So You Think You Can Multiply?

A History of Multiplication

Stephen Lucas

Department of Mathematics and Statistics
James Madison University, Harrisonburg VA

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Outline

- Ancient Techniques
  - Definitions
  - Squares and Triangular Numbers
  - Doubling and Halving
  - Geometry
- Positional Notation
  - Positional Definition
  - Hinge and Scratch
  - Cross
  - Lattice
  - Napier's Rods and the "Modern" method
  - Genaille's Rods
- Multiplication as Addition
  - Prosthaphaeresis
  - Logarithms
  - Powers and Pascal's Triangle
- High Precision
  - Karatsuba
  - Toom-Cook
  - Schönhage-Strassen

Definitions

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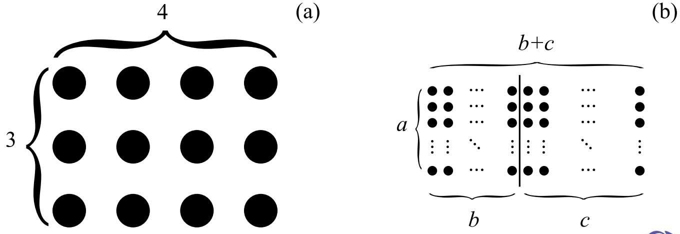

Using Squares & Triangular Numbers

Ancient Babylon:

$(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$. Subtract:

$$ab = \frac{(a+b)^2 - (a-b)^2}{4}.$$

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Teacher Resources on Line: If $T_n = 1 + 2 + \cdots + n = n(n+1)/2$, then $ab = T_a + T_{b-1} - T_{a-b}$.

Russian(?) Peasant

Doubling and Halving: If a is even, $a \times b = (a/2) \times (2b)$, and if a is odd, $a \times b = (a - 1 + 1) \times b = (a - 1) \times b + b$.

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Traditional way: list halvings of first number (round down) and doublings of second, add second numbers with odd first number. For example

$\surd$	41	59
	20	118
	10	236
$\surd$	5	472
	2	944
$\surd$	1	1888

$59 + 472 + 1888 = 2419.$

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41	59

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	41	59
1		59

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	41	59
1		59
2		118

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	41	59
1		59
2		118
4		236

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	41	59
1		59
2		118
4		236
8		472

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	41	59
1		59
2		118
4		236
8		472
16		944

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	41	59
1		59
2		118
4		236
8		472
16		944
32		1888

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	41	59
1		59
2		118
4		236
8		472
16		944
32	$41 - 32 = 9$	1888

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	41	59
1		59
2		118
4		236
8	$9 - 8 = 1$	472
16		944
32	$41 - 32 = 9$	1888

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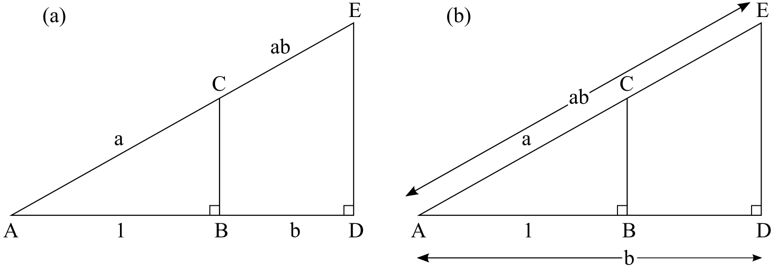
	41	59
1	$1 - 1 = 0$	59
2		118
4		236
8	$9 - 8 = 1$	472
16		944
32	$41 - 32 = 9$	1888

Geometry



Positional Definition Example

By the definition,

$$243 \times 596$$
$$= \left(2 \times 10^2 + 4 \times 10^1 + 3 \times 10^0\right) \times \left(5 \times 10^2 + 9 \times 10^1 + 6 \times 10^0\right)$$

