autonomous differential equations in $\Re^2$ and draw all phase portraits for the different cases according to eigenvalues of the coefficient matrix. We also introduce some terminology, including stability of solutions, according to the properties revealed, which leads us to detailed study of the same subject later for general differential equations in $\Re^n, n \geq 1$. In Section 3, we examine the conditions which ensure that solutions of autonomous differential equations and their linearizations have essentially the same local geometric and qualitative properties near the origin. In Section 4, we apply the results to analyze an equation of a simple pendulum. In Section 5, we generalize the ideas of a simple pendulum and study the Hamiltonian systems and gradient systems.

Chapter 5. Stability. Part I. In Section 1, we introduce the notion of stabilities in the sense of Liapunov for general differential equations in $\Re^n$, which are based on some consideration in physics and the planar differential equations studied in Chapter 4. In Section 2, we study stabilities for linear differential equations with constant coefficients and show that eigenvalues of the coefficient matrices determine stability properties. In Section 3, stabilities of linear equations with linear or nonlinear perturbations are studied using the variation of parameters formula and Gronwall's inequality. The results include some planar autonomous nonlinear differential equations studied in Chapter 4 as special cases. Therefore, some unproven results in Chapter 4 can now get a partial proof. In Section 4, linear periodic differential equations are treated. The Floquet theory from Chapter 3 is used to transform linear periodic equations into linear equations with constant coefficients and the results from Section 2 can then be applied. In Section 5, we introduce Liapunov's method for autonomous nonlinear differential equations and prove their stability properties under the assumption that there exist appropriate Liapunov functions. Thus, we can obtain stabilities without explicitly solving differential equations. In Section 6, we provide examples to demonstrate how the Liapunov theory is applied by constructing Liapunov functions in specific applications. Liapunov's method for general (nonautonomous) differential equations will be given in Chapter 9.

Chapter 6. Bifurcation. In Section 1, we use examples, including Euler's buckling beam, to introduce the concept of bifurcation of critical points of differential equations when some parameters are varied. In Section 2, we study saddle-node bifurcations and use examples to explain why saddle and node appear for this type of bifurcations. We analyze the geometric aspects of some scalar differential equations that undergo saddle-node bifurcations and use them to formulate and prove a result concerning saddle-node bifurcations for scalar differential equations. In Section 3, we study transcritical bifurcations and apply them to a solid-state laser in physics. Again, the geometric aspects of some examples are analyzed and used to formulate and prove a result concerning transcritical bifurcations for scalar differential equations. In Section 4, we study pitchfork bifurcations and apply them to Euler's buckling beam and calculate Euler's first buckling load, which is the value the buckling takes place. The hysteresis effect with applications in physics is also discussed. A result concerning pitchfork bifurcations for scalar differential equations is formulated using the geometric interpretation. In Section 5, we analyze the situations where a pair of two conjugate complex eigenvalues cross the pure imaginary axis when some parameters are varied. We introduce the Poincaré-Andronov-Hopf bifurcation theorem and apply it to van der Pol's oscillator in physics.

Chapter 7. Chaos. In Section 1, we use examples, such as some discrete maps and the Lorenz system, to introduce the concept of chaos. In Section 2, we study recursion relations, also called maps, and their bifurcation properties by finding the similarities to the bifurcations of critical points of differential equations, hence the results in Chapter 6 can be carried over. In Section 3, we look at a phenomenon called period-doubling bifurcations cascade, which provides a route to chaos. In Section 4, we introduce some universality results concerning one-dimensional maps. In Section 5, we study some properties of the Lorenz system and introduce the notion of strange attractors. In Section 6, we study the Smale horseshoe which provides an example of a strange invariant set possessing chaotic dynamics.

Chapter 8. Dynamical Systems. In Section 1, we discuss the need to study the global properties concerning the geometrical relationship between critical points, periodic orbits, and nonintersecting curves. In Section 2, we study the dynamics in $\Re^2$ and prove the Poincaré-Bendixson theorem. In Section 3, we use the Poincaré-Bendixson theorem, together with other results, to obtain existence and nonexistence of limit cycles, which in turn help us determine the global properties of planar systems. In Section 4, we apply the results to a Lotka-Volterra competition equation. In Section 5, we study invariant manifolds and the Hartman-Grobman theorem, which generalize certain results for planar equations in Chapter 4 to differential equations in $\Re^n$.

