Crazy Bases: Fractions and Twoandthree

Stephen Lucas

Department of Mathematics and Statistics
James Madison University, Harrisonburg VA

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Outline

- Integer Bases: Natural number, transforming, negative.
- Fractional Bases: Traditional, new, arithmetic.
- Base 2&3: Digits representing the same number in bases 2 and 3 simultaneously.
- *p*-adic Interlude: negative integers and fractions
- More Craziness: other simultaneous bases, irrational bases, complex bases...

Natural Number Base

Given a natural number base $b > 1$, a natural number has a unique representation $x = d_0 + d_1 b + d_2 b^2 + \cdots + d_n b^n$, each $d_i \in \{0, 1, \ldots, b-1\}$, as $(d_n d_{n-1} \ldots d_2 d_1 d_0)_b$.

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- Odlyzko (1978): base ten with $\{0, 1, 2, 3, 4, 50, 51, 52, 53, 54\}$, Matula (1982): base three with $\{0, 1, -7\}$, and complete theory (including 0).

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We also lack uniqueness:
$2 = (10.01000001\ldots)_{3/2} = (0.111\ldots)_{3/2}$.

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Eg base 3/2 with $[+2, -3]$: $(10) \xrightarrow{3} (6,1) \xrightarrow{2} (4,0,1) \xrightarrow{1} (2,1,0,1)$.
Check: $2\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^2 + 1 = \frac{27}{4} + \frac{9}{4} + 1 = \frac{40}{4} = 10.$

Eg base 10/3 with $[+3, -10]$:

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Eg base 10/3 with [+3, −10]: $(10) \xrightarrow{1} (3,0)$, $(12) \xrightarrow{1} (3,2)$,
$(222) \xrightarrow{22} (66,2) \xrightarrow{6} (18,6,2) \xrightarrow{1} (3,8,6,2)$.

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Backwards in base 10/3: $(9,9,8,2) \xrightarrow{-3} (39,8,2) \xrightarrow{-13} (138,2) \xrightarrow{-46} (462)$,

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Backwards in base 10/3: $(9,9,8,2) \xrightarrow{-3} (39,8,2) \xrightarrow{-13} (138,2) \xrightarrow{-46} (462)$, but $(9,8,8,2) \xrightarrow{-3} (0,38,8,2) \xrightarrow{-12} (0,2,128,2) \xrightarrow{-42} (0,2,2,422)$.

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Backwards in base $10/3$: $(9,9,8,2) \stackrel{-3}{\to} (39,8,2) \stackrel{-13}{\to} (138,2) \stackrel{-46}{\to} (462)$, but $(9,8,8,2) \stackrel{-3}{\to} (0,38,8,2) \stackrel{-12}{\to} (0,2,128,2) \stackrel{-42}{\to} (0,2,2,422)$. $2 \times \frac{100}{9} + 2 \times \frac{10}{3} + 422 = 450\frac{8}{9}$.

Arithmetic in Base 3/2

- Addition is as for traditional positional notation, right to left, but in base 3/2 sometimes carries two digits: $(2)_{3/2} + (1)_{3/2} = (20)_{3/2}$, $(2)_{3/2} + (2)_{3/2} + (2)_{3/2} = (210)_{3/2}$.

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- Multiplication is also straightforward, easier with lattice separating digit products from carries.

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Note as polynomial coefficients, $b - 2 = 0 \to b = 2$, $b - 3 = 0 \to b = 3$, $b^2 - 5b + 6 = (b - 2)(b - 3) = 0 \to b = 2, 3$.

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After a million applications of the carry rule,
$13 = 32.44444444444322223442441414(39)(44)(18\,217)(8\,978\,498)(26\,352\,477)(3\,348\,444\,877)(10\,311\,561\,742)444\ldots_{2\&3}$.