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$$
\begin{aligned}
&243 \times 596 \\
&= \left(2 \times 10^2 + 4 \times 10^1 + 3 \times 10^0\right) \times \left(5 \times 10^2 + 9 \times 10^1 + 6 \times 10^0\right) \\
&= (2 \times 5) \times 10^4 + (2 \times 9) \times 10^3 + (2 \times 6) \times 10^2 + (4 \times 5) \times 10^3 \\
&\quad +(4 \times 9) \times 10^2 + (4 \times 6) \times 10^1 + (3 \times 5) \times 10^2 + (3 \times 9) \times 10^1 \\
&\quad +(3 \times 6) \times 10^0
\end{aligned}
$$

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$$
\begin{aligned}
&243 \times 596 \\
&= \left(2 \times 10^2 + 4 \times 10^1 + 3 \times 10^0\right) \times \left(5 \times 10^2 + 9 \times 10^1 + 6 \times 10^0\right) \\
&= (2 \times 5) \times 10^4 + (2 \times 9) \times 10^3 + (2 \times 6) \times 10^2 + (4 \times 5) \times 10^3 \\
&\quad + (4 \times 9) \times 10^2 + (4 \times 6) \times 10^1 + (3 \times 5) \times 10^2 + (3 \times 9) \times 10^1 \\
&\quad + (3 \times 6) \times 10^0 \\
&= 10 \times 10^4 + (18 + 20) \times 10^3 + (12 + 36 + 15) \times 10^2 \\
&\quad + (24 + 27) \times 10^1 + 18 \times 10^0
\end{aligned}
$$

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$$
\begin{aligned}
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&= (2 \times 10^2 + 4 \times 10^1 + 3 \times 10^0) \times (5 \times 10^2 + 9 \times 10^1 + 6 \times 10^0) \\
&= (2 \times 5) \times 10^4 + (2 \times 9) \times 10^3 + (2 \times 6) \times 10^2 + (4 \times 5) \times 10^3 \\
&\quad + (4 \times 9) \times 10^2 + (4 \times 6) \times 10^1 + (3 \times 5) \times 10^2 + (3 \times 9) \times 10^1 \\
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&= 10 \times 10^4 + (18 + 20) \times 10^3 + (12 + 36 + 15) \times 10^2 \\
&\quad + (24 + 27) \times 10^1 + 18 \times 10^0 \\
&= 10 \times 10^4 + 38 \times 10^3 + 63 \times 10^2 + 51 \times 10^1 + 18 \times 10^0
\end{aligned}
$$

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By the definition,

$$
\begin{aligned}
&243 \times 596 \\
&= (2 \times 10^2 + 4 \times 10^1 + 3 \times 10^0) \times (5 \times 10^2 + 9 \times 10^1 + 6 \times 10^0) \\
&= (2 \times 5) \times 10^4 + (2 \times 9) \times 10^3 + (2 \times 6) \times 10^2 + (4 \times 5) \times 10^3 \\
&\quad + (4 \times 9) \times 10^2 + (4 \times 6) \times 10^1 + (3 \times 5) \times 10^2 + (3 \times 9) \times 10^1 \\
&\quad + (3 \times 6) \times 10^0 \\
&= 10 \times 10^4 + (18 + 20) \times 10^3 + (12 + 36 + 15) \times 10^2 \\
&\quad + (24 + 27) \times 10^1 + 18 \times 10^0 \\
&= 10 \times 10^4 + 38 \times 10^3 + 63 \times 10^2 + 51 \times 10^1 + 18 \times 10^0 \\
&= 1 \times 10^5 + 4 \times 10^4 + 4 \times 10^3 + 8 \times 10^2 + 2 \times 10^1 + 8 \times 10^0 \\
&= 144\,228.
\end{aligned}
$$

Positional Example Continued

Laying out the digit products:

			2	4	3
		×	5	9	6
1	0				
	1	8			
		1	2		
	2	0			
		3	6		
			2	4	
		1	5		
			2	7	
	$_1$	$_1$	$_1$	1	8
1	4	4	8	2	8