Chapter 9. Stability. Part II. In Section 1, we prove a result concerning the equivalence of “stability” (or “asymptotic stability”) and “uniform stability” (or “uniform asymptotic stability”) for autonomous differential equations. In Section 2, we use the results from Chapter 3 to derive stability properties for general linear differential equations, and prove that they are determined by the fundamental matrix solutions. The results here include those derived in Chapter 5 for linear differential equations with constant or periodic coefficients as special cases. Stability properties of general linear differential equations with linear or nonlinear perturbations are also studied using the variation of parameters formula and Gronwall’s inequality. In Section 3, we introduce Liapunov’s method for general (nonautonomous) differential equations and derive their stability properties, which extends the study of stabilities in Chapter 5 for autonomous differential equations.

Chapter 10. Bounded Solutions. In Section 1, we make some definitions and discuss the relationship between boundedness and ultimate boundedness. In Section 2, we derive boundedness results for general linear differential equations by using the results from Chapter 9. It will be seen that stability and boundedness are almost equivalent for linear homogeneous differential equations, and they are determined by the fundamental matrix solutions. For nonlinear differential equations, examples will be given to show that the concepts of stability and boundedness are not equivalent. In Section 3, we look at the case when the coefficient matrix is a constant matrix, and verify that the eigenvalues of the coefficient matrix determine boundedness properties. In Section 4, the case of a periodic coefficient matrix is treated. The Floquet theory from Chapter 3 is used to transform the equation with a periodic coefficient matrix into an equation with a constant coefficient matrix. Therefore, the results from Section 3 can be applied. In Section 5, we use Liapunov’s method to study boundedness properties for general nonlinear differential equations.

Chapter 11. Periodic Solutions. In Section 1, we give some basic results concerning the search of periodic solutions and indicate that it is appropriate to use a fixed point approach. In Section 2, we derive the existence of periodic solutions for general linear differential equations. First, we derive periodic solutions using the eigenvalues of $U(T, 0)$, where $U(t, s)$ is the fundamental matrix solution of linear homogeneous differential equations. Then we derive periodic solutions from the bounded solutions. Periodic solutions of linear differential equations with linear and nonlinear perturbations are also given. In Section 3, we look at general nonlinear differential equations. Since using eigenvalues is not applicable now, we extend the idea of deriving periodic solutions using the boundedness. First, we present some Massera-type results for one-dimensional and two-dimensional differential equations, whose proofs are generally not extendible to higher dimensional cases. Then, for general n-dimensional differential equations, we apply Horn's fixed point theorem to obtain fixed points, and hence periodic solutions, under the assumption that the solutions are equi-ultimate bounded.

Chapter 12. Some New Types of Equations. In this chapter, we use applications, such as those in biology and physics, to introduce some new types of differential equations, which are extensions and improvements of the differential equations discussed in the previous chapters. They include finite delay differential equations, infinite delay differential equations, integrodifferential equations, impulsive differential equations, differential equations with nonlocal conditions, impulsive differential equations with nonlocal conditions, and abstract differential equations. For each new type of differential equations mentioned above, we use one section to describe some of their important features. For example, for integrodifferential equations, we outline a method which can reformulate an integrodifferential equation as a differential equation in a product space; and for abstract differential and integrodifferential equations, we introduce the semigroup and resolvent operator approaches. The purpose of this chapter is to provide some remarks and references for the recent advancement in differential equations, which will help readers to access the frontline research, so they may be able to contribute their own findings in the research of differential equations and other related areas.

How to use this book?

For an upper level undergraduate course. The material in Chapters 1–7 is enough. Moreover, if there are time constraints, then some results, such as the following, can be mentioned without detailed proofs: in Chapter 2, the proofs concerning existence and existence without uniqueness of solutions, the dependence on initial data and parameters, and the maximal interval of existence; in Chapter 3, differential equations with periodic coefficients and Floquet theory; in Chapter 5, the proofs concerning Liapunov's method; in Chapters 6–7, certain proofs concerning bifurcations and chaos. (Note that Section 2.5 concerning the Fixed Point Method is optional.)

For a beginning graduate course. Chapters 1–11 provide a sufficient resource for different selections of subjects to be covered. If time permits, Chapter 12 can provide some direction for further reading and/or research in the qualitative theory of differential equations.

One more thing we would like to point out is that Chapters 6 through 12 are rather independent of each other and the instructors may choose among them to fit the last part of the course to their particular needs.

Exercises and notations. Most questions in the Exercises are quite important and should be assigned to give the students a good understanding of the subjects.

In Theorem x.y.z, x indicates the chapter number, y the section number, and z the number of the result in section y. The same numbering system holds true for Lemma x.y.z, Example x.y.z, etc.

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