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Unfortunately $32.444\ldots_3 = 13$, but $32.444\ldots_2 = 12$, so 13 (and higher) don't appear to work :-(

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\begin{gathered}
1 = 1_{2\&3},\ 2 = 2_{2\&3},\ 3 = 3_{2\&3},\ 4 = 4_{2\&3},\\
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8 = \underline{1}4\underline{1}2_{2\&3},\ 9 = \underline{1}4\underline{1}3_{2\&3},\ 10 = \underline{1}4\underline{1}4_{2\&3},\\
11 = \underline{1}4\underline{1}5_{2\&3} = \underline{1}34\underline{1}_{2\&3},\ 12 = \underline{1}340_{2\&3},
\end{gathered}
$$

and so on. Or using the carry rule directly,
$(12)_{2\&3} \stackrel{-2}{\to} (-2)(10)0_{2\&3} \stackrel{-1}{\to} \underline{1}340_{2\&3}$, and
$(43)_{2\&3} \stackrel{-7}{\to} (-7)(35)1_{2\&3} \stackrel{-6}{\to} (-6)(23)\underline{1}1_{2\&3}$
$\stackrel{-4}{\to} (-4)(14)\underline{11}1_{2\&3} \stackrel{-2}{\to} (-2)62\underline{11}1_{2\&3} \stackrel{-1}{\to} \underline{1}302\underline{11}1_{2\&3}$.

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Existence: If $y \geq 1$, $y' = (y - d)/3 \geq (y - 4)/3 \geq (1 - 4)/3 = -1$. If $y = 0$, $y' = 0, -1$. If $y = -1$, $y' = 0, -1$.

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$x' + y' < x + y$ when $3x + 4y > 5$. Checking the seven cases where this is not true all converge to $x = y = 0$.

Alternate Construction

Our proof gives a different way of finding digits: start with $x_0 = y_0 = x$. Given pair (x_i, y_i), find digit $d_i \in \{-1, 0, 1, 2, 3, 4\}$ congruent to x_i mod 2 and y_i mod 3, set $x_{i+1} = (x_i - d_i)/2$, $y_{i+1} = (y_i - d_i)/3$, until $x_k = y_k = 0$.

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- 21 mod 2 = 1, 14 mod 3 = 2: $d_1 = -1$ and $x_2 = 11$, $y_2 = 5$.

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With $x_0 = y_0 = 43$:

- $43 \bmod 2 = 1$, $43 \bmod 3 = 1$: $d_0 = 1$ and $x_1 = 21$, $y_1 = 14$.
- $21 \bmod 2 = 1$, $14 \bmod 3 = 2$: $d_1 = -1$ and $x_2 = 11$, $y_2 = 5$.
- $11 \bmod 2 = 1$, $5 \bmod 3 = 2$: $d_2 = -1$ and $x_3 = 6$, $y_3 = 2$.
- $6 \bmod 2 = 0$, $2 \bmod 3 = 2$: $d_3 = 2$ and $x_4 = 2$, $y_4 = 0$.
- $2 \bmod 2 = 0$, $0 \bmod 3 = 0$: $d_4 = 0$ and $x_5 = 1$, $y_5 = 0$.
- $1 \bmod 2 = 1$, $0 \bmod 3 = 0$: $d_5 = 3$ and $x_6 = -1$, $y_6 = -1$.
- $-1 \bmod 2 = 1$, $-1 \bmod 3 = 2$: $d_6 = -1$ and $x_7 = 0$, $y_7 = 0$.

Reading off the digits, $43 = \underline{1}302\underline{1}\underline{1}1_{2\&3}$ again.

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$x_3 = x_1$ repeating, $1/6 = \ldots 040404 1_5$.

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So $35/3 = \ldots 33345$, and $7/60 = \ldots 333.45$.

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Every positive integer greater than five eventually has the repeated digit 2 in 6-adic base 2&3.

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- $d_0 = 3$, $x_1 = (5/3 - 3)/2 = -2/3$, $y_1 = (15/4 - 3)/3 = 1/4$.
- $d_1 = 4$, $x_2 = (-2/3 - 4)/2 = -7/3$, $y_2 = (1/4 - 4)/3 = -5/4$.
- $d_2 = 1$, $x_3 = (-7/3 - 1)/2 = -5/3$, $y_3 = (-5/4 - 1)/3 = -3/4$.
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$(x_4, y_4) = (x_2, y_2)$, shifting by two digits
$5/12 = \ldots 313131.43_{2\&3}$.

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- Base $2\&3\&5$, carry $[+1, -10, +31, -30]$, digits $0 \to 29$, $43 = \ldots 88890(21)(21)1(13)_{2\&3\&5}$, $143 = \ldots 88891(14)(26)(24)4(23)_{2\&3\&5}$.

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Complex Conjugate Bases, Gaussian Integers

Carry rule [1, 4, 5] using digits $\{0, 1, 2, 3, 4\}$ has quadratic $x^2 + 4x + 5 = 0$ and bases $-2 \pm i$.

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Does Base Ten Seem Boring Now?

Questions?