Hinge Multiplication

Lay out the digit multiples, the entire second number times the digits of the first, then add. (Early Hindu texts, Medieval English)

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5	9	6		

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			2	4	3
	~~5~~	9	6		

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	1				
1	0	8			
			2	4	3
	~~5~~	~~9~~	6		

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	1	1			
1	0	8	2		
			~~2~~	4	3
	~~5~~	~~9~~	~~6~~		

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	1	1			
1	0	8	2		
			~~2~~	4	3
	~~5~~	~~9~~	~~6~~	6	
		5	9		

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	2	0			
	1	1			
1	0	8	2		
			~~2~~	4	3
	~~5~~	~~9~~	~~6~~	6	
		~~5~~	9		

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Lay out the digit multiples, the entire second number times the digits of the first, then add. (Early Hindu texts, Medieval English)

		3			
	2	0			
	1	1	6		
1	0	8	2		
			~~2~~	4	3
	~~5~~	~~9~~	~~6~~	6	
		~~5~~	~~9~~		

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		3			
	2	0	2		
	1	1	6		
1	0	8	2	4	
			~~2~~	~~4~~	3
	~~5~~	~~9~~	~~6~~	~~6~~	
		~~5~~	~~9~~		

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		3			
	2	0	2		
	1	1	6		
1	0	8	2	4	
			~~2~~	~~4~~	3
	~~5~~	~~9~~	~~6~~	~~6~~	6
		~~5~~	~~9~~	9	
			5		

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		1			
		3	5		
	2	0	2		
	1	1	6		
1	0	8	2	4	
			~~2~~	~~4~~	3
	~~5~~	~~9~~	~~6~~	~~6~~	6
		~~5~~	~~9~~	9	
			~~5~~		

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		1	2		
		3	5		
	2	0	2		
	1	1	6	7	
1	0	8	2	4	
			~~2~~	~~4~~	3
	~~5~~	~~9~~	~~6~~	~~6~~	6
		~~5~~	~~9~~	~~9~~	
			~~5~~		

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		1	2		
		3	5		
	2	0	2	1	
	1	1	6	7	
1	0	8	2	4	8
			~~2~~	~~4~~	~~3~~
	~~5~~	~~9~~	~~6~~	~~6~~	~~6~~
		~~5~~	~~9~~	~~9~~	
			~~5~~		

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1	4_1	4_1	8_1	2	8
		1	2		
		3	5		
	2	0	2	1	
	1	1	6	7	
1	0	8	2	4	8
			~~2~~	~~4~~	~~3~~
	~~5~~	~~9~~	~~6~~	~~6~~	~~6~~
		~~5~~	~~9~~	~~9~~	
			~~5~~		

Scratch Multiplication

As for hinge, but add new products above the line. Very popular in Medieval Europe.

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		2	4	3
5	9	6		

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1	0				
			2	4	3
	~~5~~	9	6		

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$$\begin{array}{cccccc} & 1 & & & & \\ 1 & \not{0} & 8 & & & \\ \hline & & & 2 & 4 & 3 \\ & \not{5} & \not{9} & 6 & & \end{array}$$

Scratch Multiplication

As for hinge, but add new products above the line. Very popular in Medieval Europe.

	1	9			
1	~~0~~	~~8~~	2		
			~~2~~	4	3
	~~5~~	~~9~~	~~6~~		

Scratch Multiplication

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	1	9			
1	~~0~~	~~8~~	2		
			~~2~~	4	3
	~~5~~	~~9~~	~~6~~	6	
		5	9		

Scratch Multiplication

As for hinge, but add new products above the line. Very popular in Medieval Europe.

	3				
	~~1~~	9			
1	~~0~~	~~8~~	2		
			~~2~~	4	3
	~~5~~	~~9~~	~~6~~	6	
		~~5~~	9		

Scratch Multiplication

As for hinge, but add new products above the line. Very popular in Medieval Europe.

	4				
	~~3~~	2			
	~~1~~	~~9~~	8		
1	~~0~~	~~8~~	~~2~~		
			~~2~~	4	3
	~~5~~	~~9~~	~~6~~	6	
		~~5~~	~~9~~		

Scratch Multiplication

As for hinge, but add new products above the line. Very popular in Medieval Europe.

	4	3			
	~~3~~	~~2~~	0		
	~~1~~	~~9~~	~~8~~		
1	~~0~~	~~8~~	~~2~~	4	
			~~2~~	~~4~~	3
	~~5~~	~~9~~	~~6~~	~~6~~	
		~~5~~	~~9~~		

Scratch Multiplication

As for hinge, but add new products above the line. Very popular in Medieval Europe.

	4	3			
	~~3~~	~~2~~	0		
	~~1~~	~~9~~	~~8~~		
1	~~0~~	~~8~~	~~2~~	4	
			~~2~~	~~4~~	3
	~~5~~	~~9~~	~~6~~	~~6~~	6
		~~5~~	~~9~~	9	
			5		

Scratch Multiplication

As for hinge, but add new products above the line. Very popular in Medieval Europe.

		4			
	4	~~3~~	5		
	~~3~~	~~2~~	~~0~~		
	~~1~~	~~9~~	~~8~~		
1	~~0~~	~~8~~	~~2~~	4	
			~~2~~	~~4~~	3
	~~5~~	~~9~~	~~6~~	~~6~~	6
		~~5~~	~~9~~	9	
			~~5~~		

Scratch Multiplication

As for hinge, but add new products above the line. Very popular in Medieval Europe.

		4	8		
	4	~~3~~	~~5~~		
	~~3~~	~~2~~	~~0~~		
	~~1~~	~~9~~	~~8~~	1	
1	~~0~~	~~8~~	~~2~~	~~4~~	
			~~2~~	~~4~~	3
	~~5~~	~~9~~	~~6~~	~~6~~	6
		~~5~~	~~9~~	~~9~~	
			~~5~~		

Scratch Multiplication

As for hinge, but add new products above the line. Very popular in Medieval Europe.

		4	8		
	4	~~3~~	~~5~~		
	~~3~~	~~2~~	~~0~~	2	
	~~1~~	~~9~~	~~8~~	~~1~~	
1	~~0~~	~~8~~	~~2~~	~~4~~	8
			~~2~~	~~4~~	~~3~~
	~~5~~	~~9~~	~~6~~	~~6~~	~~6~~
		~~5~~	~~9~~	~~9~~	
			~~5~~		

Scratch Multiplication

As for hinge, but add new products above the line. Very popular in Medieval Europe.

		4	8		
	4	~~3~~	~~5~~		
	~~3~~	~~2~~	~~0~~	2	
	~~1~~	~~9~~	~~8~~	~~1~~	
1	~~0~~	~~8~~	~~2~~	~~4~~	8
—	—	—	—	—	—
			~~2~~	~~4~~	~~3~~
	~~5~~	~~9~~	~~6~~	~~6~~	~~6~~
		~~5~~	~~9~~	~~9~~	
			~~5~~		

$$243 \times 596 = 144\,828$$

Cross Multiplication

In terms of digits, $abc \times def = ad \times 10^4 + (ae + bd) \times 10^3 + (af + be + cd) \times 10^2 + (bf + ce) \times 10^1 + cf \times 10^0$.

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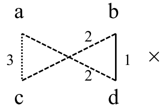
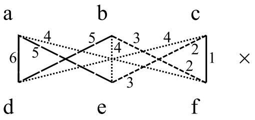


ac	ad	bd
	bc	



ad	bd	cd	ce	cf
	ae	be	bf	
		af		

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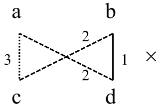
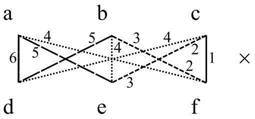


ac	ad	bd
	bc	



ad	bd	cd	ce	cf
	ae	be	bf	
		af		

Same effort as hinge, different order of digits. Recommended for mental arithmetic.

Lattice

Hinge separates digit multiples from carries, scratch and cross don't. Lattice is like hinge, but easier.

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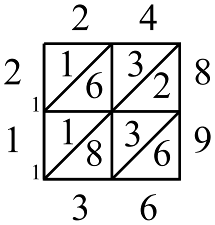
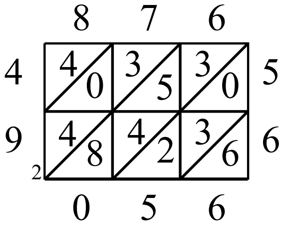
For example, $24 \times 89 = 2136$ and $876 \times 56 = 49\,056$.





Napier's Rods

To make digit products easier, in 1617 John Napier built rods engraved with the digit multiplication table.

0	1	2	3	4	5	6	7	8	9
0/0	0/2	0/4	0/6	0/8	1/0	1/2	1/4	1/6	1/8
0/0	0/3	0/6	0/9	1/2	1/5	1/8	2/1	2/4	2/7
0/0	0/4	0/8	1/2	1/6	2/0	2/4	2/8	3/2	3/6
0/0	0/5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5
0/0	0/6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4
0/0	0/7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3
0/0	0/8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2
0/0	0/9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1

Napier's Rods Example

Consider 878×944.

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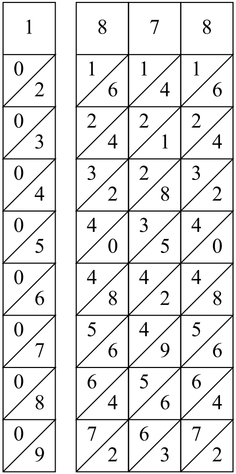

3512

7902

Napier's Rods Example

Consider 878×944.

1	8	7	8	
0/2	1/6	1/4	1/6	
0/3	2/4	2/1	2/4	
0/4	3/2	2/8	3/2	3512
0/5	4/0	3/5	4/0	
0/6	4/8	4/2	4/8	
0/7	5/6	4/9	5/6	
0/8	6/4	5/6	6/4	
0/9	7/2	6/3	7/2	7902

$$
\begin{array}{cccccc}
 & & & 8 & 7 & 8 \\
 & & \times & 9 & 4 & 4 \\
\hline
 & & 3 & 5 & 1 & 2 \\
 & 3 & 5 & 1 & 2 & \\
7 & 9 & 0 & 2 & & \\
\hline
8_1 & 2 & 8 & 8 & 3 & 2
\end{array}
$$

or

$$
\begin{array}{cccccc}
 & & & 8 & 7 & 8 \\
 & & \times & 9 & 4 & 4 \\
\hline
7 & 9 & 0 & 2 & & \\
 & 3 & 5 & 1 & 2 & \\
 & & 3 & 5 & 1 & 2 \\
\hline
8_1 & 2 & 8 & 8 & 3 & 2
\end{array}
$$

The Modern Method

Just like Napier's Rods, but you have to do digit multiplications and intermediate carries yourself.

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$$\begin{array}{rrrr} & & {\scriptstyle 3} & \\ & 8 & 7 & 8 \\ \times & 9 & 4 & 4 \\ \hline & & & 2 \end{array}$$

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$$\begin{array}{rrrr} & {\scriptstyle 3} & {\scriptstyle 3} & \\ & 8 & 7 & 8 \\ \times & 9 & 4 & 4 \\ \hline & & 1 & 2 \end{array}$$

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$$\begin{array}{rrrr} & {\scriptstyle 3} & {\scriptstyle 3} & \\ & 8 & 7 & 8 \\ \times & 9 & 4 & 4 \\ \hline 3 & 5 & 1 & 2 \end{array}$$

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Just like Napier's Rods, but you have to do digit multiplications and intermediate carries yourself. For example

$$\begin{array}{cccc} & 3 & & \\ & \not{3} & 3 & \\ & 8 & 7 & 8 \\ \times & 9 & 4 & 4 \\ \hline 3 & 5 & 1 & 2 \\ & & 2 & \end{array}$$

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$$\begin{array}{cccc}
 & {\scriptstyle 3} & & \\
{\scriptstyle 3} & {\scriptstyle \cancel{3}} & {\scriptstyle 3} & \\
 & 8 & 7 & 8 \\
\times & 9 & 4 & 4 \\
\hline
3 & 5 & 1 & 2 \\
 & 1 & 2 &
\end{array}$$

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$$
\begin{array}{ccccc}
 & & 3 & & \\
 & 3 & \not{3} & 3 & \\
 & & 8 & 7 & 8 \\
 & \times & 9 & 4 & 4 \\
\hline
 & 3 & 5 & 1 & 2 \\
3 & 5 & 1 & 2 &
\end{array}
$$

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$$
\begin{array}{ccccc}
 & 7 & 3 & & \\
 & \not{3} & \not{3} & 3 & \\
 & & 8 & 7 & 8 \\
 & \times & 9 & 4 & 4 \\
\hline
 & 3 & 5 & 1 & 2 \\
3 & 5 & 1 & 2 & \\
 & & 2 & &
\end{array}
$$

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$$
\begin{array}{ccccc}
 & 7 & 3 & & \\
7 & \cancel{3} & \cancel{3} & 3 & \\
 & & 8 & 7 & 8 \\
 & \times & 9 & 4 & 4 \\
\hline
 & 3 & 5 & 1 & 2 \\
3 & 5 & 1 & 2 & \\
 & 0 & 2 & &
\end{array}
$$

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$$
\begin{array}{cccccc}
 & & 7 & 3 & & \\
 & 7 & \cancel{3} & \cancel{3} & 3 & \\
 & & & 8 & 7 & 8 \\
 & & \times & 9 & 4 & 4 \\
\hline
 & & 3 & 5 & 1 & 2 \\
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7 & 9 & 0 & 2 & & \\
\hline
\end{array}
$$

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		7	3		
	7	~~3~~	~~3~~	3	
			8	7	8
		×	9	4	4
		3	5	1	2
	3	5	1	2	
7	9	0	2		
					2

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	7	~~3~~	~~3~~	3	
			8	7	8
		×	9	4	4
		3	5	1	2
	3	5	1	2	
7	9	0	2		
				3	2

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$$
\begin{array}{cccccc}
 & & 7 & 3 & & \\
 & 7 & \not{3} & \not{3} & 3 & \\
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 & & 3 & 5 & 1 & 2 \\
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\hline
 & & 8 & 8 & 3 & 2
\end{array}
$$

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$$
\begin{array}{cccccc}
 & & \scriptstyle 7 & \scriptstyle 3 & & \\
\scriptstyle 1 & \scriptstyle 7 & \scriptstyle \cancel{3} & \scriptstyle \cancel{3} & \scriptstyle 3 & \\
 & & & 8 & 7 & 8 \\
 & & \times & 9 & 4 & 4 \\
\hline
 & & 3 & 5 & 1 & 2 \\
 & 3 & 5 & 1 & 2 & \\
7 & 9 & 0 & 2 & & \\
\hline
 & 2 & 8 & 8 & 3 & 2
\end{array}
$$

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		7	3		
1	7	~~3~~	~~3~~	3	
			8	7	8
		×	9	4	4
		3	5	1	2
	3	5	1	2	
7	9	0	2		
8	2	8	8	3	2

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$$
\begin{array}{cccccc}
 & & 7 & 3 & & \\
1 & 7 & \not{3} & \not{3} & 3 & \\
 & & & 8 & 7 & 8 \\
 & & \times & 9 & 4 & 4 \\
\hline
 & & 3 & 5 & 1 & 2 \\
 & 3 & 5 & 1 & 2 & \\
7 & 9 & 0 & 2 & & \\
\hline
8 & 2 & 8 & 8 & 3 & 2
\end{array}
\qquad
\begin{array}{cccccc}
 & & & 8 & 7 & 8 \\
 & & \times & 9 & 4 & 4 \\
\hline
 & & 3 & 5_3 & 1_3 & 2 \\
 & 3 & 5_3 & 1_3 & 2 & \\
7 & 9_7 & 0_7 & 2 & & \\
\hline
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\end{array}
$$

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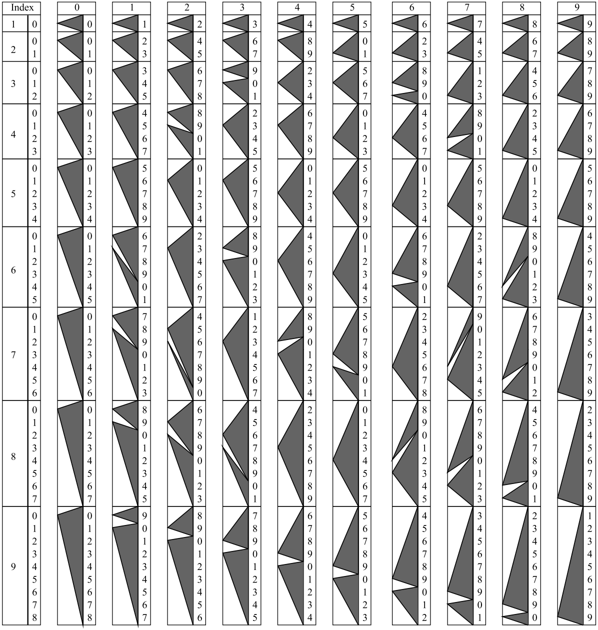
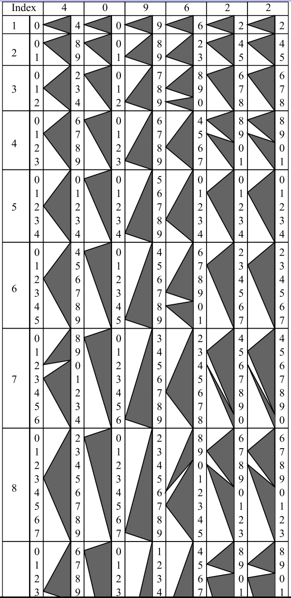
I prefer the second: product and sum carries are with the associated numbers.

Genaille's Rods 1891, Napier's rods without carries



Genaille's Rods Example



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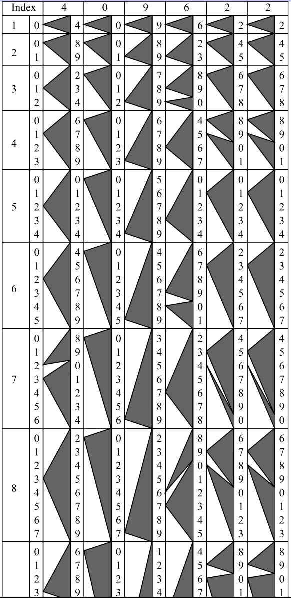

$$
\begin{array}{ccccccccc}
 & & & 4 & 0 & 9 & 6 & 2 & 2 \\
 & & \times & & & & 3 & 8 & 8 \\
\hline
 & & 3 & 2 & 7 & 6 & 9 & 7 & 6 \\
 & 3 & 2 & 7 & 6 & 9 & 7 & 6 & \\
1 & 2 & 2 & 8 & 8 & 6 & 6 & & \\
\hline
1 & 5 & 8_1 & 9_2 & 3_2 & 3_2 & 3_1 & 3 & 6
\end{array}
$$

Prosthaphaeresis

$$\cos a \cos b = \frac{1}{2}(\cos(a+b) + \cos(a-b)).$$

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$xy = \frac{1}{2}(\cos(148^\circ 0' 57'') + \cos(-19^\circ 18' 55'')) \times 10^3 = \frac{1}{2}(-0.848194503 + 0.943712787) \times 10^3 = 0.047759142 \times 10^3 = 47.759142$.

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Logarithms

Napier (1614): if $y = \log x$ then $x/10^7 = (1 - 10^7)^y$. Then $\log 10^7 = 0$, logs increase as the number decreases, and $\log xy = \log x + \log y$.

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Slide Rule (Oughtred 1622): Rulers with logarithmic scales add lengths to multiply numbers.

Pascal's Triangle and Powers of Eleven

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

Pascal's Triangle and Powers of Eleven

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

$$11^0 = 1$$
$$11^1 = 11$$
$$11^2 = 121$$
$$11^3 = 1331$$
$$11^4 = 14\,641$$
$$11^5 = 161\,051$$
$$11^6 = 1\,771\,561$$

Pascal's Triangle and Powers of Eleven

							1							
						1		1						
					1		2		1					
				1		3		3		1				
			1		4		6		4		1			
		1		5		10		10		5		1		
	1		6		15		20		15		6		1	

$$
\begin{aligned}
11^0 &= 1\\
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11^3 &= 1331\\
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11^5 &= 161\,051\\
11^6 &= 1\,771\,561
\end{aligned}
$$

Using carries, Pascal's triangle rows give powers of eleven.

Explanation

	a	b	c	d	
			1	1	$\times$
	a	b	c	d	
a	b	c	d		
a	$a+b$	$b+c$	$c+d$	d	

so

	a		b		c		d	
a		$a+b$		$b+c$		$c+d$		d

Explanation

	a	b	c	d	$\times$
			1	1	
	a	b	c	d	
a	b	c	d		
a	$a+b$	$b+c$	$c+d$	d	

each number is the sum of the pair diagonally above.

so

$$\begin{array}{ccccccccc} & a & & b & & c & & d & \\ a & & a+b & & b+c & & c+d & & d \end{array}$$

Generalization

Start with one, then if each digit is b times upper left plus a times upper right, each row is a power of $a \times 10 + b$.

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$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 2 & & 7 & & & & \\
 & & & 4 & & 28 & & 49 & & & \\
 & & 8 & & 84 & & 294 & & 343 & & \\
 & 16 & & 224 & & 1176 & & 2744 & & 2401 & \\
32 & & 560 & & 3920 & & 13\,720 & & 24\,010 & & 16\,807
\end{array}
$$

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Start with one, then if each digit is b times upper left plus a times upper right, each row is a power of $a \times 10 + b$. For example 27:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 2 & & 7 & & & & \\
 & & & 4 & & 28 & & 49 & & & \\
 & & 8 & & 84 & & 294 & & 343 & & \\
 & 16 & & 224 & & 1176 & & 2744 & & 2401 & \\
32 & & 560 & & 3920 & & 13\,720 & & 24\,010 & & 16\,807
\end{array}
$$

$$27^0 = 1,\ 27^1 = 27,\ 27^2 = 729,\ 27^3 = 19\,683,\ 27^4 = 531\,441,$$
$$27^5 = 14\,348\,907.$$

Karatsuba

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Practically better than traditional method with more than ~ 400 (decimal) digits.

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This version is $O(n^{\log_3 5}) \approx O(n^{1.465})$, but has a larger constant than Karatsuba. Better with more than 700 digits.

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Best with more than about ten to forty thousand digits.

Conclusion

So, just how would you like to multiply now?